Measure and Integration

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Lecture-33

5.8-Integrable function

So, we have been looking at the Lebesgue integral of non-negative simple functions from which we also went ahead to define the Lebesgue integral of a non-negative measurable function.

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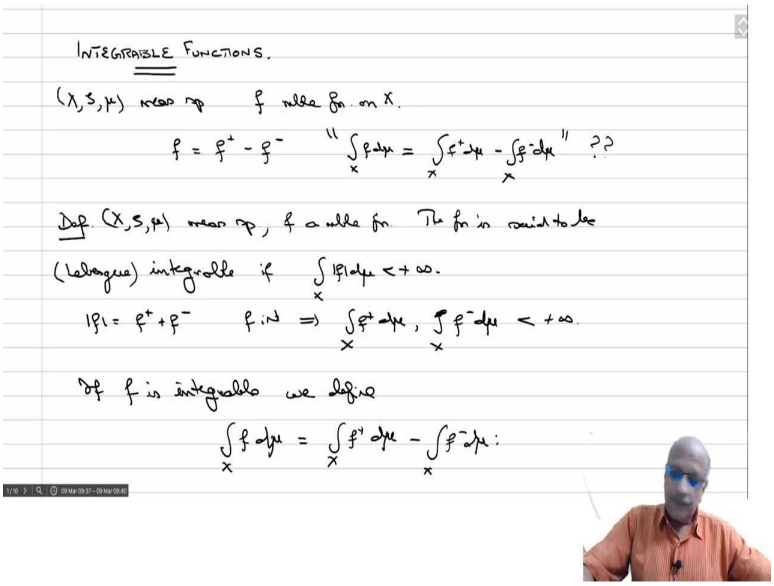

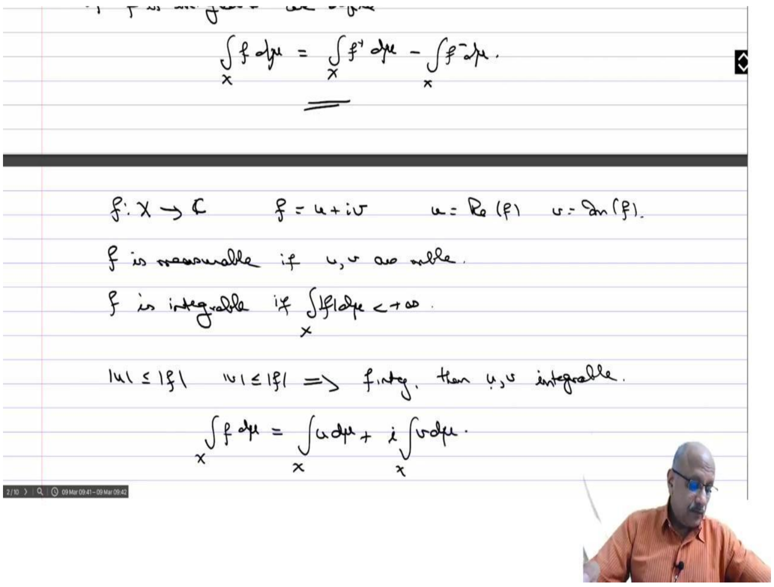

So, now, we will move on and deal with arbitrary measurable functions and therefore, we cannot do it for all functions of course, so, we want to now look at integrable functions so, these are functions for which an integral can be defined. So, (X, S, μ) measure space and you consider f measurable function on X. So, now, we can of course, write $f = f^+ - f^-$ and since each of these is non-negative, and you expect the integral to be linear therefore, the only possible definition you can give ``$\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$" ?

But there is a catch if both these happen to be infinite then f integral of f plus and integral of f minus if they are both infinite then we will not be able to define this difference in a meaningful ways and therefore, at least one of them has to be finite in this so, we make the following definition.

Definition: (X, S, μ) is a measure space and f a measurable function. The function f is said to be (Lebesgue) integrable, we will just say integrable in future, if

$$\int_X |f| d\mu < +\infty.$$

Now, $|f| = f^+ + f^-$. So, if f is integrable $\Rightarrow \int_X f^+ d\mu < +\infty$, $\int_X f^- d\mu < +\infty$.

So, if f is integrable, then we define $\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu$.

So, at this point we can consider all complex valued functions also. So, $f: X \to \mathbb{C}$ is a complex valued function, let us write it as in terms of the real and imaginary part. So,

$f = u + iv$, $u = Re(f)$, $v = Im(f)$. Therefore, both u and v are real valued functions. So, we say that f is measurable if u and v are measurable, f is integrable if |f| is integrable, same definition.

