Measure and Integration
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Lecture No-36
Absolute Continuity

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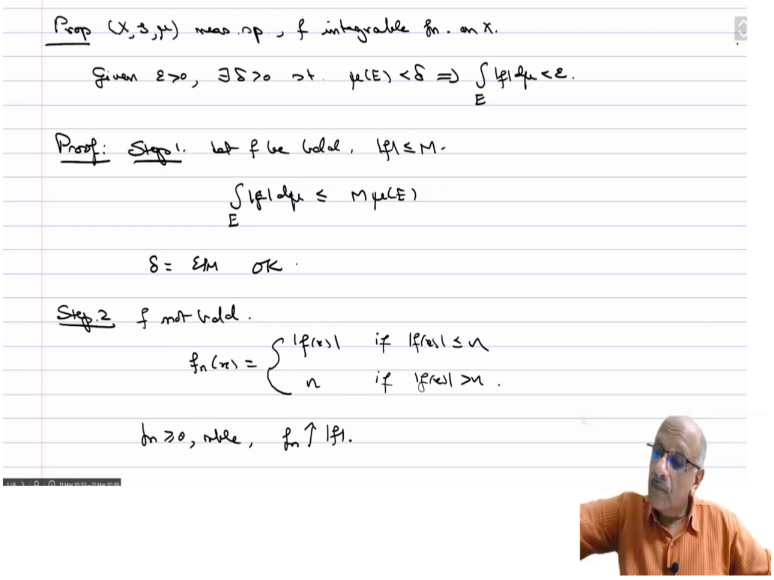

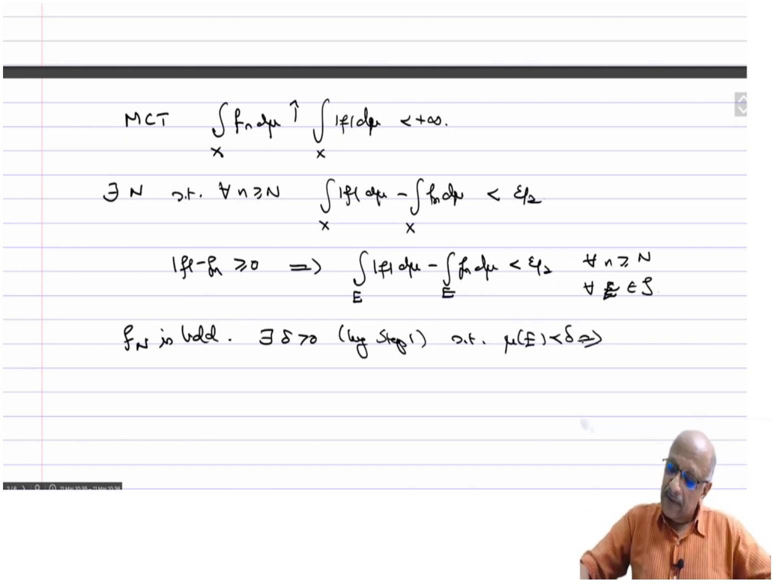

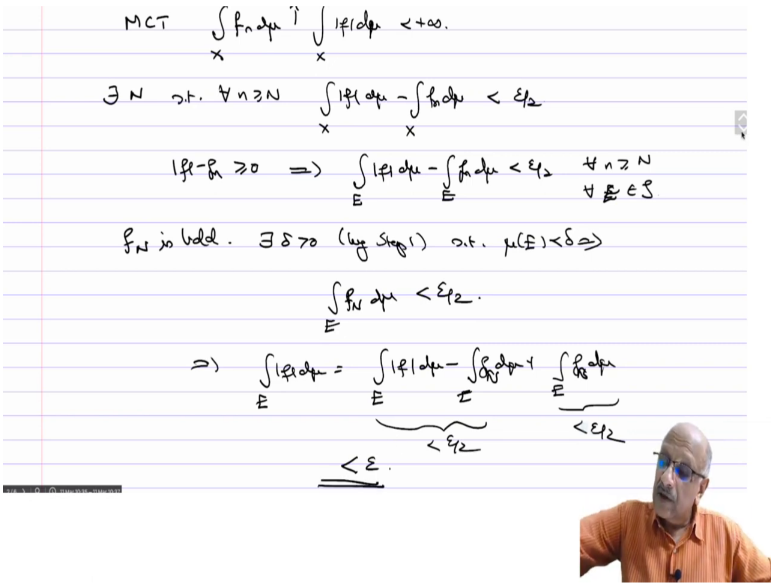

Proposition: (X, S, μ)-measure space, f integrable function on X. Given $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta \Rightarrow \int_E |f| d\mu < \epsilon$.

