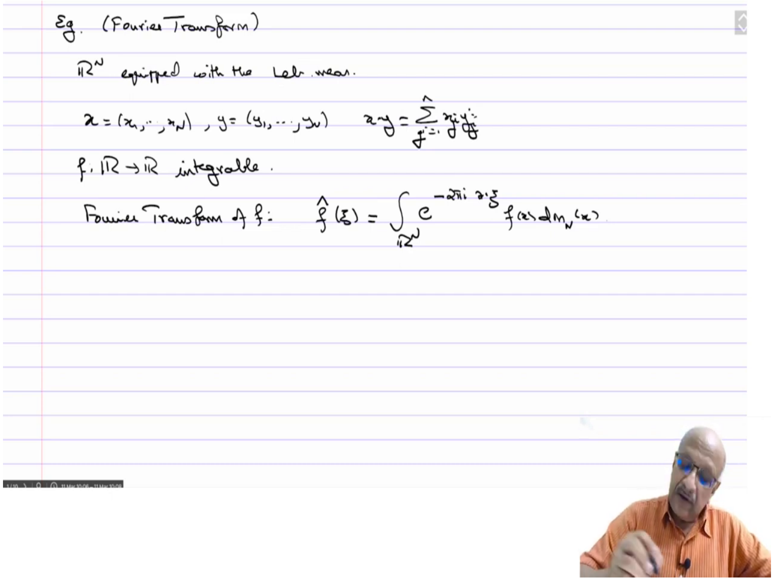
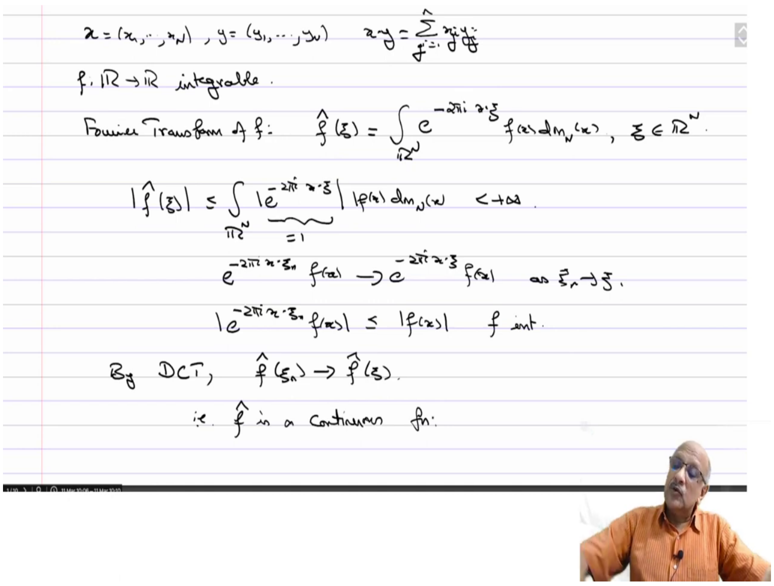
**Measure and Integration**
**Professor S. Kesavan**
**Department of Mathematics**
**The Institute of Mathematical Sciences**
**Lecture No-35**
**Dominated convergence theorem**

(Refer Slide Time: 00:07)




We will now look at a few examples of the application of the dominated convergence theorem. So, the first example.

**Example:** (Fourier transform) So, $\mathbb{R}^N$ equipped with Lebesgue measure. So, given 2 vectors

$$x = (x_1, \ldots, x_N),\ y = (y_1 \ldots, y_N),\ x \cdot y = \sum_{i=1}^{N} x_i y_i.$$

So, let $f: \mathbb{R} \to \mathbb{R}$ integrable, then the Fourier transform so, this is denoted by $\hat{f}$,

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} e^{-2\pi i\, x \cdot \xi} f(x) dm_N(x),\ \xi \in \mathbb{R}^N.$$

