Measure and Integration
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Lecture-34
Dominated convergence theorem

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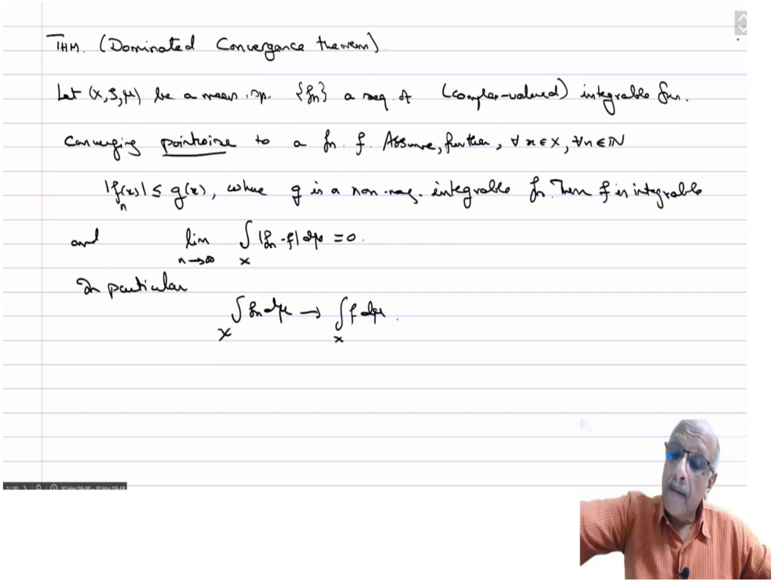