So, if f is integrable then $|u| \leq |f|$ and $|v| \leq |f|$ and therefore, this implies that if f is integrable then u and v are also integrable. And now, we can define

$$\int_X f d\mu = \int_X u d\mu + i \int_X v d\mu.$$

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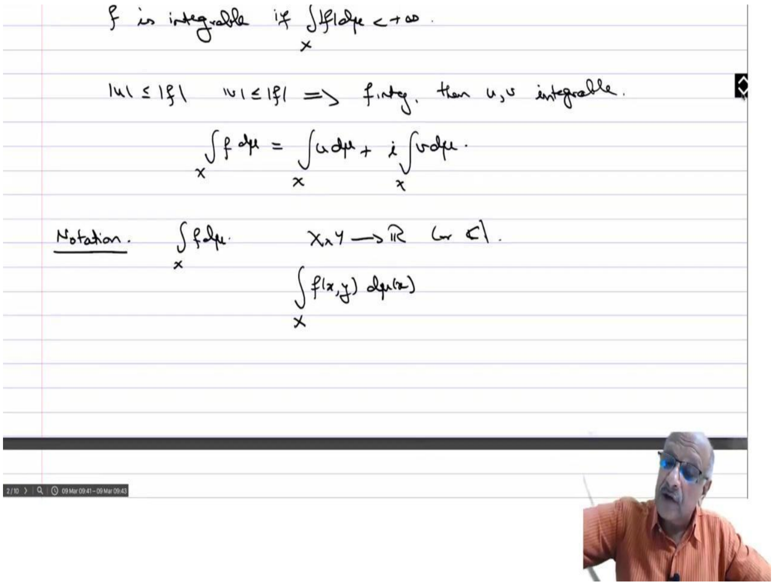

Notation: so, as long as there is no confusion, namely, there is only one space and so on and we will always write $\int_X f d\mu$. But in case you have say two variables $\mathbb{R}$ *or* $\mathbb{C}$ etc. and you have a function of two variables $X \times Y \to \mathbb{R}$ *or* $\mathbb{C}$ and we want to define an integration of this with respect to one of the variables. So, the measure space would be (X, S, μ). In that case we write the integral $\int_X f(x, y) d\mu(x)$ and this will say the variable with respect to which we are integrating, that is just a question of notation.

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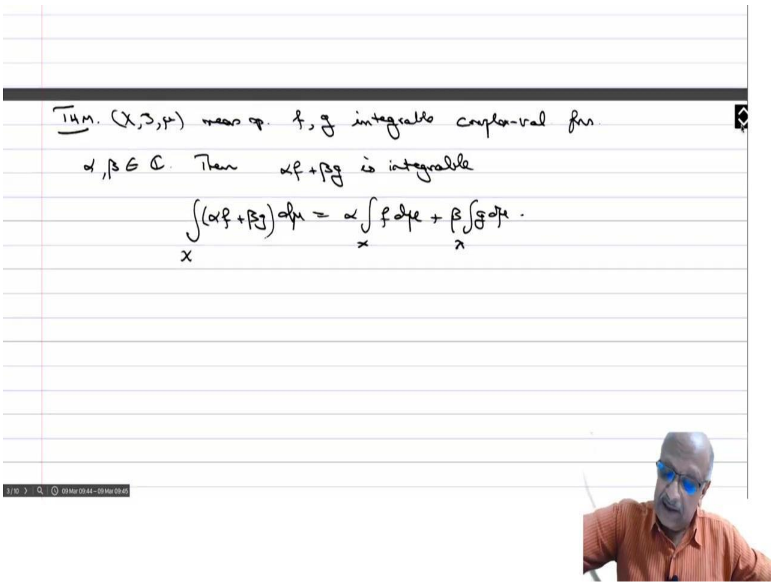

Theorem: (X, S, μ) measure space f and g integrable complex valued functions and $\alpha,\ \beta \in \mathbb{C}$. Then $\alpha f + \beta g$ is integrable and

$$\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu .$$

So, this is the full linearity of the Lebesgue integrals.