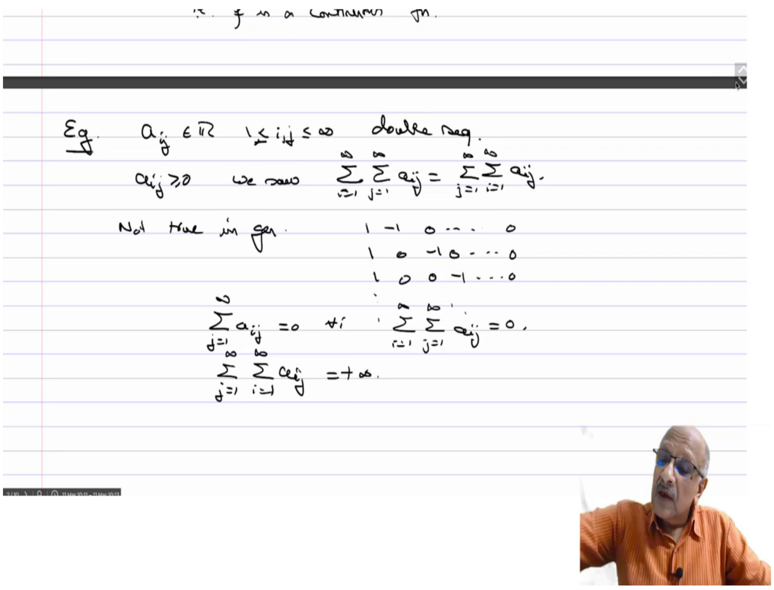
So, first of all it is well defined because

$$|\hat{f}(\xi)| \leq \int_{\mathbb{R}^N} |e^{-2\pi i\, x \cdot \xi}||f(x)|\, dm_N(x) < \infty.$$

So, $\hat{f}$ is well defined and it is a bounded function and namely the L infinity norm of f hat is less than or equal to the L1 norm of the function f. The integral of the function mod f. So, now, we also have that e power minus 2 pi i x dot xi n f of x will converge to e power minus 2 pi i x dot xi f of x as x n converges to xi.

So, this is for xi in Rn and mod of e power minus 2 pi i x dot xi f of x is less than equal to again modulus of this is equal to 1 and this is less than equal to mod fx and f is integrable. Therefore, by the dominated convergence theorem we get f hat xi n converges to f hat xi that is $\hat{f}$ is a continuous function. So, given an integrable function the Fourier transform produces a continuous and bounded function. In fact, it is a bounded and uniformly continuous function which can be proved also. So, that is one example.

(Refer Slide Time: 04:26)



**Example:** So, if $a_{ij}$ are all non-negative, we saw

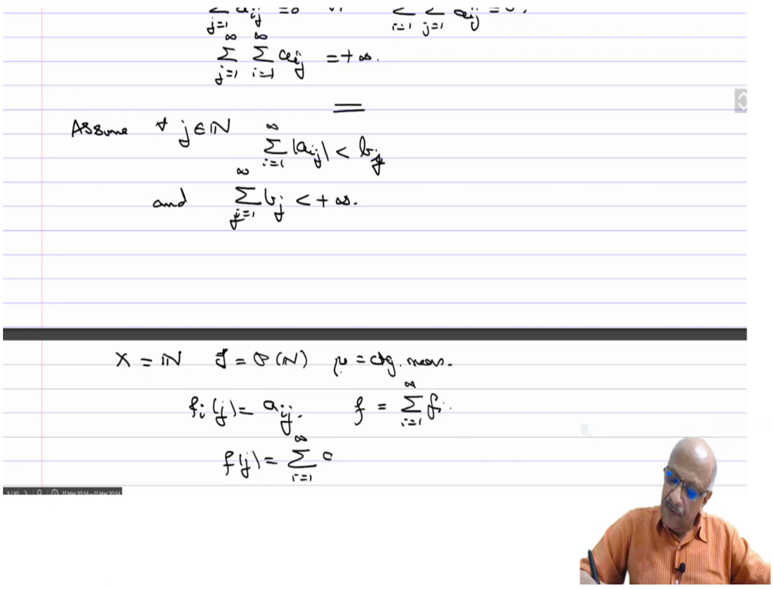
$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}.$$