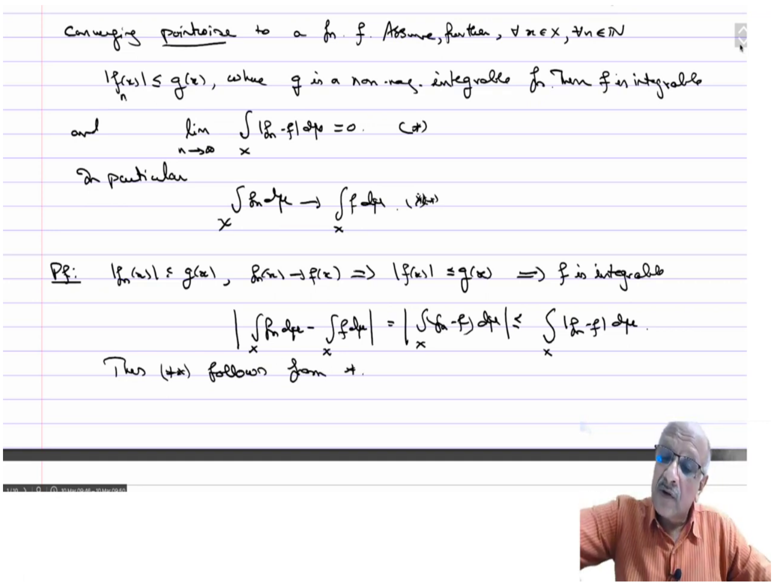

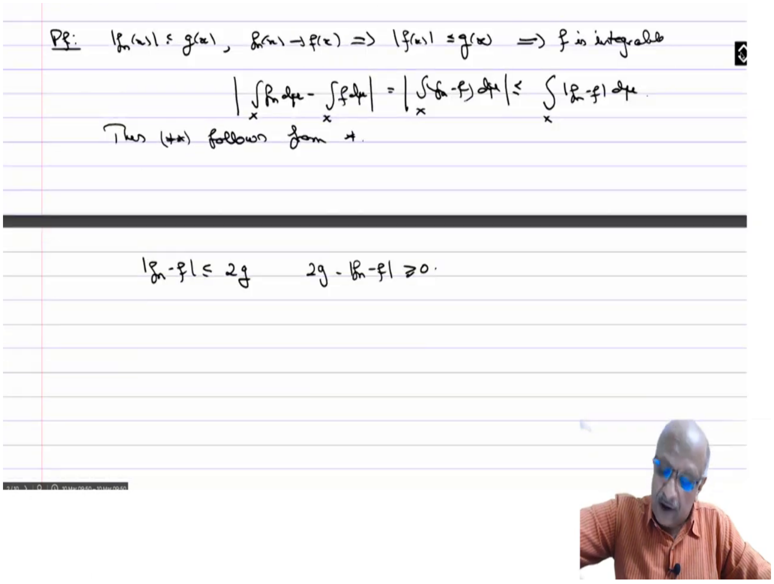

We will now prove a very important theorem without exaggeration. This can be called really the high point of this entire course, it is a most important and most applied theorem in measure theory in my opinion. So, we are now going to prove the following theorem. This is called the dominated convergence theorem.

Theorem: (Dominated convergence theorem**)** Let (X, S, μ) be a measure space, $\{f_n\}$ a sequence of (complex valued) integrable functions converging pointwise to a function f. Assume further for all $x \in X$, $\forall\, n \in \mathbb{N}$, $|f_n(x)| \le g(x)$, where g is a nonnegative integrable function. Then f is integrable and

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0. \text{ ------------ } (*)$$

In particular, $\int_X f_n d\mu \to \int_X f d\mu$. --------- (**)

proof. So, $|f_n(x)| \le g(x)$ and $f_n(x) \to f(x) \Rightarrow |f(x)| \le g(x)$ and since g is integrable , we get that f is integrable. And also, you have that

$$\left|\int_X f_n d\mu - \int_X f d\mu\right| = \left|\int_X (f_n - f) d\mu\right| \le \int_X |f_n - f| d\mu$$

Therefore, (**) follows from (*). So, that is enough to prove star, integral mod fn minus f goes to 0. So, you know $|f_n - f| \le 2g$, *i.e.*, $2g - |f_n - f| \ge 0$.

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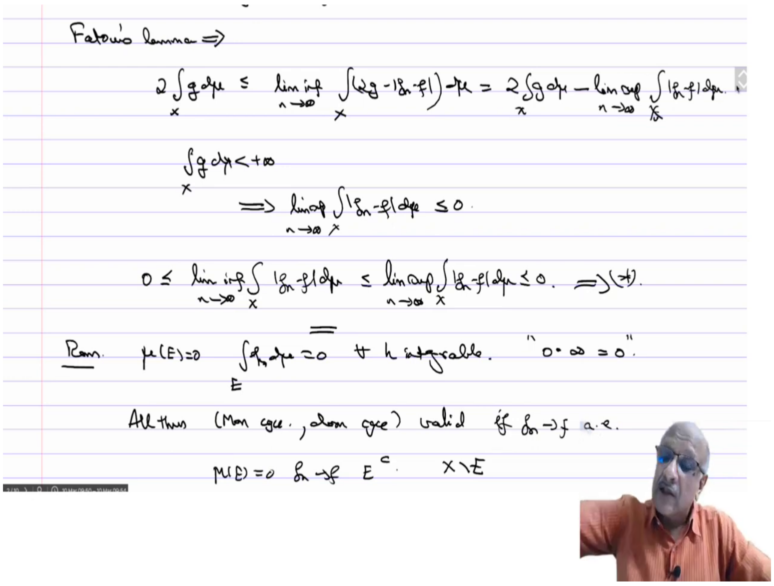

So, Fatou's lemma gives:

$$2\int_X g d\mu \leq \lim_{n\to\infty} \inf \int_X (2g - |f_n - f|) d\mu = 2\int_X g d\mu - \lim_{n\to\infty} \sup \int_X |f_n - f| d\mu.$$

Now, $\int_X g d\mu$ is finite and therefore, this implies that $\lim_{n\to\infty} \sup \int_X |f_n - f| d\mu \leq 0.$

The proof is over.