proof: **step 1:** Let f be bounded. So, $|f| \leq M$. Then $\int_E |f| d\mu < M\mu(E)$ and therefore, if you take $\delta = \frac{\epsilon}{M}$, we are okay.

step 2: If f is not bonded therefore, you take a

$$f_n(x) = |f(x)| \text{ if } |f(x)| \leq n,$$

$$= n \quad \text{if } |f(x)| > n.$$

Then f_n is non-negative, measurable, and $f_n \uparrow |f|$.

So, by the monotone convergence theorem, you have that

$$\int_X f_n d\mu \uparrow \int_X |f| d\mu < \infty.$$

Therefore, given any epsilon there exists a capital N such that for all $n \geq N$, you have

$$\int_X |f| d\mu - \int_X f_n d\mu < \frac{\epsilon}{2}.$$

But since $|f| - f_n \geq 0 \Rightarrow \int_E |f|d\mu - \int_E f_n d\mu < \frac{\epsilon}{2}, \forall\, n \geq N, \forall\, E \in S.$

Now, f_N is bounded. So, there exists (by step 1) $\delta > 0$ such that $\mu(E) < \delta \Rightarrow \int_E f_N d\mu < \frac{\epsilon}{2}$.

$$\Rightarrow \int_E |f|d\mu = \int_E |f|d\mu - \int_E f_N d\mu + \int_E f_N d\mu < \epsilon.$$

That proves this theorem.

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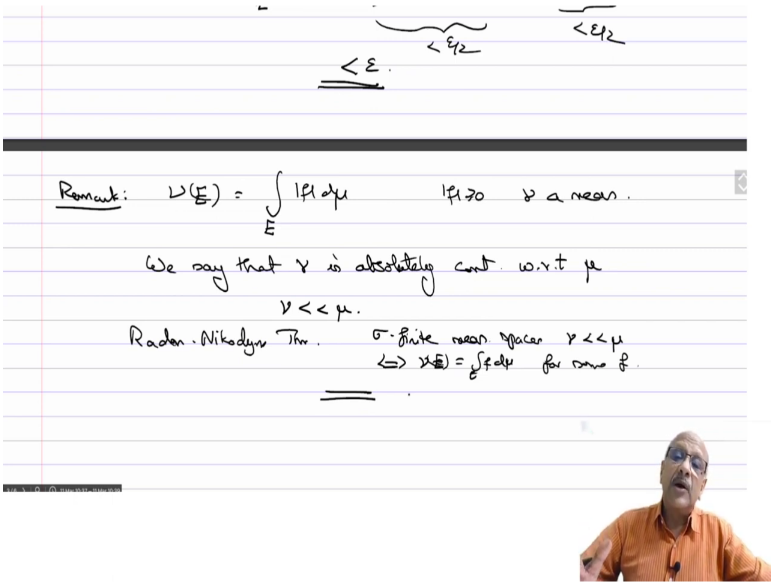

Remark: So, if you define $\nu(E) = \int_E |f|d\mu$. We already saw mod f is non-negative measurable function therefore ν is a measure. So, we say that ν is absolutely continuous with respect to μ and we write $\nu << \mu$. So, if $|f|$ is integrable the ν is absolutely continuous with respect to μ and the Radon Nikodym theorem I have referred to already. In say, the converse so for σ-finite measure spaces ν absolutely continuous with this μ if and only if

$$\nu(E) = \int_E f d\mu, \text{ for sum f.}$$

So, this is an if and only if statement and we will see this later much later in this course. So, every measure which is absolutely continuous with respect to a given measure will occur in this form and f will be a non-negative and if nu E is finite measure then this will be an integrable function, otherwise it is a general function.

So, our next aim is to have on the we have been discussing abstract Lebesgue integration and our interest of course, is about the Lebesgue measures in Rm or Rm et cetera. So, on our R for instance, you have two types of integration. Now, you have the Lebesgue integration coming from the Lebesgue measure and you also have the classical Riemann integration. So, we like to compare these two and also that will help us to compute the Lebesgue integral of some functions. We will see that next time.