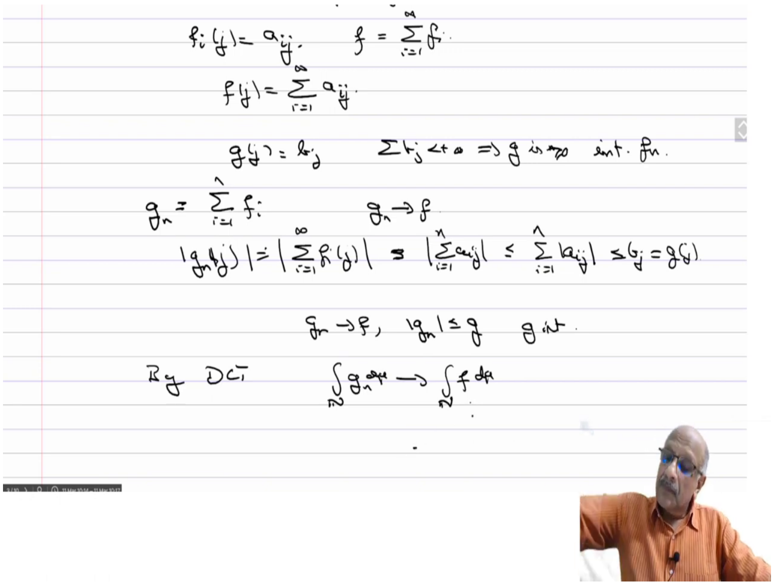
But it is not true in general.

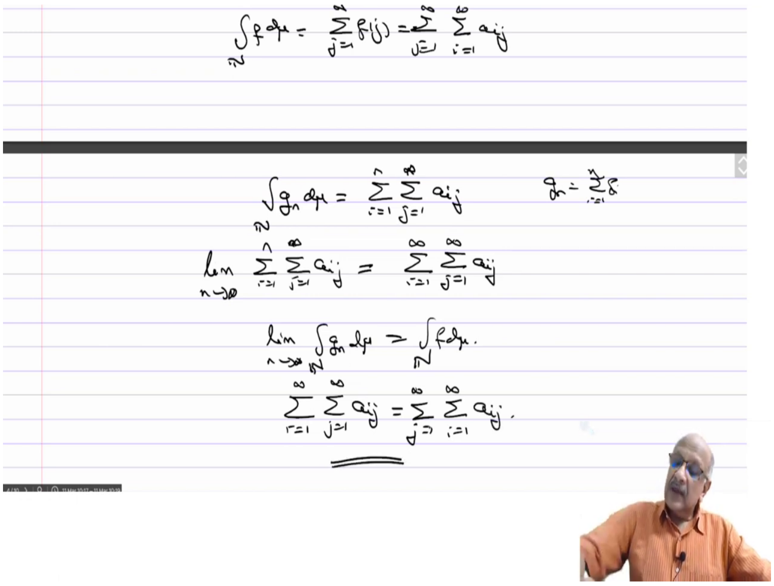
You take you have 1 minus 1 0, 0, 0, 1 minus 1 0, 0, 1, 0, 0 minus 1 then 0 and so on. Let us go on like this. Then if you look at $\sum_{j=1}^{\infty} a_{ij} = 0, \forall i$. So, $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = 0$.

But $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij} = \infty$. Because the first column gives you infinity and then you are subtracting one at a time and therefore, this is giving you infinity and therefore, this does not agree in general. So, we are now going to give you a sufficient condition when the result is in fact true.

(Refer Slide Time: 06:40)







So, let us assume so, assume for each $j \in \mathbb{N}$, you have $\sum_{i=1}^{\infty} |a_{ij}| < b_i$ *and* $\sum_{j=1}^{\infty} b_j < +\infty$.

So, now, let us take $X = \mathbb{N}$, $S = P(\mathbb{N})$, μ-counting measure. And let us take $f_i(j) = a_{ij}$ and

$f = \sum_{i=1}^{\infty} f_i$. So, $f(j) = \sum_{i=1}^{\infty} a_{ij}$. Define $g(j) = b_j$. Then $\sum_{j=1}^{\infty} b_j < \infty \Rightarrow g$ is a nonnegative integrable function. Let us define $g_n = \sum_{i=1}^{n} f_i$. Then gn tends to f and you have

$$|g_n(j)| = \sum_{i=1}^{\infty} f_i(j) = \sum_{i=1}^{n} a_{ij} \leq b_j = g(j).$$

mod gn and any j equals mod sigma i equals 1 to n fi, fj and that is less than equal to mod sigma i equals 1 to n aij and that is less than equal to sigma i equals 1 to n mod aij and that is therefore, less than equal to bj equal to g of j. So, gn converges to f and gn is bounded by g and g is integral. So, gn converges to f mod gn less than equal to g and g integrable.