So, this is the dominated convergence theorem.

Remark: If $\mu(E) = 0$ then $\int_E h d\mu = 0,$ for any function any, any h , for all h integrable.

Now, in proving this of course, initially for simple functions or also we have this we always use the convention that 0 into infinity is 0 this is for consistency. So, the function is infinite if the set has measure 0 it is this if the (())(07:46) is measure infinity and the function is 0, then also the integral is 0 we have used this convention tacitly I am just indicating it to you now.

Now, you all theorems monotone convergence, dominated convergence valid if fn goes to f almost everywhere. So, you take E μ E equal to 0 and fn goes to f on E complement, then you work with x minus E and prove these theorems and adding the integral over E does not change anything because the integral over E is automatically 0. So, all theorems are valid even if fn converges f almost everywhere.

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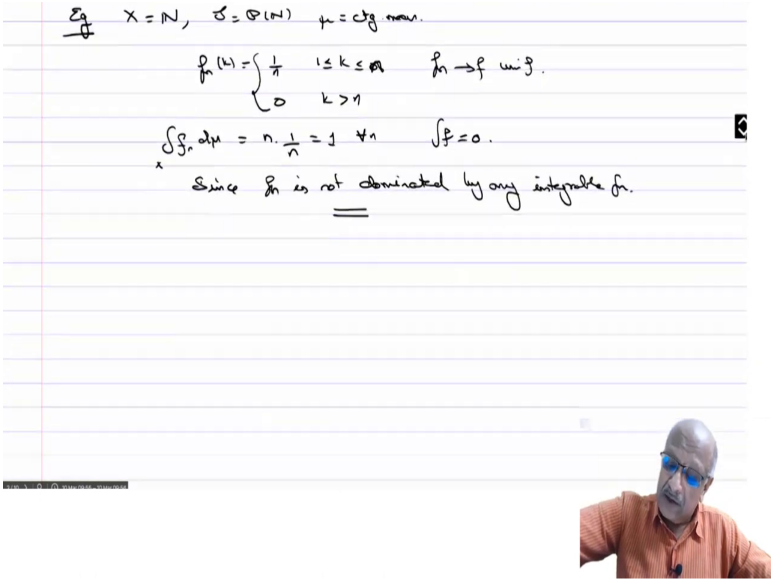

Example: you have x equals N x equals power set of n mu equals counting measure then you define fn of k equals 1 by n if 1 less than equal to k less than equal to n, 0 if k is bigger than n we have already seen this then we know that fn goes to f uniformly. However, integral fn d mu over x is this a simple function which is 1 by n up to k. So, it is n time up to n sorry equal to 1 for all n and integral f equal to 0.

So, you have that integral fn does not converge to integral f this is because since fn is not dominated by any integrable function. Because as n tends to infinity you have 1 by n is increasing up to n. So, you cannot have a function. So, if you want a function to dominate the entire thing it should be nonzero on the entire real line with constant value and that is not possible.

So, that is why this is not. So, this is shows that the domination of integral g is important and also in the proof you will see that you could get this lim sup plus and equal to in less than equal to 0 because you could cancel that integral g on both sides and that was because the integral of g is positive. So, we have used the hypothesis and we have shown by this example that this hypothesis is important.

