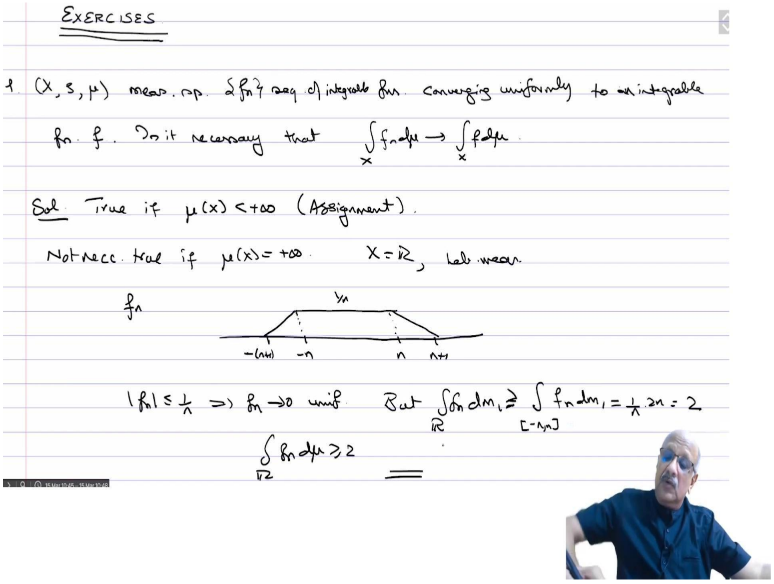
Measure and Integration

Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences

Lecture No-40

(Refer Slide Time: 0:01)



We now do some exercises:

1. (X, S, μ) measure space, $\{f_n\}$ sequence of measurable functions converging uniformly to an integrable function to an integrable function f. Is it necessary that

$$\int_X f_n d\mu \to \int_X f d\mu.$$

Solution: True if $\mu(X) < \infty$ (assignment). So, not necessarily true if $\mu(X) = +\infty$.

So, we have to give a counterexample. So, let us take $X = \mathbb{R}$, Lebesgue measure. So, fn is the function which is given like this and draw the picture you can so, this is minus of n plus 1, this minus n this is n and this is n plus 1.

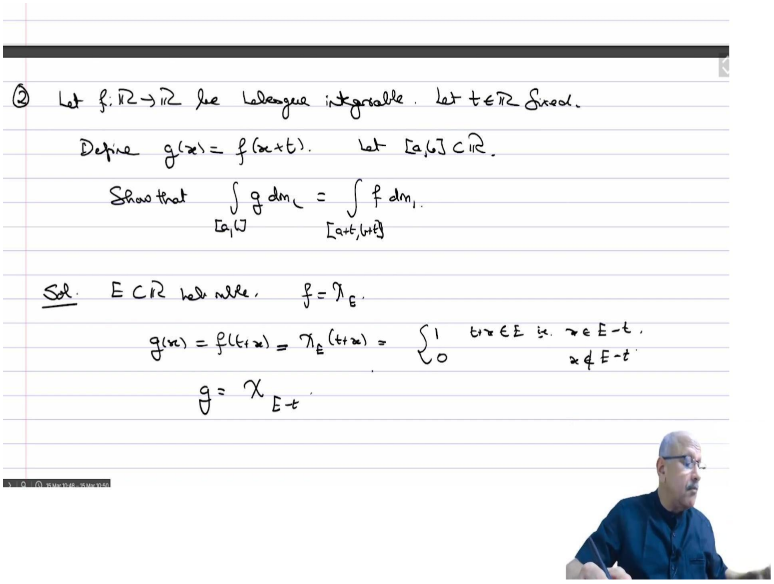
So, the function is equal to 1 by n from minus n to n and then goes to 0 here, so, fn is a function with compact support therefore, fn is definitely integrable and then

