Measure and Integration
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Lecture No- 4
1.4 Outer measure

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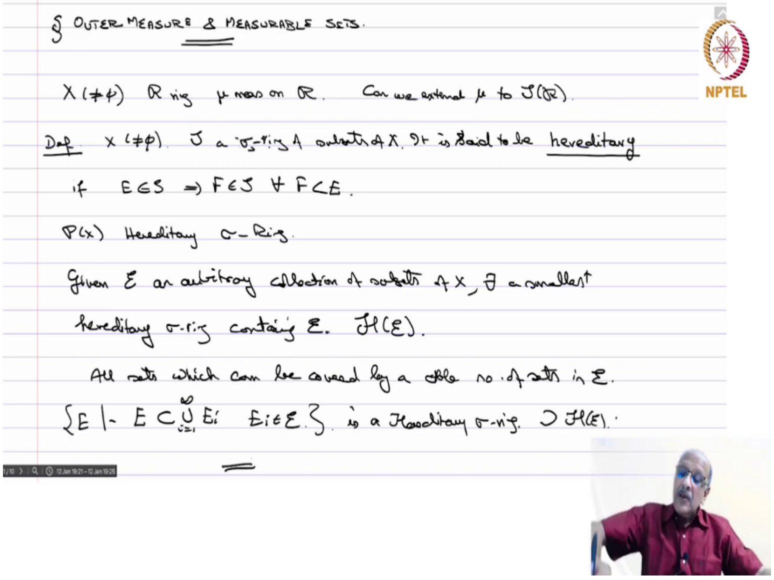

Our next topic is outer measure and measurable sets. So, we already stated this problem before X is a non-empty set, R is a ring on X and μ is a measure on R. Then we wanted to know if we can extend μ to something bigger. For instance, R is a ring. So, in particular it is a collection of subsets. So, can we extend μ to $S(R)$. $S(R)$ is the smallest sigma ring, which contains R, because we are dealing with measure countable activity and so on.

So, it is always better to work with sigma ring or sigma algebra because countable unions, countable intersections are close there rather than with the ring. So, once we, but to construct something it is easier on a ring and therefore we want to know if there is some general method to extend it to the whole thing. So, for this, we have a certain, very nice abstract procedure, which we will in fact employ when constructing the limit measures later on.

Definition: X non-empty set, S- σ ring of subsets of X. It is set to be hereditary if $E \in S$ implies $F \in S$ for every $F \subset E$.

So, if the father has a property, the son, children get it also. So, that is the heritage property. So, here, if you have a set E which is an S, then all subsets automatically belong to S. Now of course, $P(X)$ is a hereditary sigma ring.

And now if you, as usual, if you take any intersection of hereditary sigma rings, it is automatically hereditary. So, it is very easy to check because the intersection of sigma rings is a sigma ring and the hereditary property. Again, it is very easy to check that it is. So, given E, an arbitrary collection of subsets of X, there exists a smallest hereditary sigma ring containing E: $H(E)$, This is called the hereditary sigma ring generated by E.

So, that is the notation. Now, if you take all sets, which can be covered by a countable number of sets in E that means, so this,

$\{A\colon A \subset \cup_{i=1}^{\infty} A_i,\ A_i \in E\}$ is a hereditary sigma ring, and therefore it contains $H(E)$.

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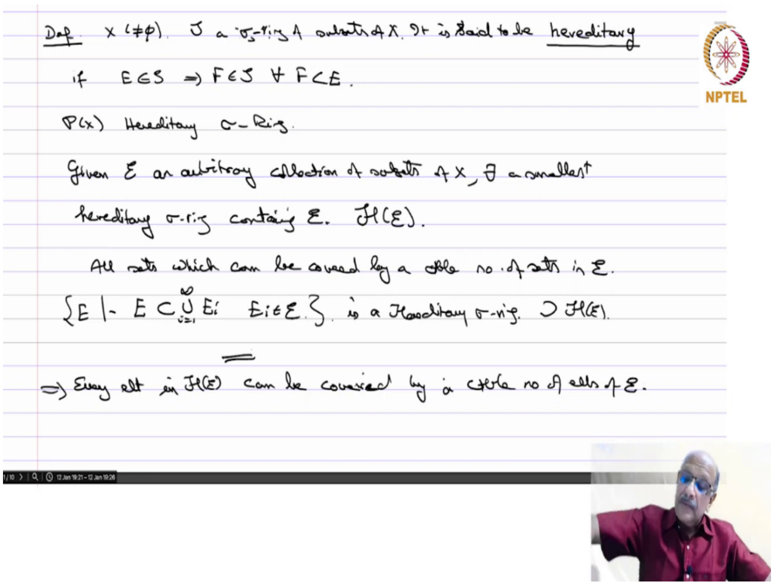

So, the conclusion is every element in $H(E)$ can be covered by a countable number of elements of E.