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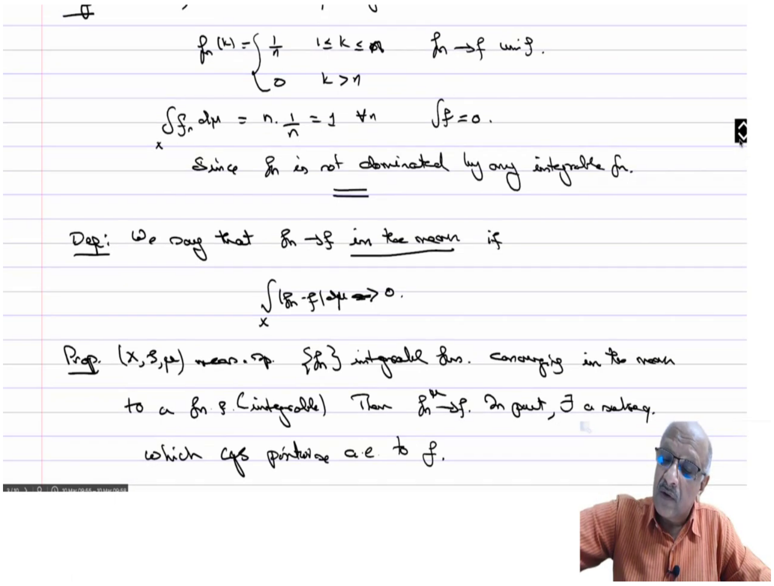

Definition: We say that $f_n \to f$ in the mean if $\int_X |f_n - f| d\mu \to 0$.

Proposition: (X, S, μ) measure space, $\{f_n\}$ integrable functions converging in the mean to a function f (integrable). Then $f_n \xrightarrow{\mu} f$. In particular, there exists a subsequence which converges point wise almost everywhere to f.

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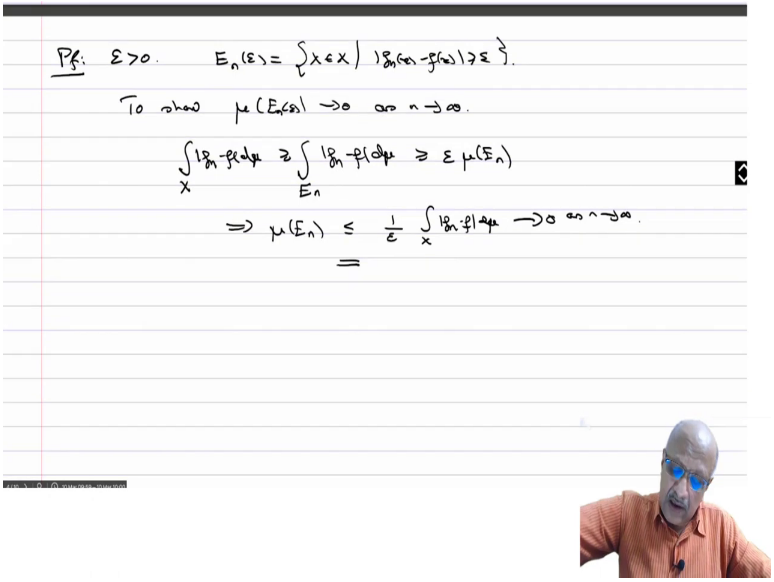

proof: So, $\epsilon > 0$. So, $E_n(\epsilon) = \{x \in X: |f_n(x) - f(x)| \geq \epsilon\}$.

To show: $\mu(E_n(\epsilon)) \to 0 \text{ as } n \to \infty$.

So, we have $\int_X |f_n - f| d\mu \geq \int_{E_n} |f_n - f| d\mu \geq \epsilon \mu(E_n)$

$$\Rightarrow \mu(E_n) \leq \frac{1}{\epsilon} \int_X |f_n - f| d\mu \to 0 \text{ as } n \to \infty.$$

So, now we give another result which is the generalization of the dominated convergence theorem.

