Measure and Integration
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Lecture No-37
Integration on the real Line

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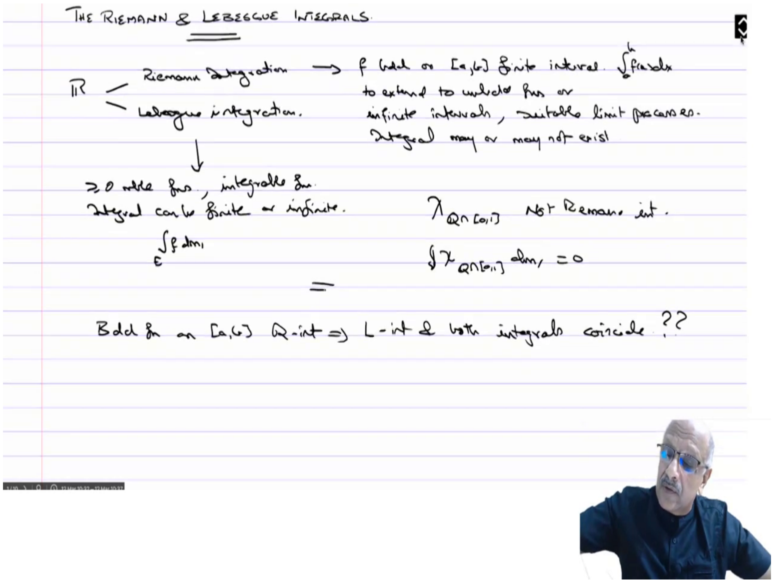

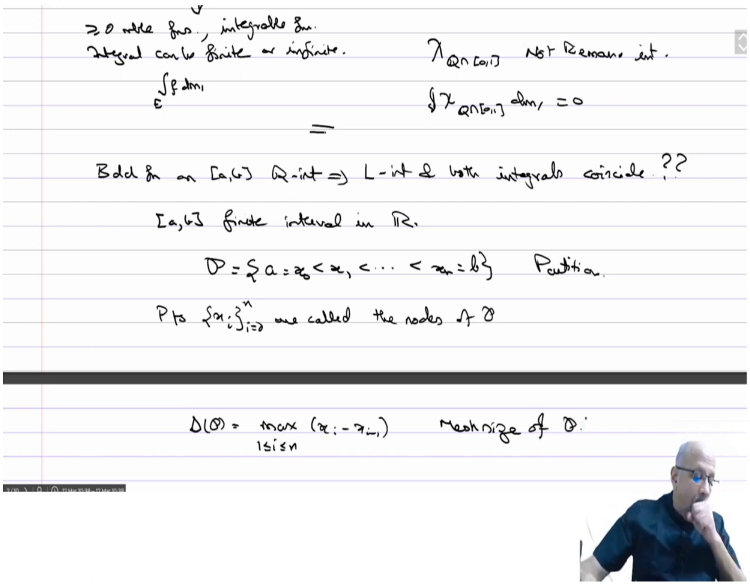

So, we now compare the Riemann and Lebesgue integrable. So, on the real line R we have two integrations, one is Riemann integration and this is the Lebesgue integration. First is the Riemann integral definition. So, Riemann integral f bounded on $[a, b]$ of finite interval and if it exists you call it $\int_a^b f(x)dx$ this notation for the Riemann integral.

Now, if you want to extend to unbounded functions or infinite intervals you may have suitable limit processes and the integral may or may not exist. Lebesgue integral on the other hand is defined for all non-negative measurable functions and integrable functions and integral can be finite or infinite. If it is finite we say Lebesgue integrable of course, the Lebesgue integrable is denoted by integral f d m 1 over the say E on which you are integrating.

Now, for instance, we have seen in the past that there exists a Riemann sequence of Riemann integrable functions whose limit may not be the Riemann integrable. On the other hand, the typical example of this is the characteristic function of Q intersection 0, 1 So, this is Riemann integrable sorry this is not Riemann integrable.

On the other hand this is in fact, the characteristic function of a countable set and the Lebesgue integrable is just a measure of the countable set which is 0. So, but integral chi Q intersection 0, 1 dm 1 equal to 0. So, it exists. So, we have, now we asked the question that if you have a bounded function which is Riemann integrable, is it also Lebesgue integrable? And if so, are the two integrals equal.