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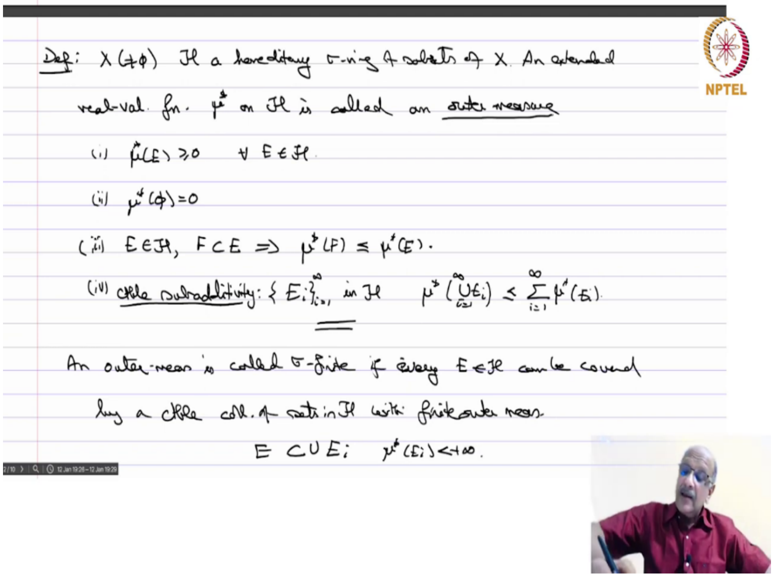

So, now we come to a definition, another definition.

Definition: So, X non-empty set, and H a hereditary σ-ring of subsets of X. An extended real valued function, μ^* on H is called an outer measure if

(1) $\mu^*(E) \geq 0, \forall E \in H,$

(2) $\mu^*(\phi) = 0,$

(3) $E \in H,\ F \subset E \Rightarrow \mu^*(F) \leq \mu^*(E),$

(4) Countable subadditivity: $\{E_i\}_{i=1}^{\infty}$ in $H \Rightarrow \mu^*(\cup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \mu^*(E_i).$

So, this looks like a measure, but there are some differences.

First 2 properties are just the properties of a measure. And then third is a consequence of the properties of measure, which you are now putting in here, because that was got from the a countable additivity, which we do not have now. And it is a countable additive, which involves disjoint sets and so on. Now, we are saying it is subadditive, which is also a property of a measure.

So, it has some properties of a measure, but it is not necessarily a measure.

So, then an outer measure is called σ-finite, if every $E \in H$ can be covered by a countable collection of sets in H, with finite out of measure. So, $E \subset \cup_{i=1}^{\infty} E_i$, $\mu * (E_i) < \infty \ \forall i$.

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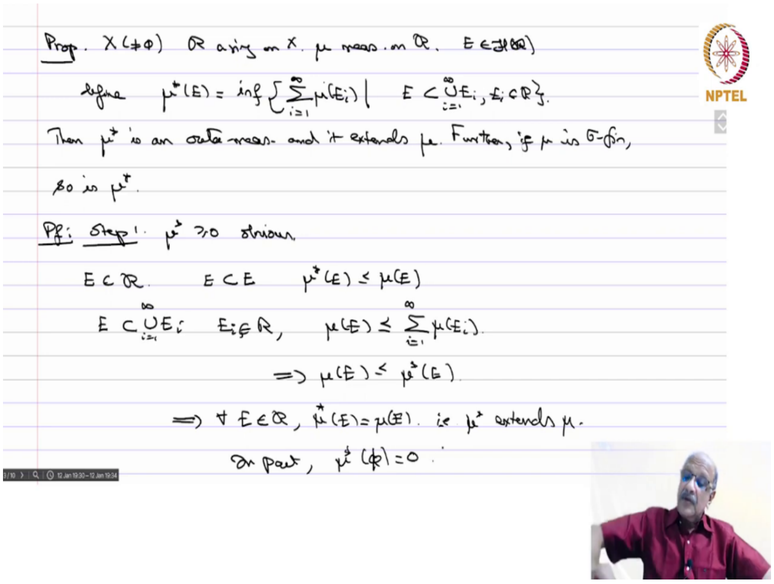

Just as we had sigma finite measures, we can have sigma finite outer measures. So, why are we defining this? Outer measures occur very naturally when we try to extend the measure. So, we have the following proposition.