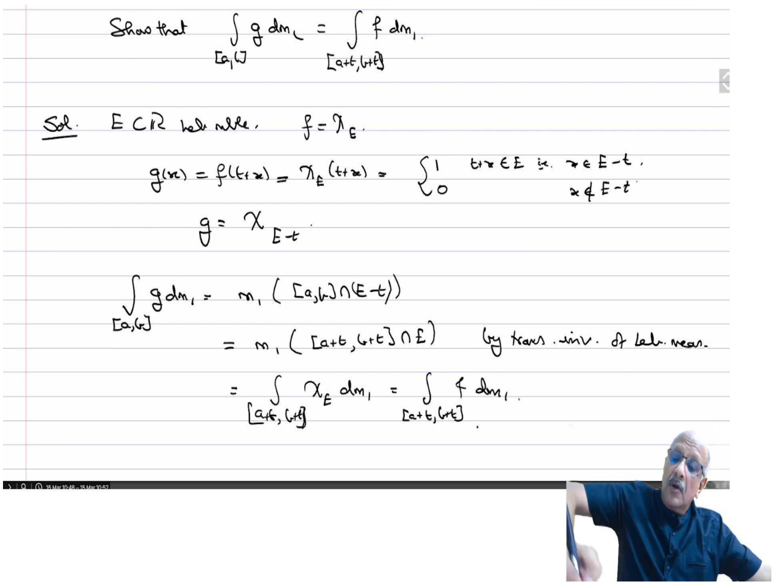
$$|f_n| \leq \frac{1}{n} \Rightarrow f_n \to 0 \; unif.$$

But $\int_{\mathbb{R}} f_n dm_1 \geq \int_{[-n,n]} f_n dm_1 = \frac{1}{n}.2n = 2 \Rightarrow \int_{\mathbb{R}} f_n dm_1 \geq 2.$

Therefore, it does not converge to 0 and therefore is not necessarily true. But if it is a finite metric space then uniform convergence does imply whatever this result is, and that is very easy to check. Just apply the definition and you can do it like that.

(Refer Slide Time: 3:54)





(2) Let $f: \mathbb{R} \to \mathbb{R}$ be Lebesgue integrable. Let $t \in \mathbb{R}$ be fixed. Define

$$g(x) = f(x + t).$$

Let $[a, b] \subset \mathbb{R}$. Show that $\int_{[a,b]} f dm_1 = \int_{[a+t,b+t]} g dm_1$.

Solution: So, let $E \subset \mathbb{R}$ Lebesgue measurable and let $f = \chi_E$. So,

$$g(x) = f(t + x) = \chi_E(t + x) = 1, \; if \; x \in E - t,$$

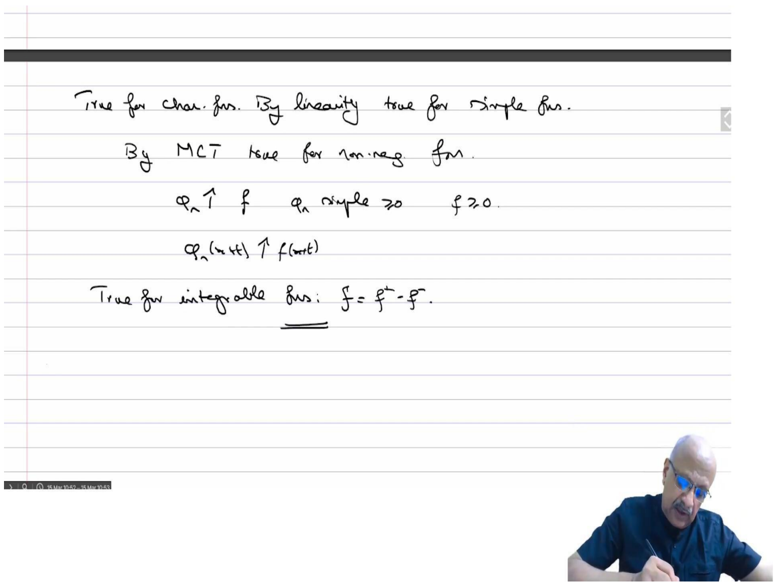
$$= 0, \; if \; x \notin E - t.$$

So, this means that $g(x) = \chi_{E-t}$. So,

$$\int_{[a+t,b+t]} g dm_1 = m_1([a, b] \cap (E - t)) = m_1([a + t, b + t] \cap E)$$