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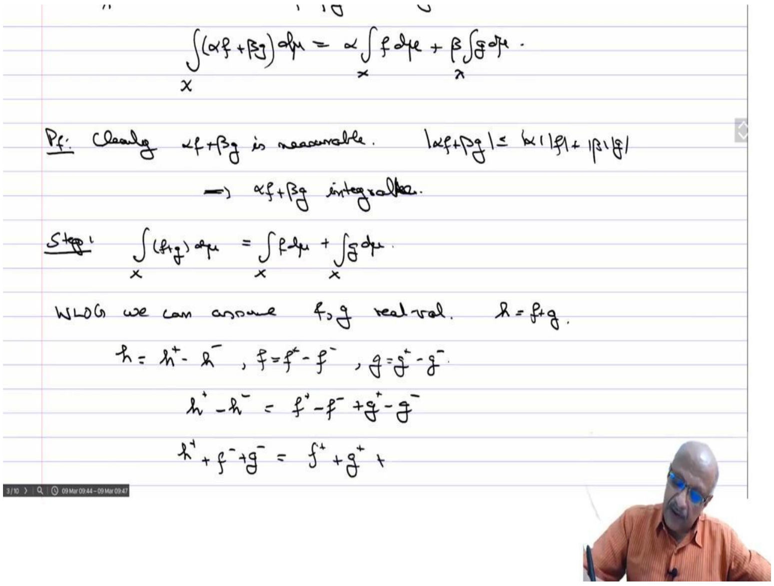

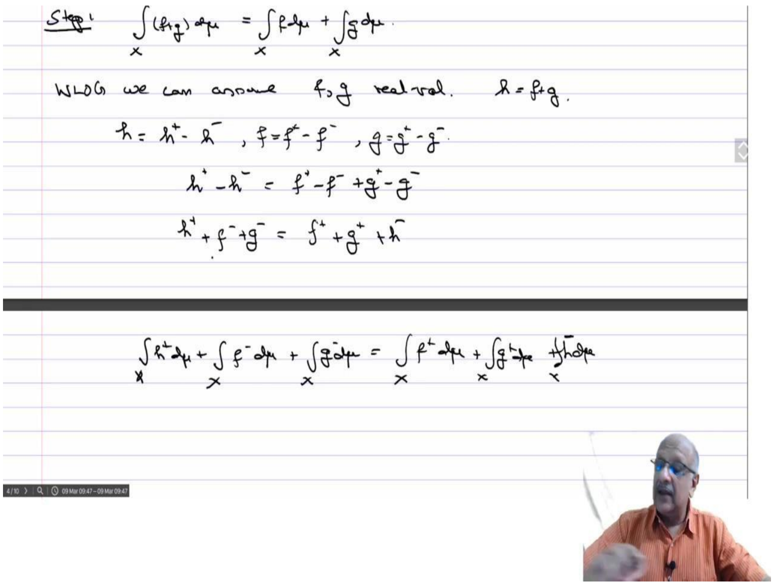

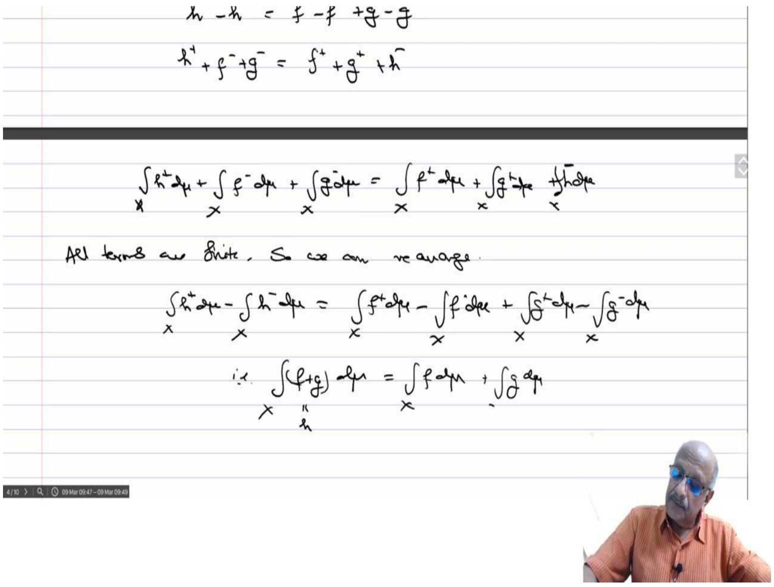

proof: Clearly, $\alpha f + \beta g$ is measurable and $|\alpha f + \beta g| \leq |\alpha||f| + |\beta||g|$. Therefore $\alpha f + \beta g$ is integrable.

step 1: $\int_X (f + g)d\mu = \int_X f d\mu + \int_X g d\mu$.