Proposition: X is a non-empty set, and R a ring on X, and μ is a measure on R. So, now you take $E \in H(R)$, the smallest hereditary sigma ring containing R.

Define $\mu^*(E) = \inf\{\sum_{i=1}^{\infty} \mu(E_i) : E \subset \cup_{i=1}^{\infty} E_i,\ E_i \in R\}$. Then μ^* is an outer measure and it extends to μ . Further, if μ is sigma finite, so is μ^*.

proof. **Step 1.** So, $\mu^* \geq 0$ is obvious because you are taking an infimum of non-negative things and therefore it is non-negative.

Now, let us take $E \in R$, then $E \subset E$. So, therefore you have $\mu^*(E) \leq \mu(E)$.

So, now on the other hand, if you are given $E \subset \cup_{i=1}^{\infty} E_i$, then by subadditivity of a measure

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i) \Rightarrow \mu(E) \leq \mu^*(E).$$

$$\Rightarrow \forall E \in R,\ \mu^*(E) = \mu(E), \text{ i.e., } \mu^* \text{ extends to } \mu.$$

In particular, $\mu^*(\phi) = 0$.

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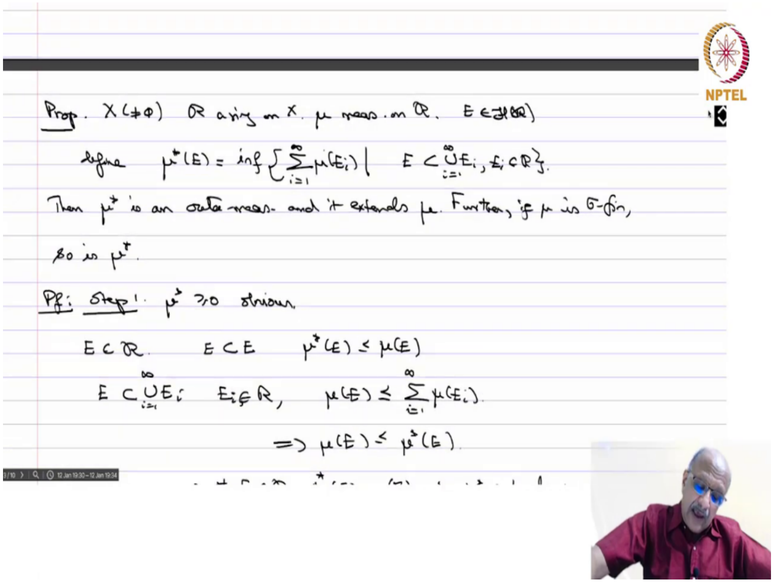