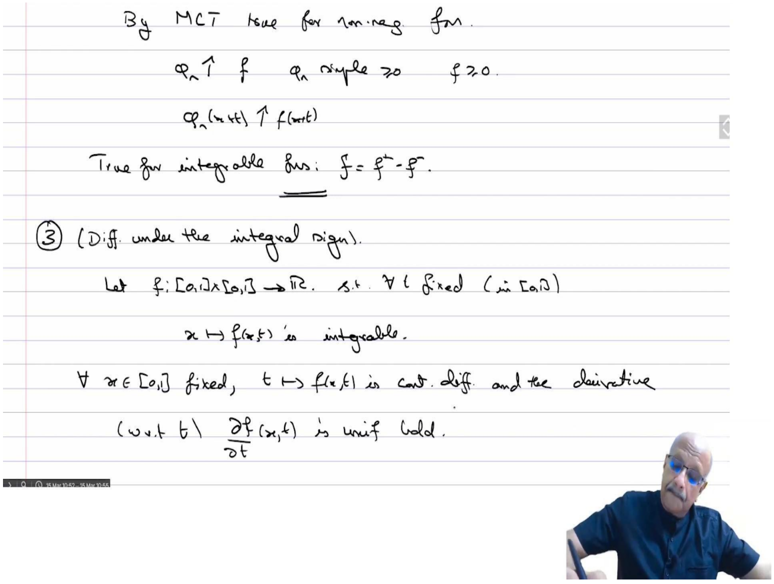
$$= \int_{[a+t,b+t]} \chi_E dm_1 = \int_{[a+t,b+t]} f dm_1.$$

(Refer Slide Time: 7:18)



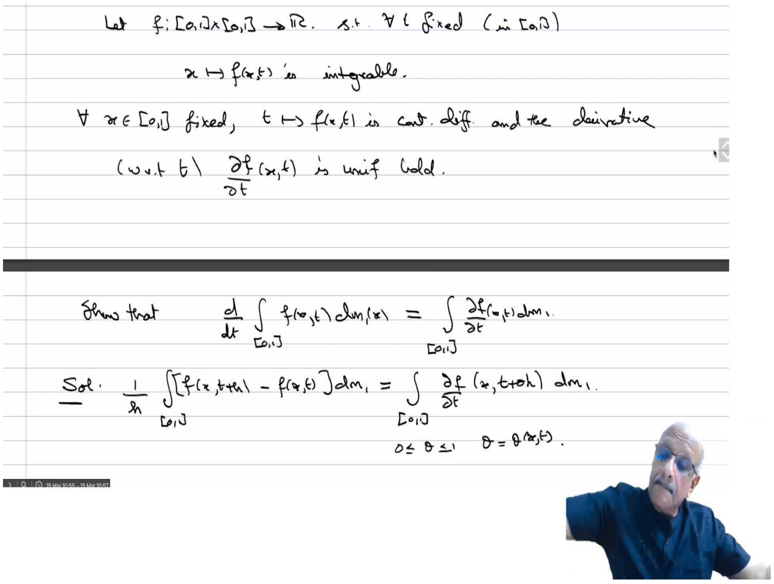
So, true for characteristic functions by linearity true for simple functions then by monotone convergence theorem true for non-negative functions, because every non-negative measurable function is the increasing limit of the of simple functions. So, given any non-negative function f you can take $\phi_n \uparrow f$, ϕ_n simple non negative, f is also non negative then what do you then of course, phi n of x plus t will increase the f of x plus t and then everything is true then by the monotone convergence theorem you can easily prove the result and then prove for integrable functions because, you take $f = f^{+} - f^{-}$, so, you use this so, we all are most of our work is there only for the functions which are characteristic functions.

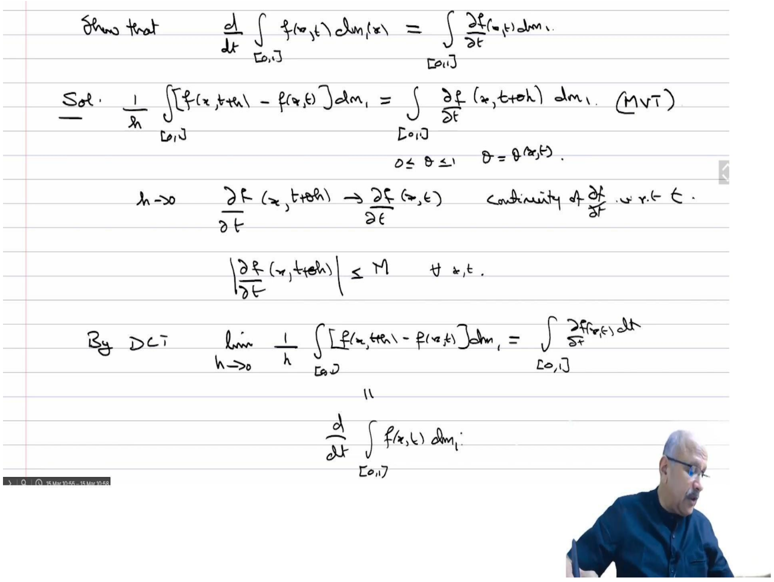
(Refer Slide Time: 8:54)



(3) (Differentiation under the integrand sign): Let $f: [0,1] \times [0,1] \rightarrow \mathbb{R}$, such that for every t fixed you have (of course, in $[0,1]$) $x \rightarrow f(x,t)$ is integrable. For every $x \in [0,1]$ fixed $t \rightarrow f(x,t)$ is continuously differentiable and the derivative (with respect to t) which we define by $\frac{\partial f}{\partial t}(x,t)$ is uniformly bounded.

