Measure and Integration

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Lecture - 32
Fatou's lemma

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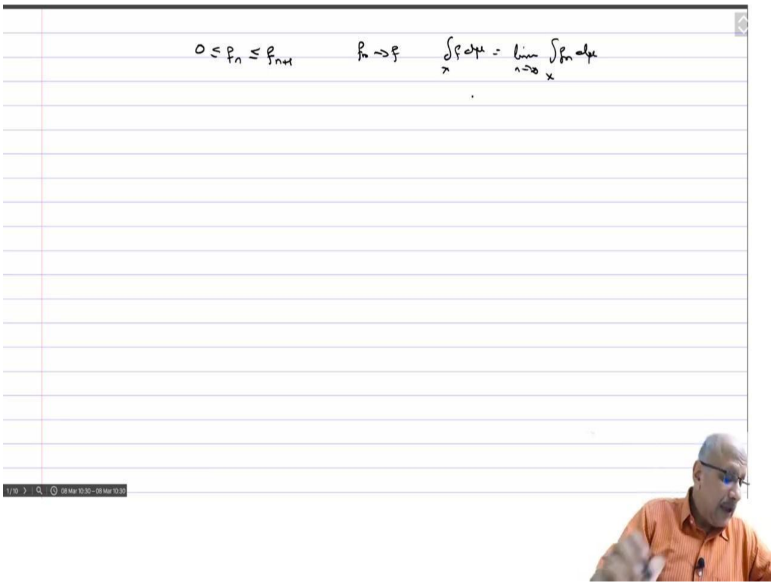

So, last time we looked at the monotone convergence theorem, which said that if you have a sequence of non-negative monotonic functions monotonically increasing sequence of functions sets then you have fn goes to fn then you have integral f d Mu over X is limit n tending to infinity integral fn d Mu over X.

So, this is a really simple theorem to apply: you need a monotonically increasing sequence of non-negative functions and then you are through. Of course, as I said both sides could be infinite and if 1 side is finite then the other side is finite and the two equal.

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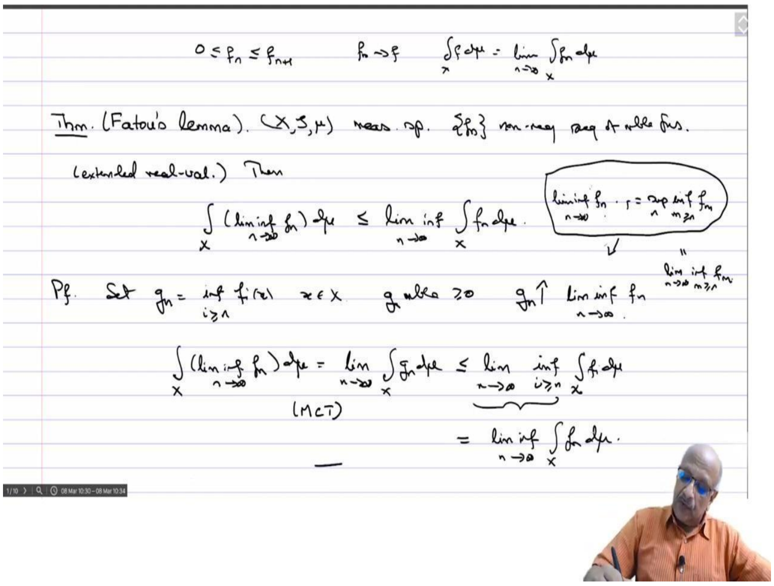

So, next we have a similar result based on the monotonic theorem again a very useful result is called Fatou's Lemma.

Theorem: (Fatou's lemma) So, (X, S, μ) measure space, and $\{f_n\}$ non-negative sequence of measurable functions. Then $\int_X (\lim_{n\to\infty} \inf f_n) d\mu \leq \lim_{n\to\infty} \inf \int_X f_n d\mu$.

proof: Set $g_n = \inf_{i\geq n} f_i(x),\ x \in X$. Then of course g_n is measurable non-negative and $g_n \uparrow \lim_{n\to\infty} \inf f_n$.