Therefore, by dominated convergence theorem, you have that $\int_{\mathbb{N}} g_n d\mu \rightarrow \int_{\mathbb{N}} f d\mu$.

So, $\int_{\mathbb{N}} f_n d\mu = \sum_{j=1}^{\infty} f(j) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$. Now, $\int_{\mathbb{N}} g_n d\mu = \sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij}$.

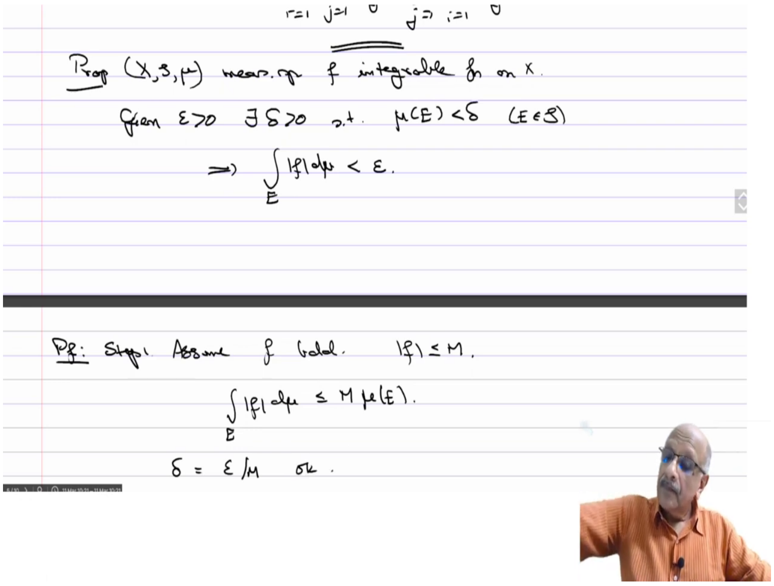
Therefore, $\lim_{n\rightarrow\infty} \sum_{i=1}^{n}\sum_{j=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij}$. Therefore, you have so,

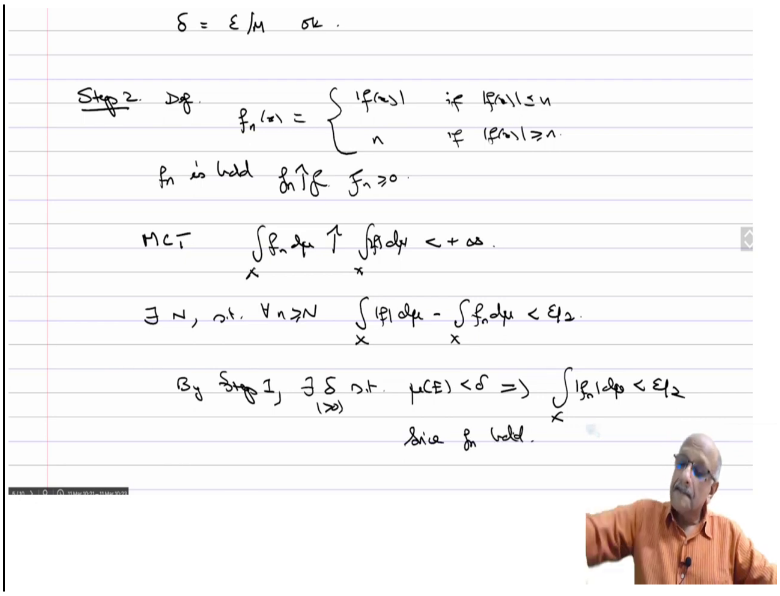
$$\lim_{n\rightarrow\infty} \int_{\mathbb{N}} g_n d\mu = \int_{\mathbb{N}} f_n d\mu.$$