(Refer Slide Time: 10:36)





Show that $\frac{d}{dt}\int\limits_{[0,1]} f(x,t)dm_1(x) = \int\limits_{[0,1]} \frac{\partial f}{\partial t}(x,t)dm_1$.

Solution: So, let us look at the difference

$$\frac{1}{h}\int\limits_{[0,1]} [f(x,t+h) - f(x,t)]dm_1 = \int\limits_{[0,1]} \frac{\partial f}{\partial t}(x,t+h)dm_1 \text{ (MVT)}$$

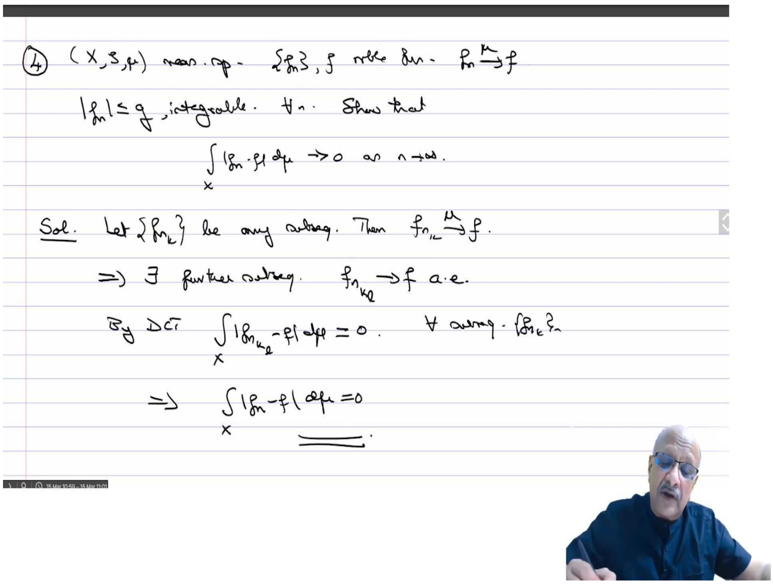
Now, as $h \to 0, \frac{\partial f}{\partial t}(x,t+\theta h) \to \frac{\partial f}{\partial t}(x,t+h)\ and\ |\frac{\partial f}{\partial t}(x,t+\theta h)| \leq M,\ \forall\, x,t.$

Therefore, by the dominated convergence theorem

$$\lim_{h\to 0} \frac{1}{h}\int\limits_{[0,1]} [f(x,t+h) - f(x,t)]dm_1 = \int\limits_{[0,1]} \frac{\partial f}{\partial t}(x,t)dm_1.$$

So, a very nice application of the dominated convergence theorem to show that the differentiation under the integral sign always requests some proof and therefore, this is 1 set of sufficient conditions which will tell you that you have this condition.

(Refer Slide Time: 14:05)



(4) (X, S, μ) measure space, f_n, f measurable functions and f_n converges in measure to f, $|f_n| \le g$, integrable for all n. Show that

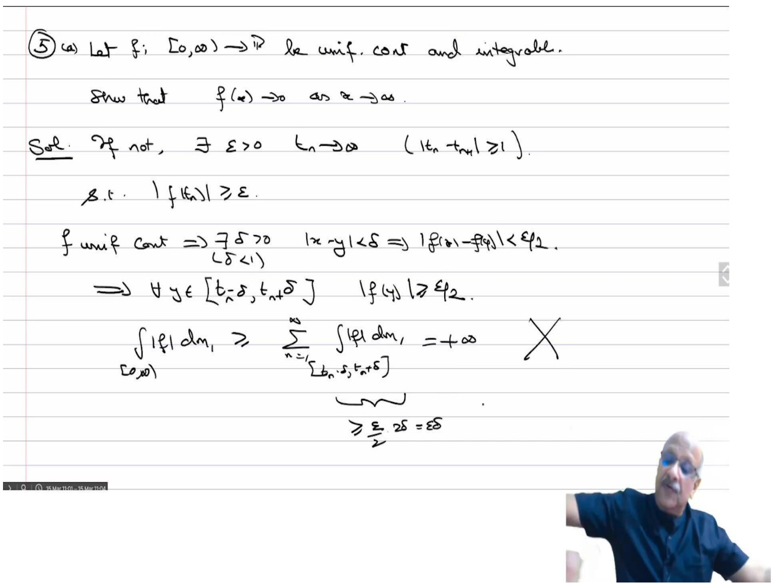
$$\int_X |f_n - f| d\mu \to 0 \; as \; n \to \infty.$$

solution: Let f_{n_k} be any sub sequence. Then f_{n_k} converges in measure to f.