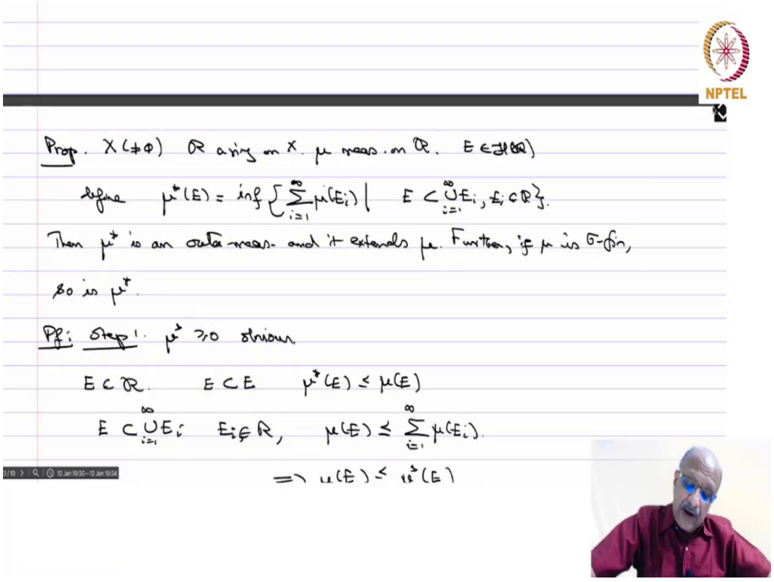

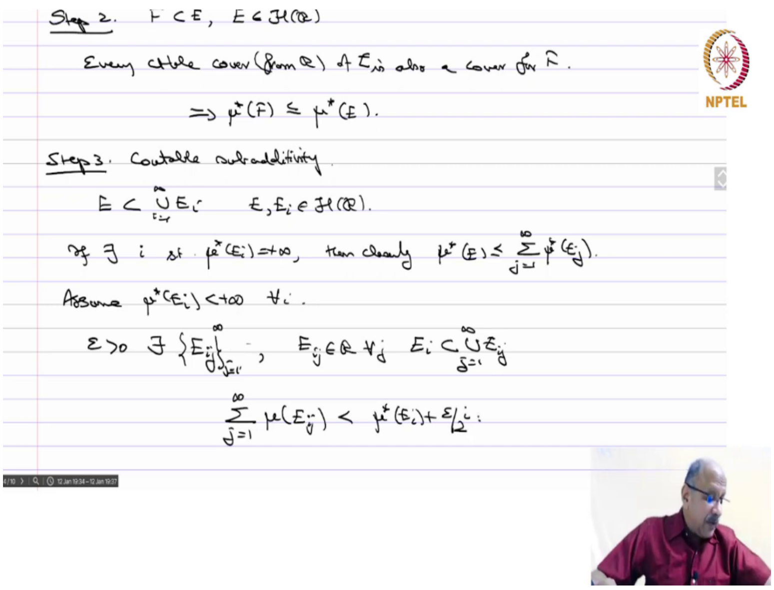

Step 2: Suppose $F \subset E$, $E \in H(R)$. Then of course, it is a hereditary sigma ring. So, F is also in $H(R)$. So, every countable cover (from R) of E is also a cover for F. So, you are number of covers for F is more than the number of covers for E, and therefore automatically this in implies that $\mu*(F) \leq \mu*(E)$.

Step 3: We prove countable subadditivity: So, let us take $E \subset \cup_{i=1}^{\infty} E_i$, $E, E_i \in H(R)$. So, if there exists an i such that $\mu*(E_i) = \infty$, then clearly $\mu*(E) \leq \sum_{i=1}^{\infty} \mu*(E_i)$.

So, therefore assume $\mu*(E_i) < \infty$, $\forall i$. Given any $\epsilon > 0$, there exists $E_{ij} \in R$, j equals 1 to infinity, such that $E_i \subset \cup_{i=1}^{\infty} E_{ij}$ and you have $\sum_{j=1}^{\infty} \mu(E_{ij}) < \mu*(E_i) + \frac{\epsilon}{2^i}$.

This is just the definition of the infimum. So, you are defining it to be the infimum over all countable covers. So, I can find a countable cover, which is bounded by mu star E, plus any small quantity I like, and I am going to call that quantity epsilon by 2 power i.

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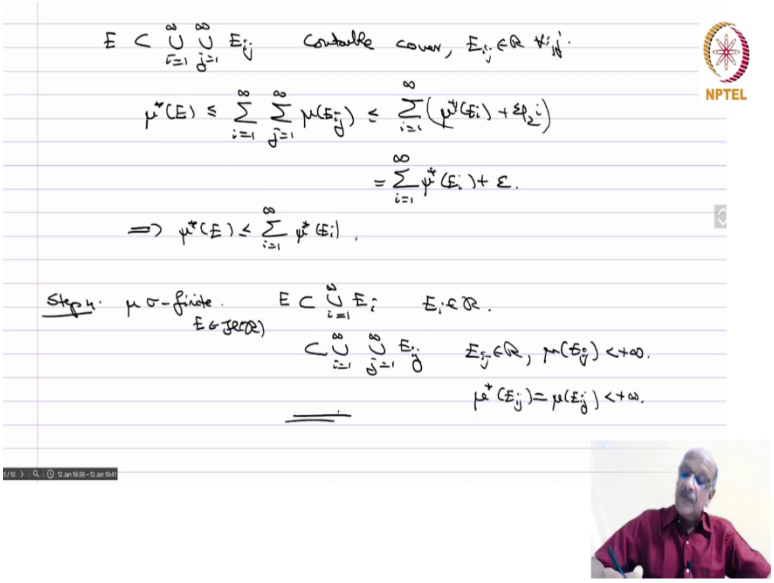