Therefore, by the monotone convergence theorem, you have

$$\int_X (\lim_{n\to\infty} \inf f_n) d\mu = \lim_{n\to\infty} \int_X g_n d\mu \leq \lim_{n\to\infty} \inf_{i\geq n} \int_X f_n d\mu = \lim_{n\to\infty} \inf \int_X f_n d\mu.$$

So, this proves Fatou's lemma.

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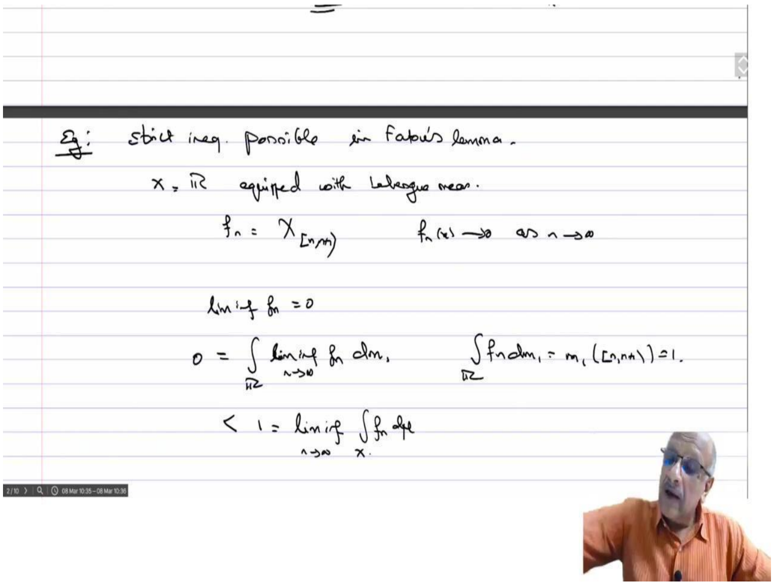

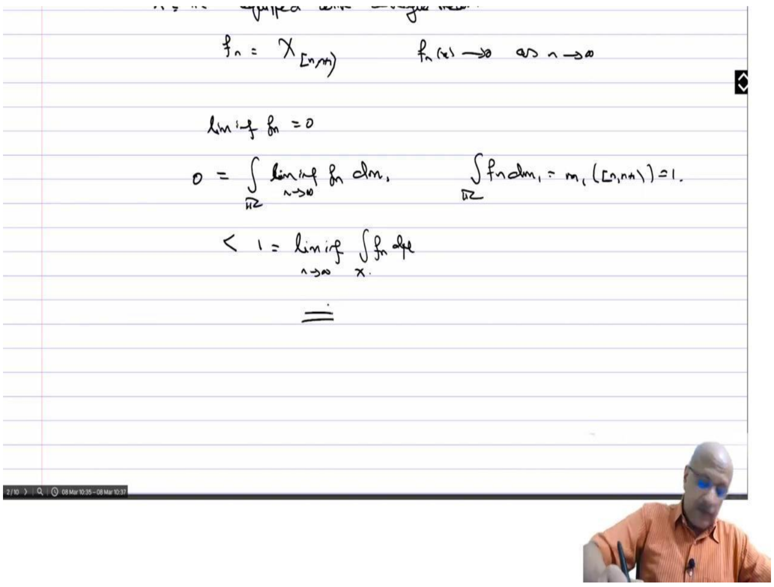

Example: Strict inequality is possible in Fatou's lemma. So, let us take $X = \mathbb{R}$ equipped with Lebesgue measure and then you say $f_n = \chi_{[n,n+1]}$. So, then $f_n(x) \to 0 \; as \; n \to \infty$.

Therefore, $\lim\limits_{n\to\infty} \inf f_n = 0$. So, you have $0 = \int_{\mathbb{R}} \lim\limits_{n\to\infty} \inf f_n dm_1$. But if you look at

$\int_{\mathbb{R}} f_n dm_1 = m_1([n, n+1]) = 1$ and therefore,

$$0 = \int_{\mathbb{R}} \lim_{n\to\infty} \inf f_n dm_1 < 1 = \lim_{n\to\infty} \inf \int_X f_n d\mu.$$

So, you have that it is possible to have strictly inequality in Fatou's lemma.