$\Rightarrow \exists$ further sub sequence $f_{n_{k_l}}$ converges to f a.e.

Then by the dominated convergence theorem, you have $\int_X |f_{n_{k_l}} - f| d\mu = 0 \; \forall$ subsequence $\{f_{n_k}\}$. $\Rightarrow \int_X |f_n - f| d\mu = 0.$

(Refer Slide Time: 16:54)



(5) (a) Let $f\colon [0, \infty] \rightarrow \mathbb{R}$ be uniformly continuous and integrable. Show that

$$f(x) \rightarrow 0 \; as \; x \rightarrow \infty.$$

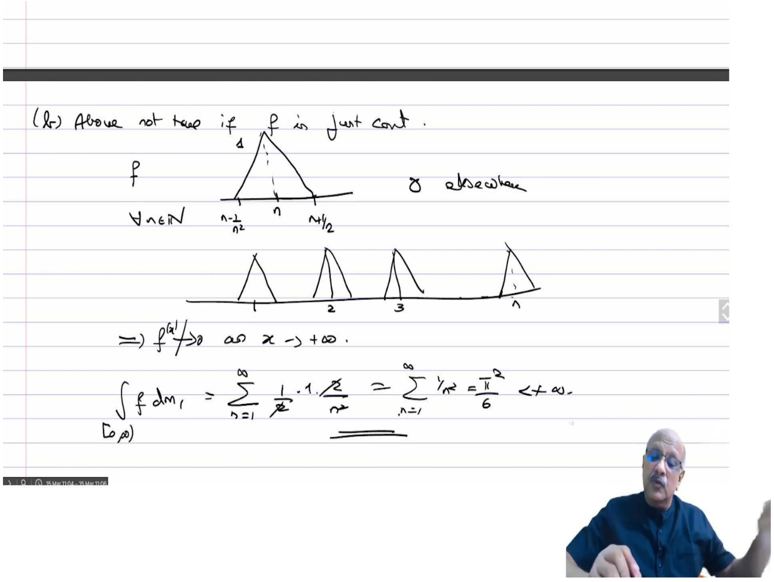
solution: If not then you can find an epsilon greater than 0 and $t_n \rightarrow \infty$ (and you can also take $|t_n - t_{n-1}| \geq 1)$ $s.t.$ $|f(t_n)| \geq \epsilon$. Now, f is uniformly continuous $\Rightarrow$ there exists a $\delta > 0$ $s.t.$ $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$.

$$\Rightarrow \forall\, y \in [t_n - \delta, t_n + \delta], \; |f(y)| \geq \frac{\epsilon}{2}.$$

But then $\int_{[0,\infty]} |f| dm_1 \geq \sum_{n=1}^{\infty} \int_{[t_n - \delta,\, t_n + \delta]} |f| dm_1 = +\infty$.

So, this is not possible and therefore, $f(x) \rightarrow 0 \; as \; x \rightarrow \infty$.

(Refer Slide Time: 19:48)



(b) Above is not true if f is just continuous. So, let us take, so, we have to give an example. So, let us take a function fn f in the following function. So, f is such that each n you take n minus 1 by n square, n plus 1 by n square and then it is the function is height 1 here and then it goes to 0 and 0 elsewhere, so this is for every n in N, so, what does the function look like? So, it looks like this 1 and then here is 2, 3 and so on. So, the function is like this. So, this implies that f does not go to 0, fx as x tends to plus infinity but what is integral of dm1 a non-negative function is nothing but sigma n equals 1 to infinity.

The area of all these triangles the height is 1, base is 1 by 2, height is 1. And I mean 1 by 2 into height is 1 and base is 2 by n squared which is sigma 1 by n squared which we know is pi squared by 6 which is of course, finite therefore, f is integrable but it does not vanish at infinity. So, we will continue with the exercises next time.