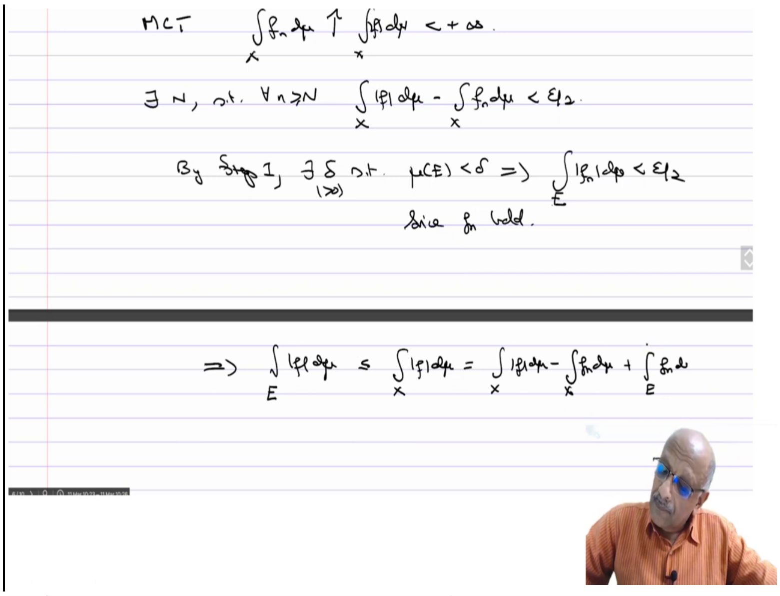
Hence $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} a_{ij}$.

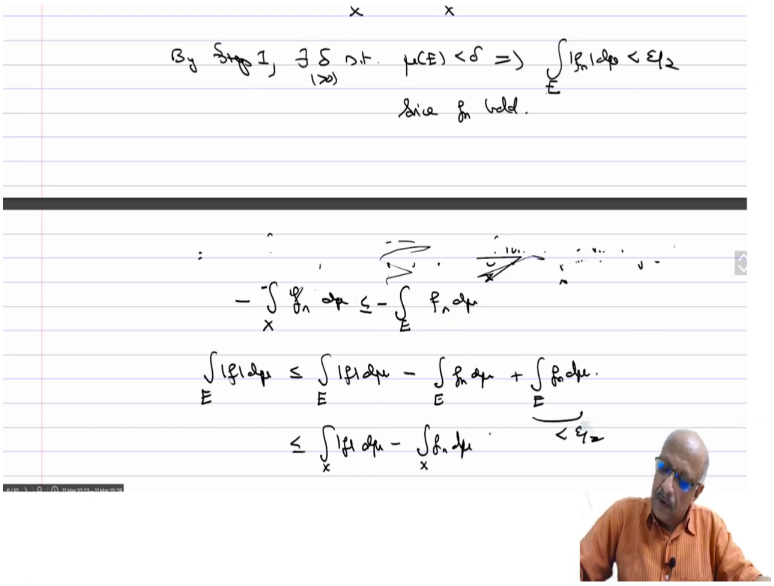
So, this hypothesis which we made here allows us to interchange the order of summation in general of course, this is not true (())(12:49). So, this comes from the dominated convergence theorem. So, the non-negative case came from the monotone convergence theorem and the general case comes from the dominated convergence theorem

(Refer Slide Time: 13:00)






**Proposition:** $(X, S, \mu)$ measure space, f integrable function, given $\epsilon > 0$, $\exists\, \delta > 0\ s.t.\ \mu(E) < \delta\ (E \in S)$,

$$\Rightarrow \int_E |f| d\mu < \epsilon.$$

*proof:* **Step 1:** Assume f bounded. So, $|f| \leq M$ and therefore,

$$\int_E |f| d\mu < M\mu(E).$$

Therefore, if you choose $\delta = \frac{\epsilon}{M}$, then we are through.

**Step 2:** Define,

$$|f_n(x)| = |f(x)|, \; if \; |f(x)| \leq n,$$

$$= n, \; if \; |f(x)| > n.$$

So, there is a cut off function. So, you take the function when it reaches a threshold of value n you cut it off and then fix it this as n then of course, fn is now bounded it is bounded by n and fn in fact increases to F and fn is non-negative.

So, by the monotone convergence theorem, we have $\int_X f_n d\mu \uparrow \int_X |f| d\mu < \infty$. Therefore, there exists a capital N such that for all $n \geq N$, you have $\int_X |f| d\mu - \int_X f_n d\mu < \frac{\epsilon}{2}$. So, in by step 1 there (())(16:23) a delta such that $\mu(E) < \delta \Rightarrow \int_E |f_n| d\mu < \frac{\epsilon}{2}$, since f_n bonded. And that tells you that $\int_E |f| d\mu = \int_E |f| d\mu - \int_E f_n d\mu + \int_E f_n d\mu \leq \int_X |f| d\mu - \int_X f_n d\mu < \epsilon$.