Then you have $E \subset \cup_{i=1}^{\infty}\cup_{j=1}^{\infty}E_{ij}$. So, this is a countable cover with $E_{ij} \in R$, for all i,j and

$$\mu * (E) \leq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(E_{ij}) \leq \sum_{i=1}^{\infty}(\mu * (E_i) + \frac{\epsilon}{2^i}) = \sum_{i=1}^{\infty}\mu * (E_i) + \epsilon.$$

$$\Rightarrow \mu * (E) \leq \sum_{i=1}^{\infty}\mu * (E_i).$$

Step 4: We now take μ is σ-finite, $E \subset \cup_{i=1}^{\infty}E_i$, $E_i \in R$, $E \in H(R)$. So, you can write it as $E \subset \cup_{i=1}^{\infty}\cup_{j=1}^{\infty}E_{ij}$, $\mu(E_{ij}) < \infty$.

So, you have a countable cover of E, with the Eij and, but $\mu * (E_{ij}) = \mu(E_{ij}) < \infty$.

So, this completes the proof of the proposition. So, all questions have been proved.

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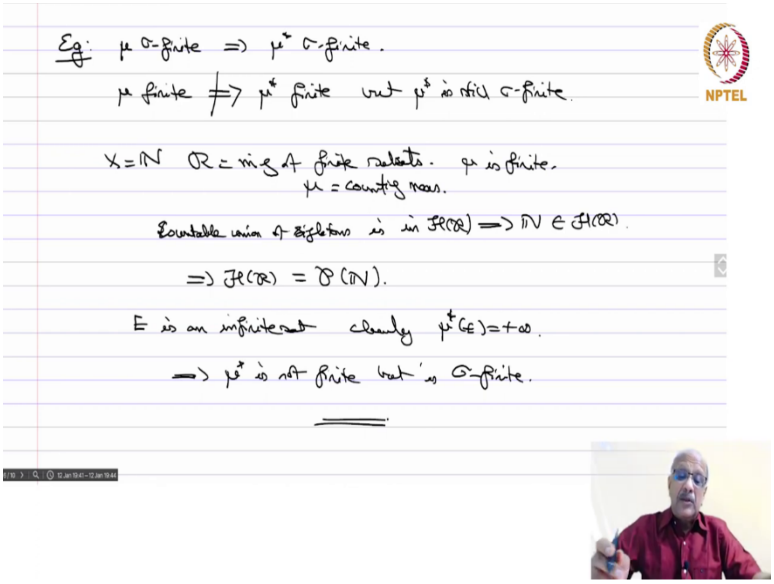

So, now an example.

Example: So we said μ σ- finite $\Rightarrow$ $\mu *$ σ- finite. So, the question is μ finite, does it imply or not that $\mu *$ is finite but $\mu *$ is still sigma finite ? Because of the previous proposition, therefore finiteness does not extend, but sigma finiteness exists. So, what is an example for this?

So, let us take $X = \mathbb{N}$, R= equals ring of finite subsets, μ is counting measure. Then of course, μ is finite. Countable number union of singletons is in $H(R)$ because $H(R)$ is a sigma ring, which contains R singletons are all in R. Therefore, any a countable N will be in singleton. So, this implies $\mathbb{N} \in H(R)$. So, it is a hereditary sigma ring. So, this implies that

$H(R) = P(\mathbb{N})$. Now, if E is an infinite set, clearly $\mu * (E) = \infty$. Because it is, you have to cover it by an infinite number of sets, countable numbers. Each one will have at least measure 1. And therefore, the infimum will also be plus infinity. So, $\mu * (E)$ of E will always be plus infinity. So, this implies that $\mu * (E)$ is not finite, but it is sigma finite.

So, we have a very natural way in which given a measure on a ring, there is a hereditary sigma ring and an outer measure. Now, given a hereditary sigma ring and an outer measure, we will see how to extract and measure out of it. That will be the thing, which we will see next.