Measure and Integration
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Lecture-31
Monotone convergence theorem

So, we started with simple non-negative simple functions and then non-negative functions, we defined what was the integral. So, we would like to know how to compute these integrals though we have a formal definition. So, given some measures, we would like to know how to compute these various integrals.

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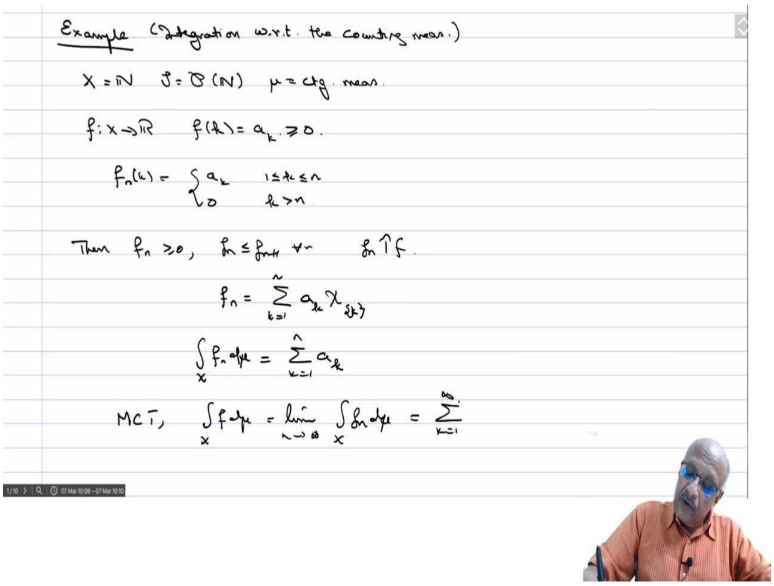

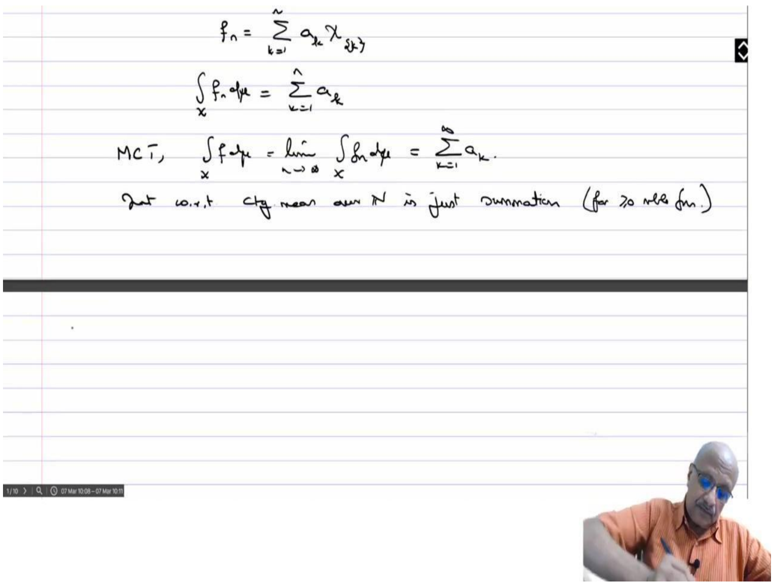

So, we do some examples now.

Example: (integration with respect to the counting measure). So, $X = \mathbb{N}$, $S = P(\mathbb{N})$, $\mu =$ counting measure. So, now, you define n the function $f: X \to \mathbb{R}$, so, $f(k) = a_k \geq 0$. So, now, let us define

$$f_n(k) = a_k,\ 1 \leq k \leq n,$$

$$= 0,\ k > n.$$

Then $f_n \geq 0$, $f_n \leq f_{n+1}$ *and* $f_n \uparrow f$. Also $f_n = \sum_{k=1}^{n} a_k \chi_{\{k\}}$. And therefore, this is a non-negative simple function. Therefore $\int_X f_n d\mu = \sum_{k=1}^{n} a_k$. Then by the monotone convergence theorem (MCT), you have

$$\int_X f\ d\mu = \lim_{n \to \infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k.$$

So, integration with respect to counting measure over n is just summation for non-negative measurable functions. And again the sum of the series need not be finite, so the integral can be infinite. So, this is just a summation which you have here.