So, the next proposition is a variation of the monotone convergence theorem.

Proposition: So, (X, S, μ) measure space, $\{f_n\}$ sequence of non-negative extended real valued functions, for all x in X, assume converges $f_n(x) \to f(x)$ and $0 \le f_n(x) \le f(x)$.

Then, $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$.

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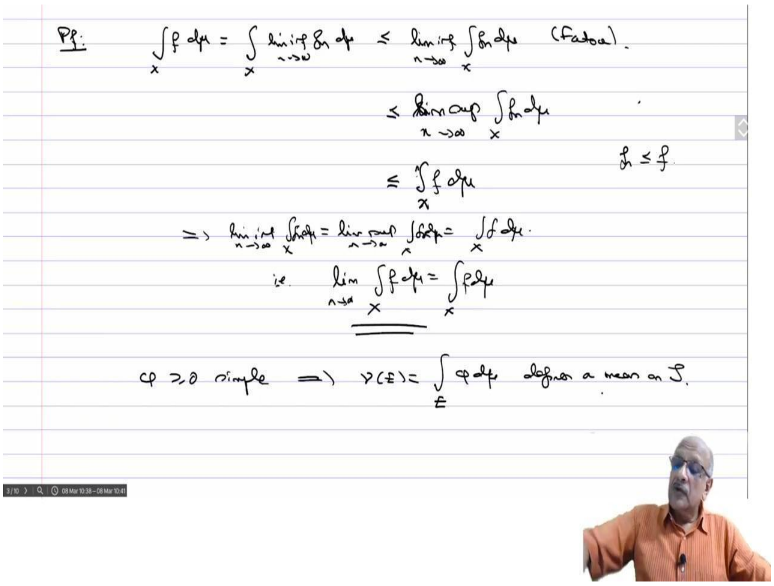

proof: So, you apply Fatou's lemma and you have that so,

$$\int_X f d\mu = \int_X \lim_{n\to\infty} \inf f_n d\mu \le \lim_{n\to\infty} \inf \int_X f_n d\mu \le \lim_{n\to\infty} \sup \int_X f_n d\mu \le \int_X f d\mu.$$

$$\Rightarrow \lim_{n\to\infty} \inf \int_X f_n d\mu = \lim_{n\to\infty} \sup \int_X f_n d\mu = \int_X f d\mu.$$

$$\Rightarrow \lim_{n\to\infty} \int_X f_n d\mu = \int_X f \, d\mu.$$

That proves the theorem.

So, earlier we showed that if $\phi \ge 0$ is a simple function $\Rightarrow \nu(E) = \int_E \phi d\mu$ defines a measure on S.

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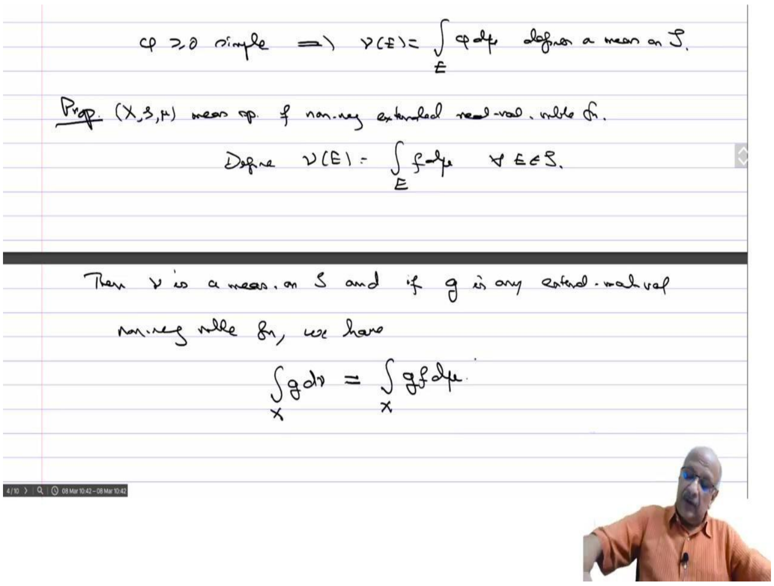