For the health of the theory, it better be so because if you have two different methods of integration and you get different results, that is a very uncomfortable situation and it is not good. So, we have to show that if you have a bounded function, so, bounded function on a, b finite interval Riemann integrable implies Lebesgue integrable and both integrals coincide. So, this is the question which we want to ask and we hope that the answer will be affirmative.

So, let us proceed. So, how do you define it? So, $[a, b]$ finite interval in $\mathbb{R}$ and then P is a partition. So, $P = \{a = x_0 < x_1 < .... < x_n = b\}$. So, the points $\{x_i\}_{i=0}^{n}$ are called the nodes of P. So, some finite number.

Then $\Delta P = \max_{1 \leq i \leq n} (x_i - x_{i-1})$. This is called the mesh size of P.

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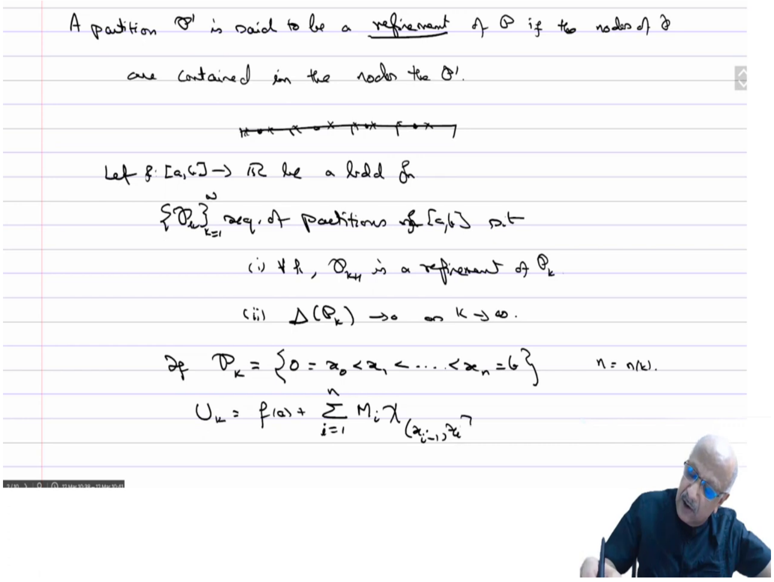