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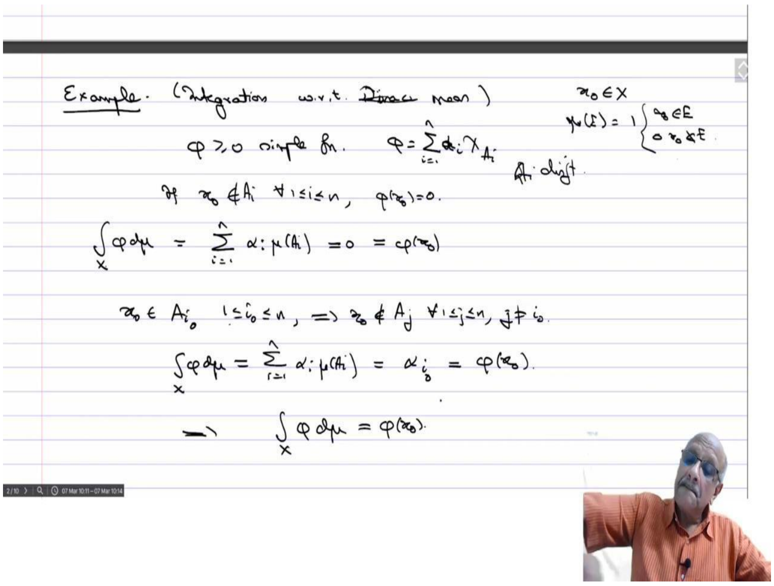

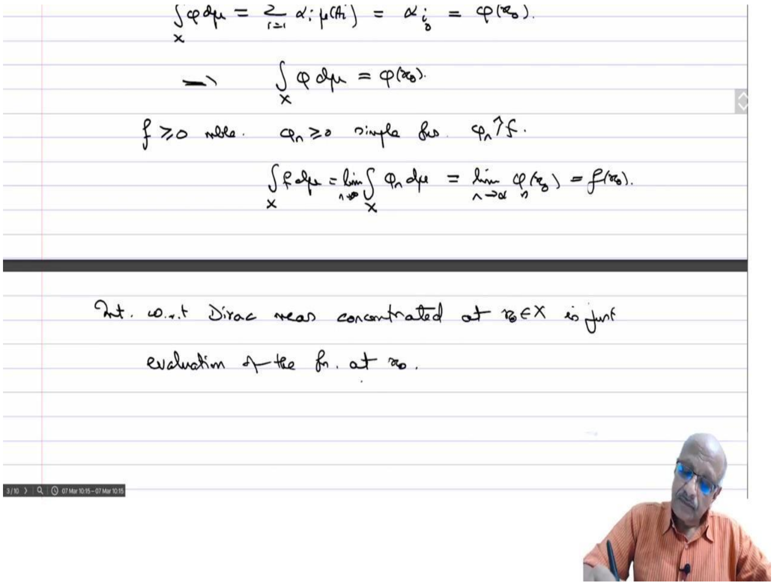

Example: (integration with respect to the Dirac measure) What is Dirac measure? You have $x_0 \in X$, and then you have $\mu(E) = 1, \ if \ x_0 \in E \ and \ \mu(E) = 0, \ if \ x_0 \notin E$. This is the Dirac measure. So, we want to know what is the integration with respect to this. So, $\phi \geq 0$ simple function. If so, $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, A_i disjoint. So, if $x_0 \notin A_i$, $\forall\, 1 \leq i \leq n, \ then\ \phi(x_0) = 0$. Also,

$$\int_X \phi d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) = 0 = \phi(x_0).$$

So, $x_0 \in A_{i_0}$, $1 \leq i_0 \leq n$ and therefore, this implies that $x_0 \notin A_j$, $\forall\, 1 \leq j \leq n, \ j \neq i_0$.

Then $\int_X \phi d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) = \alpha_{i_0} = \phi(x_0) \Rightarrow \int_X \phi d\mu = \phi(x_0)$.

Now, if f is any non-negative measurable function then ϕ_n is simple functions $\phi_n \uparrow f$, then you have that $\int_X \phi d\mu = \lim_{n \to \infty} \int_X \phi_n d\mu = \lim_{n \to \infty} \phi_n(x_0) = f(x_0)$.

