Measure and Integration
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Lecture No-38

Examples

So, last time we compared the Riemann and the Lebesgue integrals, so for a bounded function on a bounded interval, we saw that if it is Riemann integrable, then it is also Lebesgue integral and the two integrals will coincide. And it is not generally true for infinite intervals because there are other kinds of limit processes which are involved and we saw an example.

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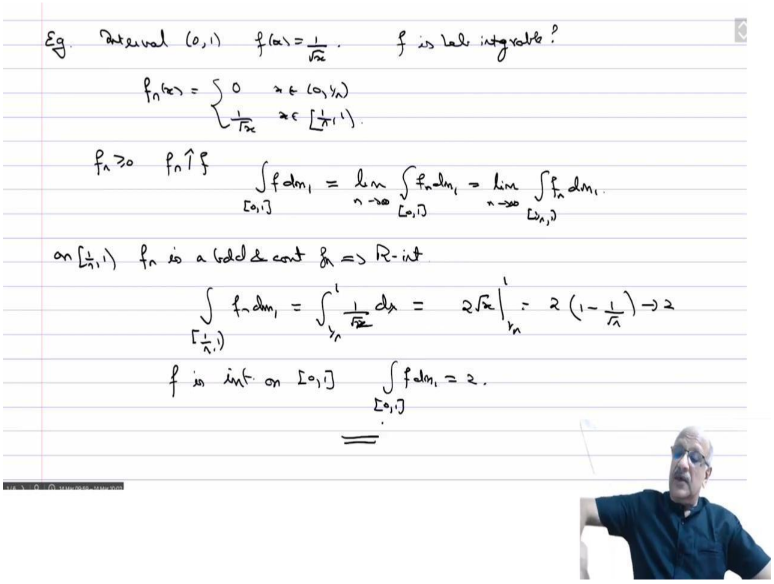

So, now let us give a couple of examples to show how to prove the integrity of the study, the integrity of a function with respect to the Lebesgue measure.

Example: So, let us take the interval (0,1) and $f(x) = \frac{1}{\sqrt{x}}$. So, we want to know if f is Lebesgue integrable? so, this is not a bounded function though the interval is bounded.

And therefore, what we do is we look at

$$f_n(x) = 0,\ x \in (0, \frac{1}{n}),$$

$$= \frac{1}{\sqrt{x}},\ x \in [\frac{1}{n}, 1].$$

Then f_n is all non-negative, and therefore f_n will increase to f. So, more and more of this interval is taken up, the functions are all increasing, and so it is very easy to see that this increases to f.

Therefore, by the monotone convergence theorem, you have that

$$\int_{[0,1]} f dm_1 = \lim_{n\to\infty} \int_{[0,1]} f_n dm_1 = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} f_n dm_1.$$

On $[\frac{1}{n}, 1]$, f_n is a bounded and continuous function. Hence it is Riemann integrable. Therefore, it is Lebesgue integral is the same as in the Riemann integral. So,

$$\int_{[\frac{1}{n},1]} f_n dm_1 = \int_{\frac{1}{n}}^{1} \frac{1}{\sqrt{x}} dx = 2(1 - \frac{1}{\sqrt{n}}) \to 2.$$

So, f is integrable and $\int_{[0,1]} f dm_1 = 2.$

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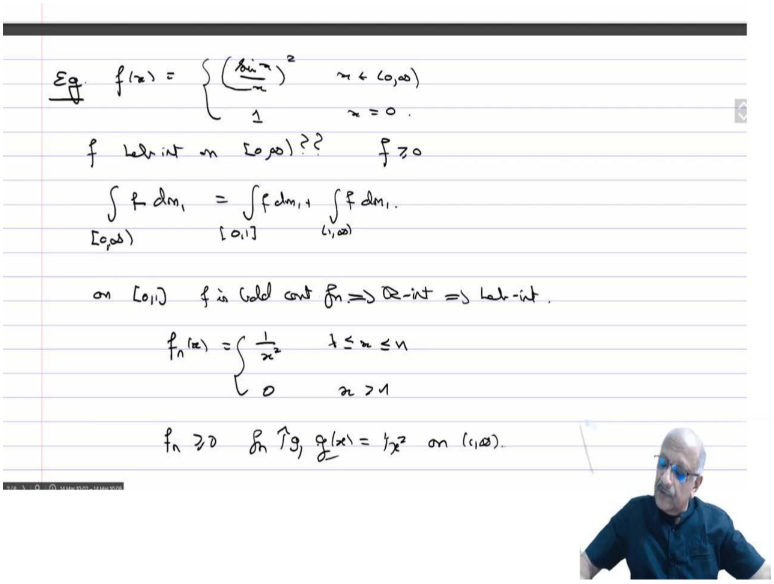