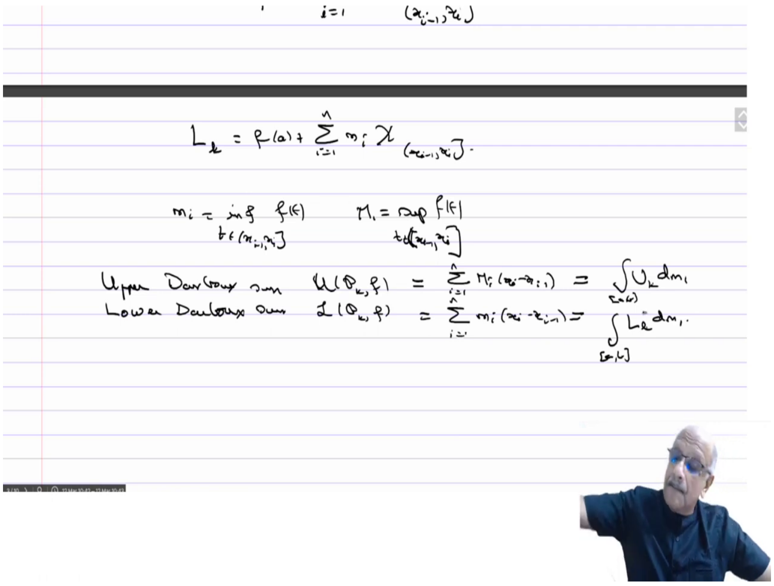

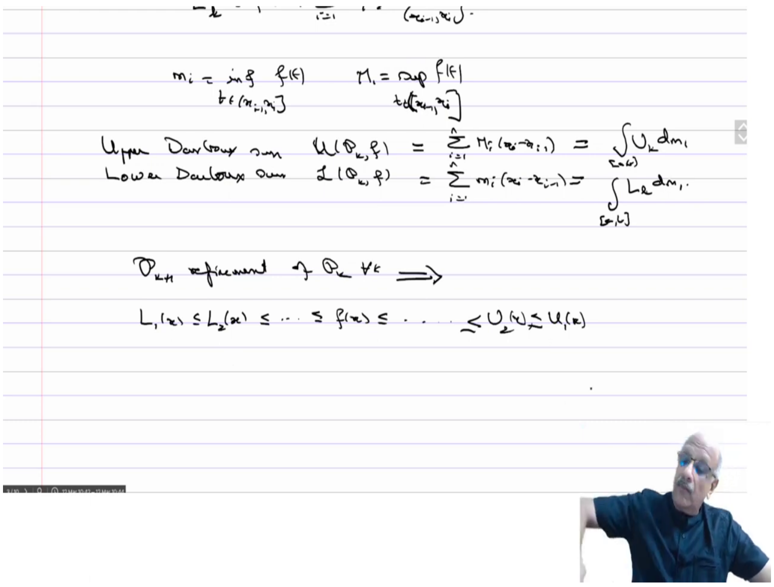

So, a partition P' is said to be refinement of P if the nodes of P are contained in the nodes of P'. So, you have a partition here.

So, now, let $f: [a, b] \to \mathbb{R}$ be a bounded function. Consider a $\{P_k\}_{k=1}^{\infty}$ sequence of partitions of $[a, b]$ such that

(1) for every k, P_{k+1} is a refinement of P_k.

(2) $\Delta P_k \to 0 \; as \; k \to \infty$.

So, define $U_k = f(a) + \sum_{i=1}^{n} M_i \chi_{(x_{i-1}, x_i]}$, $L_k = f(a) + \sum_{i=1}^{n} m_i \chi_{(x_{i-1}, x_i]}$, where

$$M_i = \sup_{t \in (x_{i-1}, x_i]} f(t), \quad m_i = \inf_{t \in (x_{i-1}, x_i]} f(t).$$

Then if you recall the definition of the Riemann integration you have the upper darboux sum and lower darboux sum.

$$U(P_k, f) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \int_{(a,b]} U_k dm_1 \; ; \; L(P_k, f) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \int_{(a,b]} L_k dm_1.$$

Now, pk plus 1 refinement of Pk for every k, so, what does that imply? That implies

$$L_1(x) \le L_2(x) \le ... \le f(x) \le ... \le U_2(x) \le U_1(x).$$

Because you are taking the supremum over smaller intervals. So, the supremum will be less and here you are doing infimum over smaller intervals. So, the infimum will be more. So, that is why U2 L2 is bigger than L1 U2 bigger than you 1 and in general for every k you will have this inequality here.

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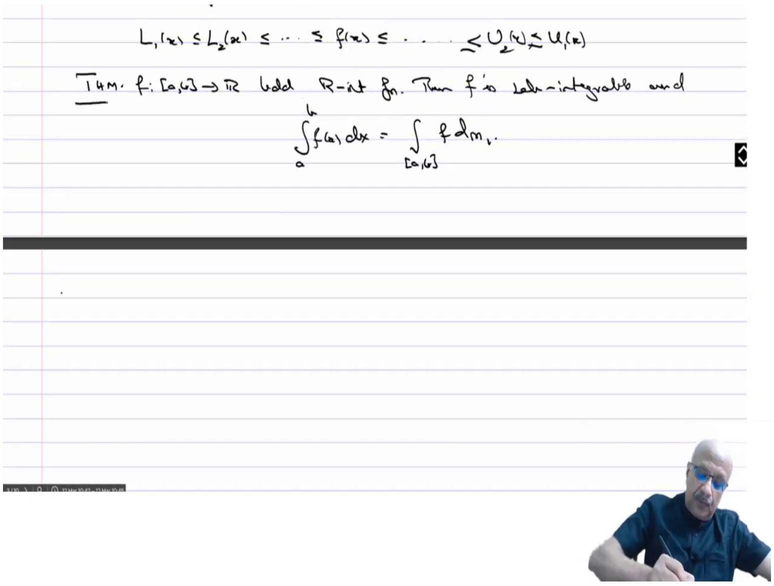