So, integration with respect to Dirac measure concentrated at x_0 is just evaluation of the function at x_0.

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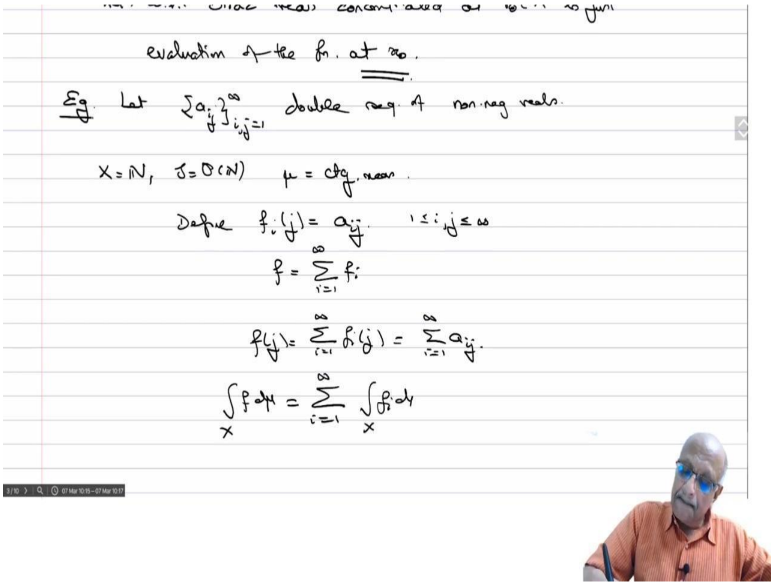

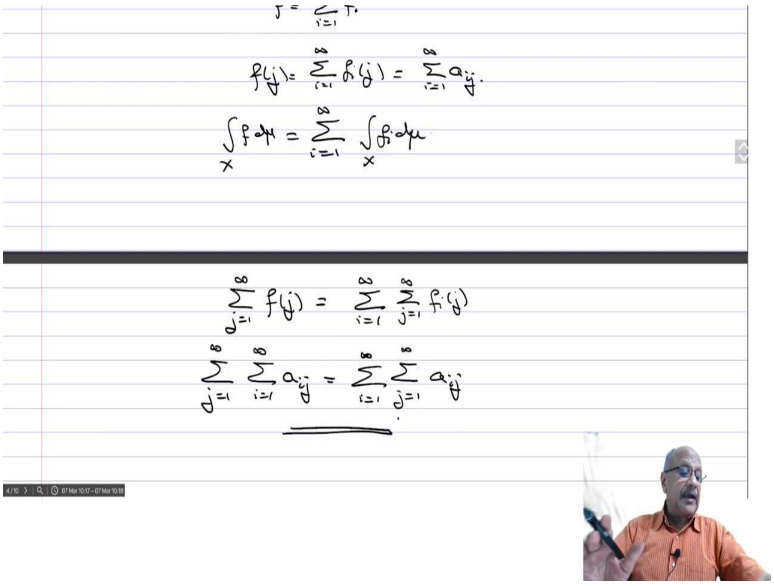

Example: So, let $\{a_{ij}\}_{i,j=1}^{\infty}$ are doubled sequences of non-negative reals. So, $X = \mathbb{N}$, $S = P(N)$, $\mu =$ counting measure. Define $f_i(j) = a_{ij}$, $1 \leq i, j \leq n$. And define

$$f = \sum_{i=1}^{\infty} f_i.$$

So, then $f(j) = \sum_{i=1}^{\infty} f_i(j) = \sum_{i=1}^{\infty} a_{ij}$. Now, we know that

$$\int_X f d\mu = \sum_{i=1}^{\infty} \int_X f_i d\mu.$$

So, $\int_X f d\mu = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

Therefore, if you have a non-negative double sequence we have proved using the methods of measure theory and it is a well-known result in theory of summation of series.

Then if you have a non-negative double sequence then you can interchange the order of summation without any problem. Either both sides will be finite and will be equal or both sides will be infinite. That is the understanding in this thing, so we have.

Now, this is not necessarily true for arbitrarily double sequences. We will see examples and that also we will use some limit theorems from measure theory to show when you can interchange the order of summation when aij's are not necessarily non-negative, we will see that example much later. So, right now, for non-negative double sequences we have that you can interchange the order of summation without any problem.