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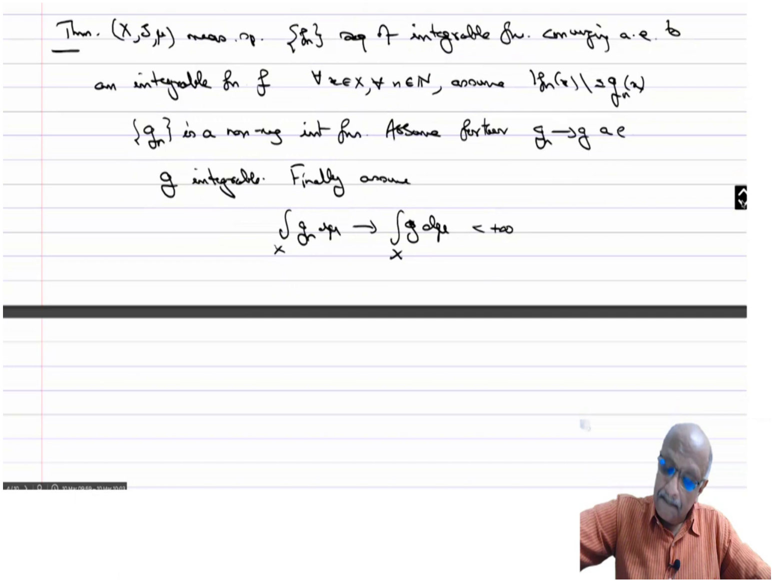

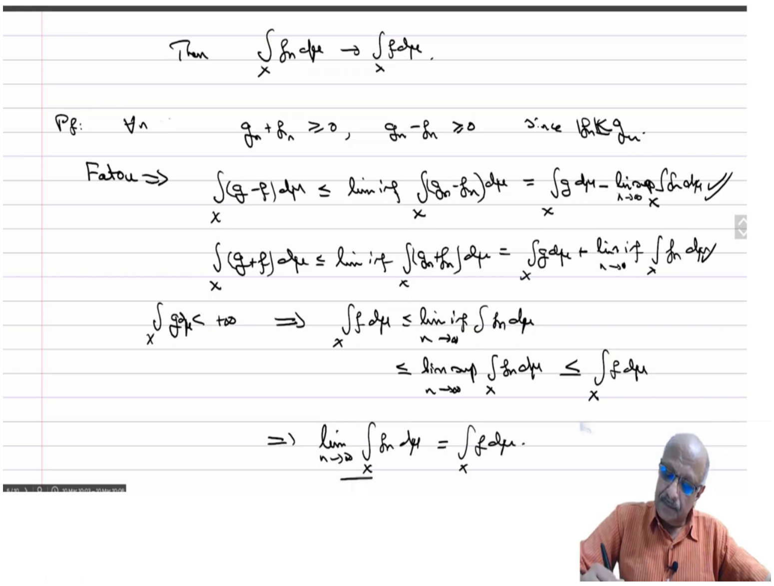

Theorem: (X, S, μ) measure space, $\{f_n\}$ integrable functions converges almost everywhere to f, for all $x \in X$, and for all $n \in \mathbb{N}$. Assume $f_n(x) \leq g_n(x)$, where g_n is nonnegative integrable functions. Assume further $g_n \to g$ almost everywhere, and g integrable. Finally, assume $\int_X g_n d\mu \to \int_X g d\mu < \infty$. Then $\int_X f_n d\mu \to \int_X f d\mu$.

proof: So, for every n, for every x we have $g_n + f_n \geq 0$, $g_n - f_n \geq 0$ since $|f_n| \leq g_n$.

So, again apply the Fatou lemma. So,

$$\int_X (g_n - f)d\mu \le \lim_{n\to\infty} \inf \int_X (g_n - f_n)d\mu = \int_X g d\mu - \lim_{n\to\infty} \sup \int_X f_n d\mu$$

$$\int_X (g_n + f)d\mu \le \lim_{n\to\infty} \inf \int_X (g_n + f_n)d\mu = \int_X g d\mu + \lim_{n\to\infty} \inf \int_X f_n d\mu.$$

So, now again we can cancel integral g is finite and therefore, we get you can cancel this.