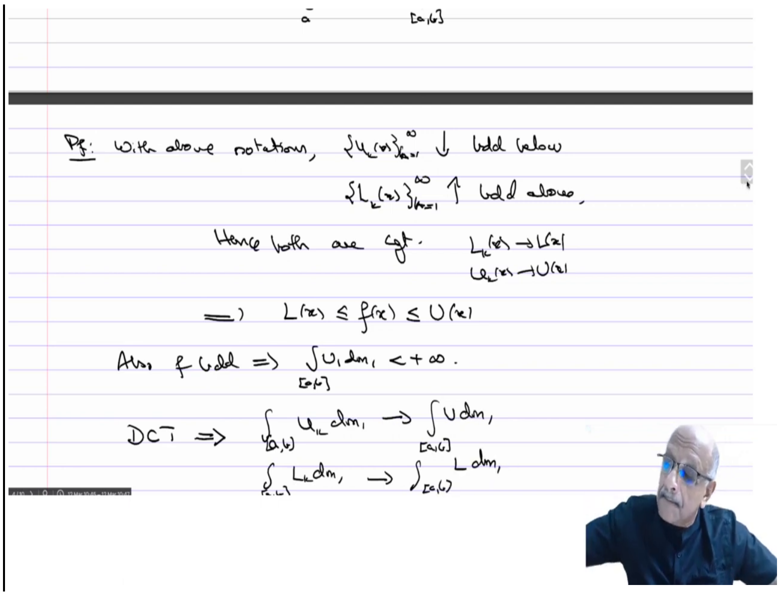

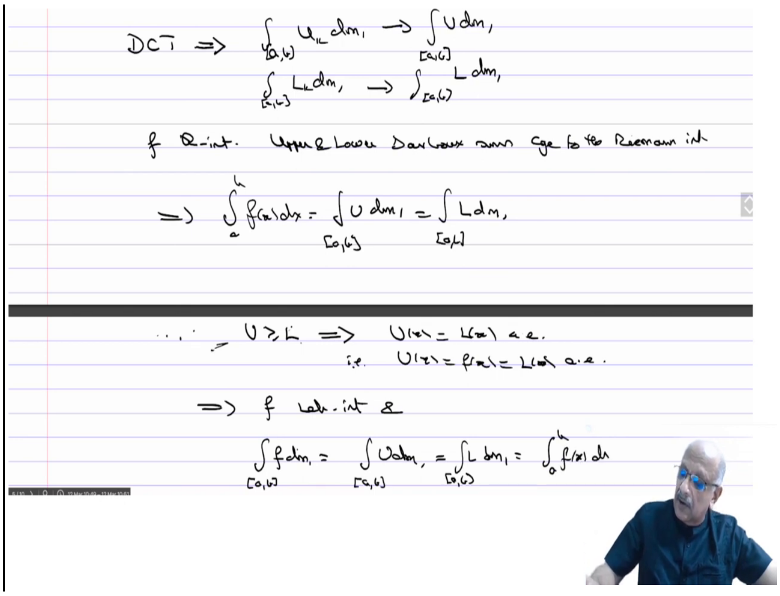

Theorem: $f: [a, b] \to \mathbb{R}$, bounded Riemann Integrable function. Then f is Lebesgue integrable and $\int_a^b f(x)dx = \int_{[a,b]} f dm_1$.

proof: With above notations you have $\{U_k(x)\}_{k=1}^{\infty}$ is decreasing sequence, bounded below and $\{L_k(x)\}_{k=1}^{\infty}$ is increasing sequence, bounded above. Hence both are convergent.

So, let us take the limit. So, $L_k(x) \to L(x)$ and $U_k(x) \to U(x)$. Then this implies that $L(x) \le f(x) \le U(x)$. Also, f bounded implies $\int_{[a,b]} U_1 dm_1$ is finite. Therefore, by dominated convergence theorem,

$$\int_{[a,b]} U_k dm_1 \to \int_{[a,b]} U \, dm_1 \ and \ \int_{[a,b]} L_k dm_1 \to \int_{[a,b]} L dm_1 .$$

Now, f is Riemann integrable. So, you have that the upper and lower and lower darboux sums converge to the Riemann integral. Therefore, what does it mean? That means

$$\int_a^b f(x)dx = \int_{[a,b]} U dm_1 = \int_{[a,b]} L dm_1.$$

But also, U is integrable because f is and therefore, you also have that $U(x) = L(x) \ a.e.$

So, $U(x) = f(x) = L(x)\ a.e.$ So, this means f is equal to U or L almost everywhere that one f is Lebesgue integrable and

$$\int_{[a,b]} f dm_1 = \int_{[a,b]} U dm_1 = \int_{[a,b]} L dm_1 = \int_a^b f(x)dx.$$

So, that is the proof.