So, without loss of generality we can assume f, g are real valued. So, let us say $h = f + g$. And then we write $h = h^+ - h^-$, $f = f^+ - f^-$ and $g = g^+ - g^-$. So,

$$h^+ - h^- = f^+ - f^- + g^+ - g^- \Rightarrow h^+ + f^- + g^- = f^+ + g^+ + h^-.$$

So, for non-negative functions, we know that you can if some of the integral is equal to the integral of the sum. So, you get

$$\int_X h^+ d\mu + \int_X f^- d\mu + \int_X g^- \, d\mu = \int_X f^+ d\mu + \int_X g^+ d\mu + \int_X h^- \, d\mu.$$

Now, everything is finite so we can rearrange. Therefore,

$$\int_X h^+ d\mu \; - \int_X h^- d\mu = \int_X f^+ d\mu - \int_X f^- \, d\mu + \int_X g^+ d\mu - \int_X g^- \, d\mu.$$

So $\quad \int_X (f + g)d\mu = \int_X f d\mu + \int_X g d\mu.$

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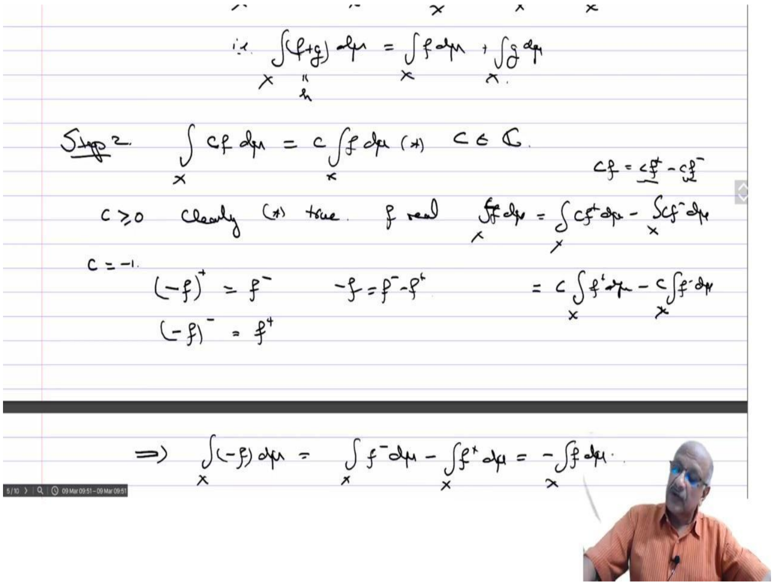

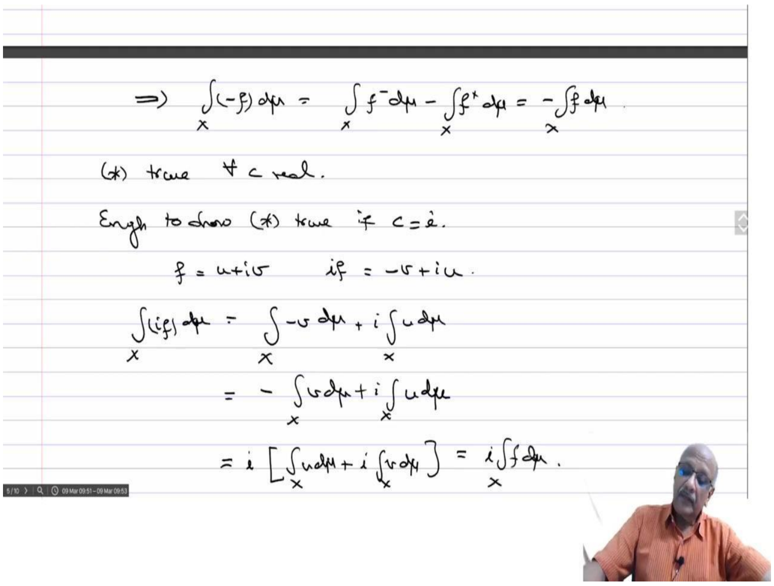

step 2: We have to show $\int_X cf d\mu = c\int_X f d\mu,\ c \in \mathbb{C}$. —-------- (*)

So, if c is non-negative that means a real number which is non-negative then clearly star is true by definition, because for non-negative functions you have and then the integral is defined using the real and imaginary parts and also for each of using the positive and negative parts.

So, for instance if f is real then you have integral f d Mu, c f, equals integral c f plus d Mu minus integral c f minus d Mu, because c f equals c f plus minus c f minus and these are the positive and negative parts of this function c f. And therefore, this is for positive functions we know that this is true d Mu minus c times integral f minus d Mu and therefore, this is true, therefore (*) is true for non-negative things.

Now, you let us take c equals minus 1, then what is minus f plus? This is nothing but f minus and minus f minus equals f plus because minus f equals f minus minus f plus. So, this is how these are the positive and negative parts and therefore you have this. So, obviously, integral minus f d Mu over X is equal to integral f minus d Mu over X minus integral f plus d Mu over X this is equal to minus integral f d Mu over X so, it is true also for this, so, this is true. Therefore, (*) true for all c real.