So, you get $\int_X f d\mu \le \lim_{n\to\infty} \inf \int_X f_n d\mu \le \lim_{n\to\infty} \inf \int_X f_n d\mu \le \int_X f d\mu.$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

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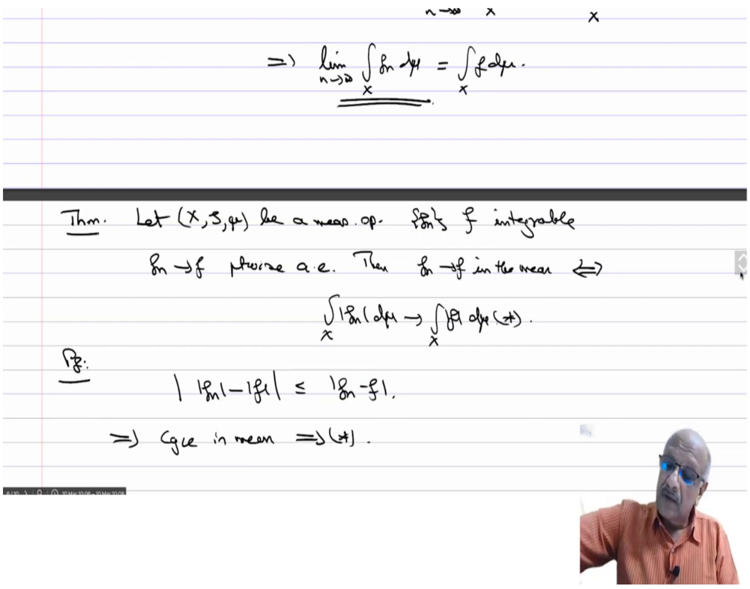

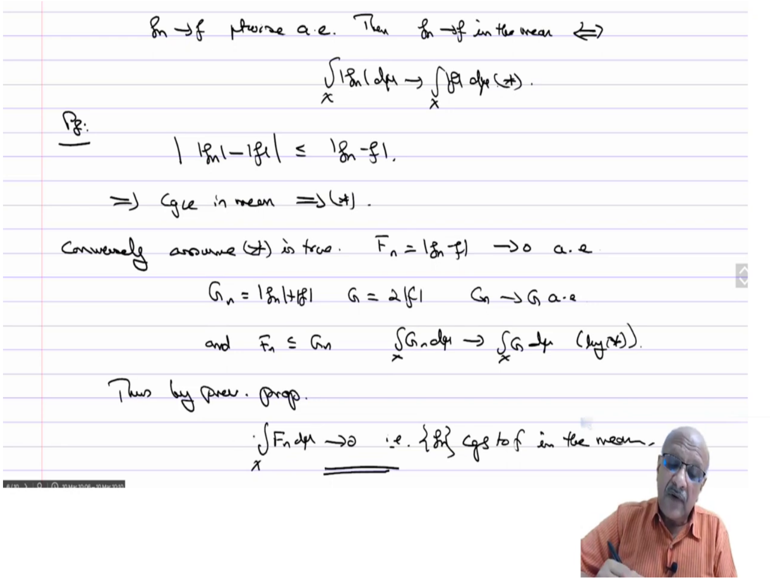

So, corollary of this result we have the following theorem.

Theorem: (X, S, μ) measure space, $\{f_n\}$, f integrable functions s.t. $f_n \to f$ pointwise almost everywhere. Then $f_n \to f$ in the mean if and only if

$$\int_X |f_n| d\mu \to \int_X |f| d\mu. \text{——--(*)}$$

proof. On one hand you have $||f_n| - |f|| \le |f_n - f|$. So, this implies that convergence in mean implies (*).

Now assume (*) is true. Now, you define $F_n = |f_n - f| \to 0$ almost everywhere. Define

$$G_n = |f_n| + |f|,\ G = 2|f.|$$

Then $G_n \to G$ almost everywhere and $F_n \le G_n$. So, $\int_X G_n d\mu \to \int_X G d\mu$ (by (*)).

Thus, by previous proposition, the dominant convergence theorem, $\int_X F_n d\mu \to 0,\ i.e.,$

$\{f_n\}$ converges to f in the mean.

So, next time we will see several examples of application of the dominated convergence theorem.