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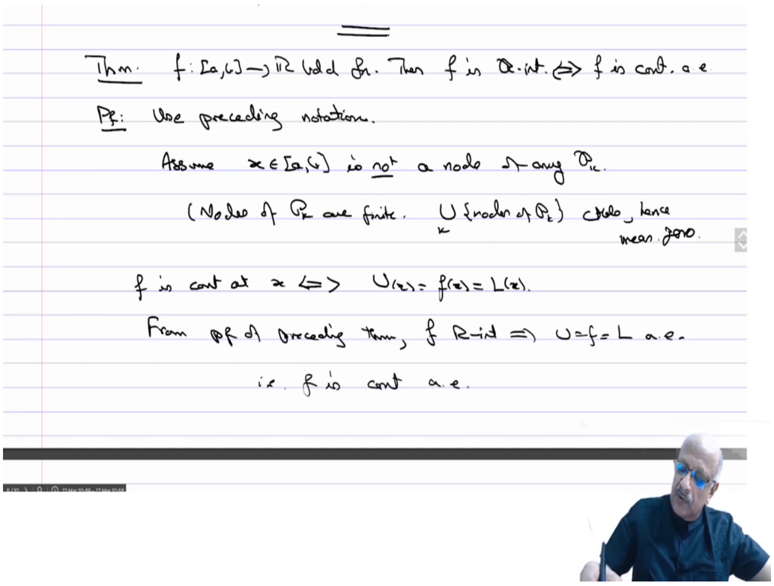