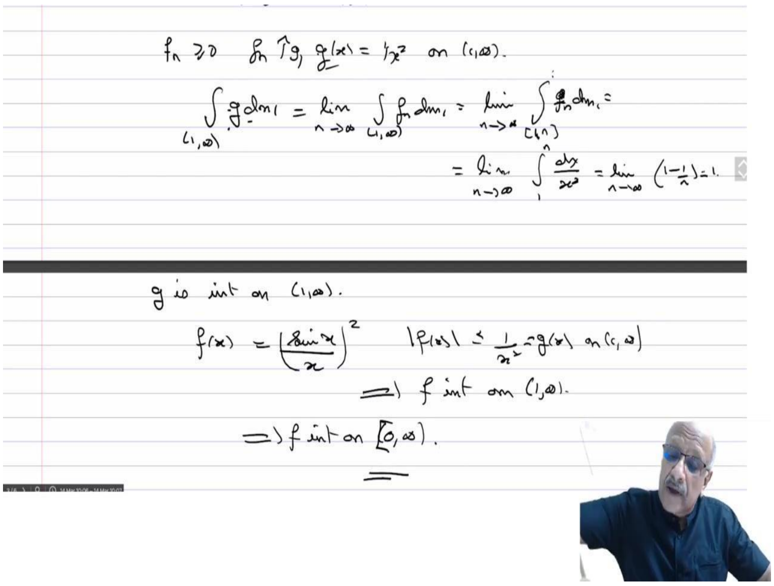

So, this idea we will do one more example. So, on should use such techniques, take away the portions which are singular and then use the traditional calculus because the Lebesgue and Riemann integrals are equal for bounded and continuous functions, continuous functions are Riemann integrable and then you use the usual rules of calculus to compute those integrals pass to the limit etc.

Example: Let $f(x) = (\frac{\sin(x)}{x})^2, \quad x \in (0, \infty),$

$$= 1, \quad x = 0.$$

Proof: So, integral f is non-negative therefore, Lebesgue integral is definitely well defined so,

$$\int_{(0,\infty)} f dm_1 = \int_{(0,1)} f dm_1 + \int_{(1,\infty)} f dm_1.$$

On $[0, 1]$, f is a bounded continuous function ⇒ Riemann integrable ⇒ Lebesgue integrable. Now, what about one infinity? So, you define

$$f_n(x) = \frac{1}{x^2},\ 1 \le x \le n,$$

$$= 0, \quad x > n.$$

So, f_n is all non-negative and in fact $f_n \uparrow g,\ g(x) = \frac{1}{x^2}\ on\ (1, \infty).$

Therefore integral

$$\int_{(1,\infty)} g dm_1 = \lim_{n \to \infty} \int_{(1,\infty)} f_n dm_1 = \lim_{n \to \infty} \int_{(1,n)} \frac{1}{x^2} dx = \lim_{n \to \infty} (1 - \frac{1}{n}) = 1.$$

So, since g is integrable and $|f(x)| \leq \frac{1}{x^2} = g(x)\ in\ (1, \infty)$. This implies that f is integrable on $(1, \infty)$, so, f is integrable on $(1, \infty)$ and therefore f is integrable on $(0, \infty)$, so, this is how we compute, we can compute the integral so, all we can study the integrability of functions.