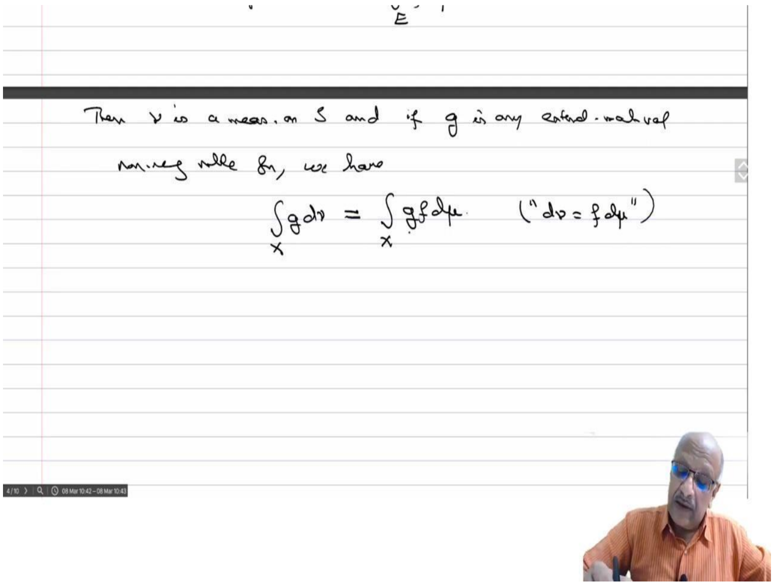

So, now, we are going to generalize this result in the following proposition:

Proposition: (X, S, μ) measure space, f non-negative extended real valued measurable function. Define $\nu(E) = \int_E f d\mu$, $\forall\, E \in S$.

Then ν is a measure on S and if g is any extended real valued non-negative measurable function we have $\int_X g d\mu = \int_X g f d\mu$. $(d\nu = f d\mu)$. .... (*)

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non-neg mble fn, we have

$$(*) \qquad \int_X g \, d\nu = \int_X g f \, d\mu \qquad (\text{"} d\nu = f \, d\mu \text{"})$$

Pf: $\nu(\phi) = 0$, $\nu \geq 0$. $E = \bigcup_{i=1}^{\infty} E_i$ $E_i \in S$, $\{E_i\}$ disjt.

$$\chi_E = \sum_{i=1}^{\infty} \chi_{E_i}.$$

$$\nu(E) = \int_E f \, d\mu = \int_X f \chi_E \, d\mu = \int_X \Big(\sum_{i=1}^{\infty} f \chi_{E_i}\Big) d\mu$$

$$= \sum_{i=1}^{\infty} \int_X f \chi_{E_i} \, d\mu = \sum_{i=1}^{\infty} \int_{E_i} f \, d\mu = \sum_{i=1}^{\infty} \nu(E_i).$$

$\Rightarrow \nu$ is a meas.

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$\Rightarrow \nu$ is a meas.

Assume $g = \chi_E$ $E \in S$.

$$\int_X g \, d\nu = \nu(E) = \int_E f \, d\mu = \int_X f \chi_E \, d\mu = \int_X f g \, d\mu.$$

(*) OK for char fn

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$g = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$ $\qquad A_i \in S$, $\alpha_i \geq 0$.

$$\int_X g \, d\nu = \sum_{i=1}^{k} \alpha_i \int_X \chi_{A_i} \, d\nu = \int_X f \sum_{i=1}^{k} \alpha_i \chi_{A_i} \, d\mu = \int_X f g \, d\mu.$$

(*) OK for ≥ 0, simple fns.