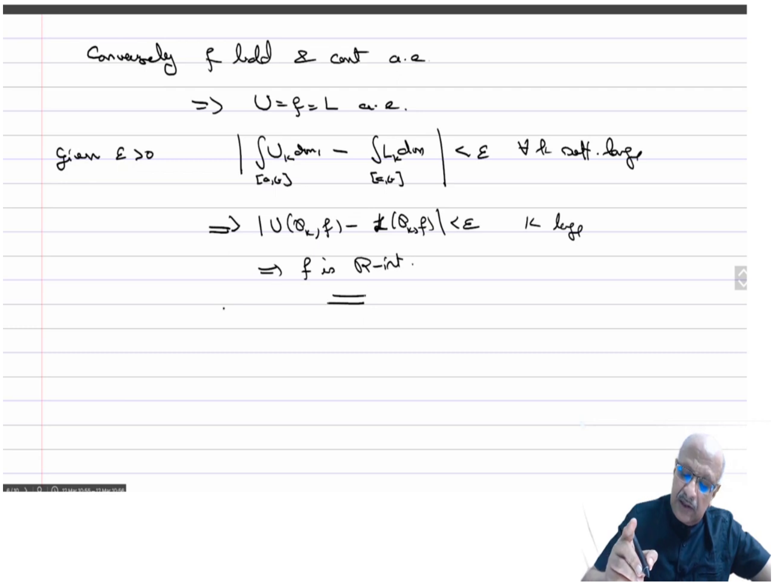

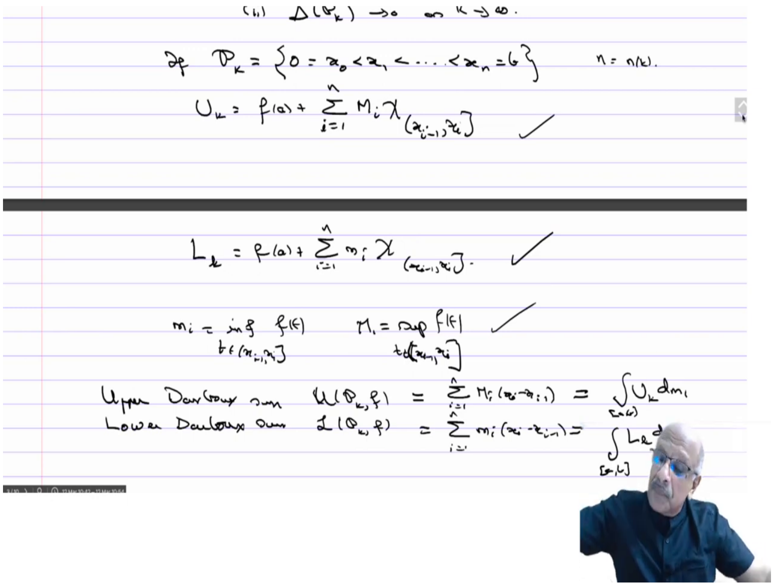

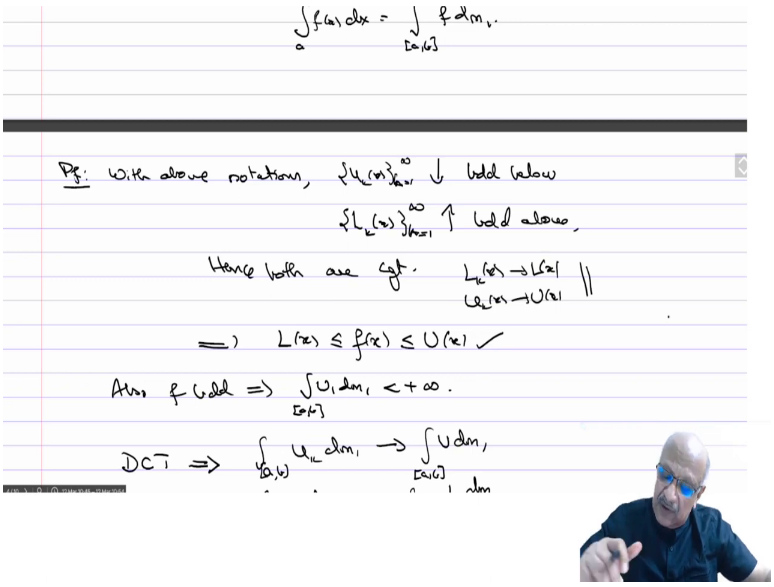

Theorem: $f: [a, b] \to \mathbb{R}$ bounded function. Then f is Riemann integrable if and only if f is continuous almost everywhere.

proof. So, again use preceding notations. So, assume $x \in [a, b]$ is not a node of any P_k. So, now nodes of Pk are finite. So, union over k 10 nodes of Pk is countable. Hence measure 0. So, we are taking x outside this set of measure 0 and now you have f is continuous at x if and only if $U(x) = f(x) = L(x)$ because what are U, L and all that, U_k comes from the maximum.

So, look at the definitions of these things Uk Lk and definitions of Mi and mi. So, you have that Uk and Lk converse to U and L and therefore here and therefore, you have this condition here. Now, if you have continuity then these three will be the same and if these three are same then obviously, the function is continuous also. So, you have the f is continuous at a point x if and only if U equals f equals Lx almost everywhere.

Now, from proof of the proceeding theorem, if f Riemann integrable this implies U equals f equals L almost everywhere that this f is continuous almost everywhere. Now, conversely f bounded and continuous almost everywhere, this implies that U equals f equal to L almost everywhere.

So, this means that integral since integral Uk converges to U and integral Lk converges to L integral L. So, those two integrals are equal and consequently given epsilon this is less than epsilon for all k sufficiently large and that is one of the conditions for Riemann integrable, that is mod U Pk f minus L Pk f is less than epsilon k large and that is implies f is Riemann integrable one of the criteria one of the alternative definitions of Riemann integrable (())(22:56) is that given any sequence of partitions you have that you can do this.

So, this proofs, so, we have proved that if a function is Riemann integrable if and only if it is continuous almost everywhere and if it is Riemann integrable and bounded then it is Lebesgue integrable is the same as Riemann integral, very good, and all this was done in a finite interval a, b.