Enough to show (*) true if $c = i$.

So, okay you have $f = u + iv$ and therefore, $if = - v + iu$. So,

$$\int_X (if)d\mu = \int_X - v d\mu + i\int_X u d\mu = -\int_X v d\mu + i\int_X u d\mu = i[\int_X u d\mu + i\int_X v d\mu] = i\int_X f d\mu.$$

And therefore, this proves for all constants and therefore, the linearity is completely proved.

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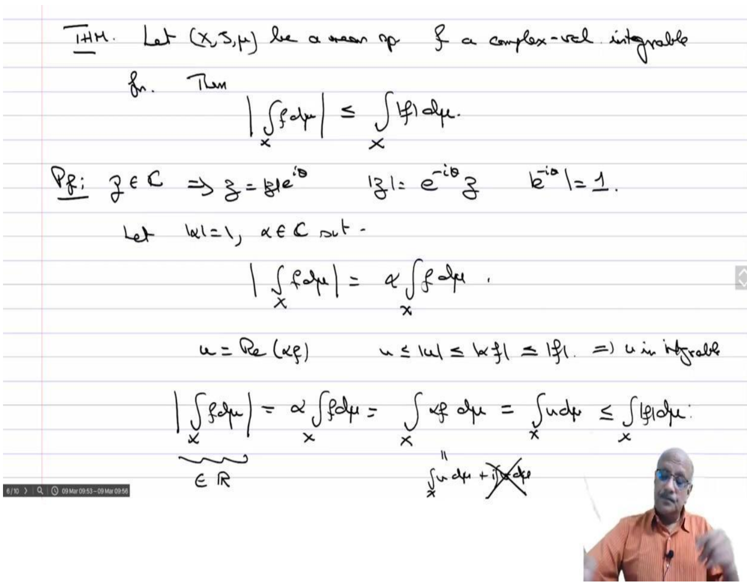

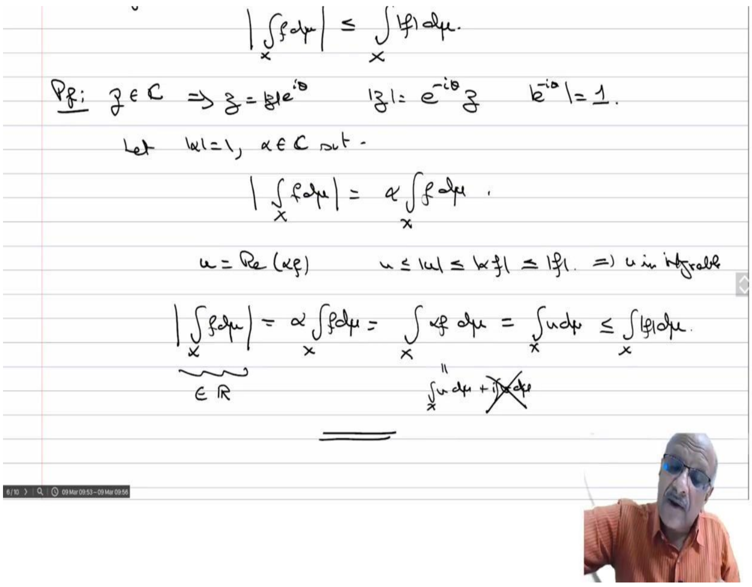

Theorem: So, (X, S, μ) be a measure space and f -a complex value integrable function. Then

$$\left|\int_X f d\mu\right| \leq \int_X |f| d\mu.$$

proof: so, if you have $z \in \mathbb{C}$, then you can write$|z| = e^{-i\theta} z$, he polar decomposition and therefore, $|z| = e^{-i\theta} z$. So, let $|\alpha| = 1,\ \alpha \in \mathbb{C}$, such that

$$\left|\int_X f d\mu\right| = \alpha \int_X f d\mu.$$

And let $u = Re(\alpha f)$. Then $u \leq |u| \leq |\alpha f| \leq |f| \Rightarrow u$ is integrable. So,

$$\left|\int_X f d\mu\right| = \alpha \int_X f d\mu = \int_X \alpha f d\mu = \int_X u d\mu \leq \int_X |f| d\mu.$$

And so, this completes this proof. And so next time we are going to look at another limit theorem like the monotone convergence theorem, the next theorem which we are going to prove will be an all important limit theorem in Lebesgue integration, it is one of the high points of the theory which we will see next time.