$g \geq 0$ mble $\exists \varphi_n$ simple $0 \leq \varphi_n \leq g$ $\varphi_n \uparrow g$.

$$\int_X \varphi_n \, d\nu = \int_X f \varphi_n \, d\mu. \qquad f\varphi_n \uparrow fg.$$

MCT $\qquad \int_X g \, d\nu = \int_X f g \, d\mu.$

proof: So, clearly $\nu(\phi) = 0$ and $\nu \geq 0$, and therefore, we have to check so, $E = \cup_{i=1}^{\infty} E_i$, $E_i \in S$, E_i disjoint. So, now $\chi_E = \sum_{i=1}^{\infty} \chi_{E_i}$. So now, if you take

$$\nu(E) = \int_E f d\mu = \int_X f\chi_E d\mu = \int_X (\sum_{i=1}^{\infty} f\chi_{E_i}) d\mu = \sum_{i=1}^{\infty} \int_X f\chi_{E_i} d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu = \sum_{i=1}^{\infty} \nu(E_i).$$

So, this implies that ν is a measure. So now, we want to establish (*). So, assume $g = \chi_E$, $E \in S$. Then $\int_X g d\mu = \nu(E) = \int_E f d\mu = \int_X f\chi_E d\mu = \int_X fg d\mu$.

So, (*) is OK for characteristic functions. Now, you take $g = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$, $A_i \in S$, $\alpha_i \geq 0$.

Then we know that if you have a finite sum of non-negative functions then the integral of the sum is the sum of the integral and alpha can come out. And therefore,

$$\int_X g d\mu = \sum_{i=1}^{k} \alpha_i \int_X \chi_{A_i} d\mu = \int_X f \sum_{i=1}^{k} \alpha_i \chi_{A_i} d\mu = \int_X fg d\mu.$$

So, (*) is Ok for non-negative, simple functions. So, now, if $g \geq 0$ is measurable, so, there exists ϕ_n simple, $0 \leq \phi_n \leq g$ and $\phi_n \uparrow g$. So,

$\int_X \phi_n d\mu = \int_X f\phi_n d\mu$. Now, phi n increases to g non-negative function, so, by the monotone convergence theorem, $\int_X g d\mu = \int_X fg d\mu$.

Therefore, this establishes the theorem.

So, this is a very important powerful technique like a proof machine: you prove something for characteristic functions, use linearity to prove it for simple functions, use a suitable limit theorem like the monotone convergence theorem to prove it for an arbitrary non-negative function.

Now, if you want to do it further, we have not come to arbitrary functions yet, but in general you know how what how the proof will go because he will take an arbitrary function break it up as f plus and f minus for each of them you prove it because you know you can do it for this and then take the difference, so that will be the idea, so, this is a very nice proof technique, which you have.

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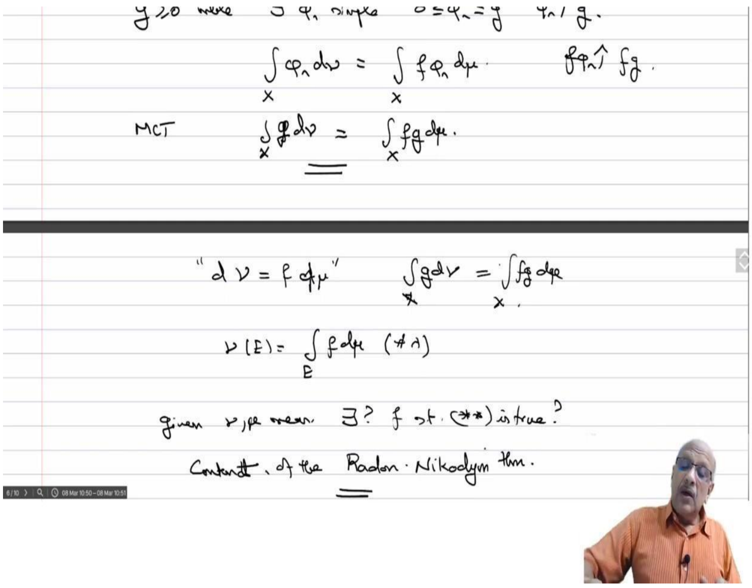

So, as I said already, we have said if the indefinite integral Nu could be witness f d Mu that means $\int_X g d\mu = \int_X f g d\mu$. Now, when can you do so, and you have

$$\nu(E) = \int_E f d\mu. \qquad (**)$$

So, given two measures ν and μ, when can you write this is the third existing episode given ν and μ measures does there exist if such a (**) is true.

So, this is the content of the Radon Nikodym theorem. When you can do this we will see that much-much later in this course. So, this is a particular case which we have seen.