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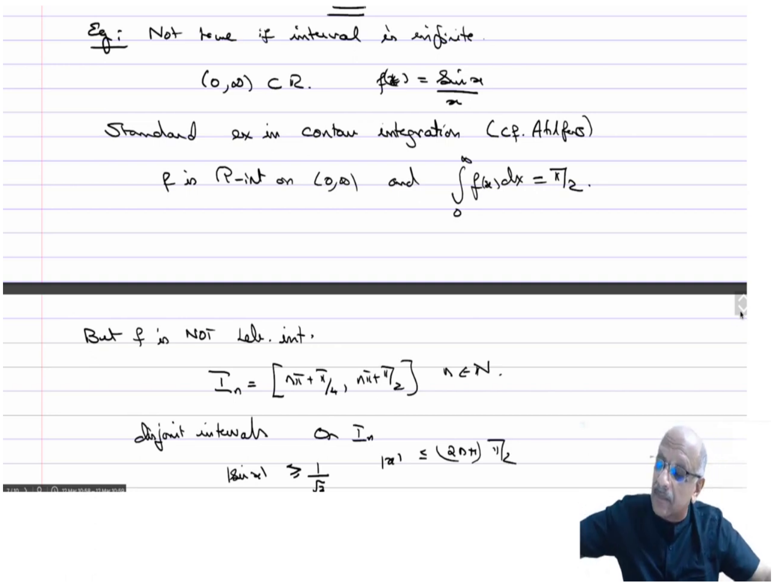

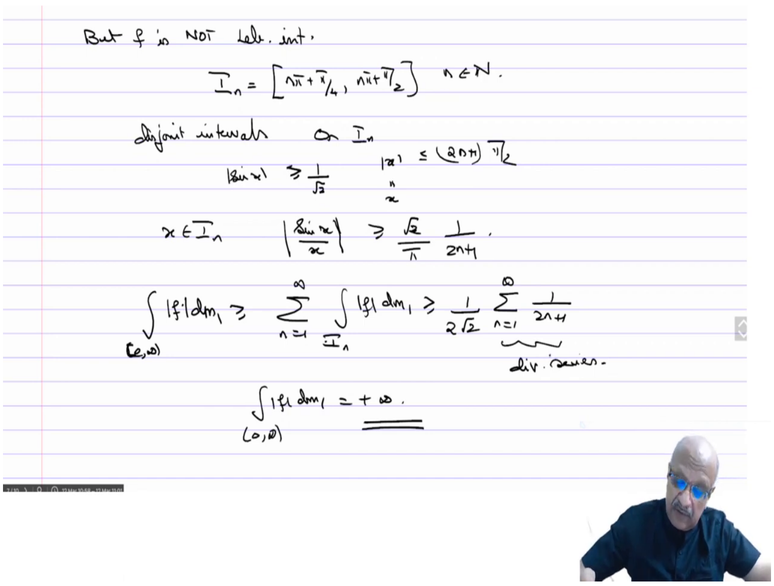

Example: Not true if the interval is infinite. Because here we are going to define the Riemann integral in terms of some suitable limit processes. So, let us give an example: let us take $(0, \infty) \subset \mathbb{R}$ and you let $f(x) = \frac{\sin(x)}{x}$.

So, this is standard exercise contour integration. You can look at any complex analysis book.

So, I will force for instance f is Riemann integrable on $(0, \infty)$ and $\int_0^\infty f(x)dx = \frac{\pi}{2}$.

But f is not Lebesgue integrable. So, we want to show that f is not Lebesgue integrable. So, we have $I_n = [n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{2}],\ n \in \mathbb{N}$.

So, these are all disjoint intervals on I_n. So you have $|\sin(x)| \geq \frac{1}{\sqrt{2}}$ and $|x| \leq (2n+1)\frac{\pi}{2}$.

So, for $x \in I_n$, we have $|\frac{\sin(x)}{x}| \geq \frac{\sqrt{2}}{\pi}\frac{1}{2n+1}$. So,

$$\int_{(0,\infty)} |f| dm_1 \geq \sum_{n=1}^{\infty} \int_{I_n} |f| dm_1 \geq \frac{1}{2\sqrt{2}} \sum_{n=1}^{\infty} \frac{1}{2n+1},$$ which is a divergent series.

And therefore, you have $\int_{(0,\infty)} |f| dm_1 = +\infty$.

So, f is not Lebesgue integrable and (())(27:42). So now, the next thing we will do is to see a couple of instances of how we use this fact about Riemann and Lebesgue integrals being the same for bounded functions on bounded intervals and use that to study the integrability of functions on the real line and then also try to compute those integrals.