Measure and Integration
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Lecture 30
Monotone convergence theorem

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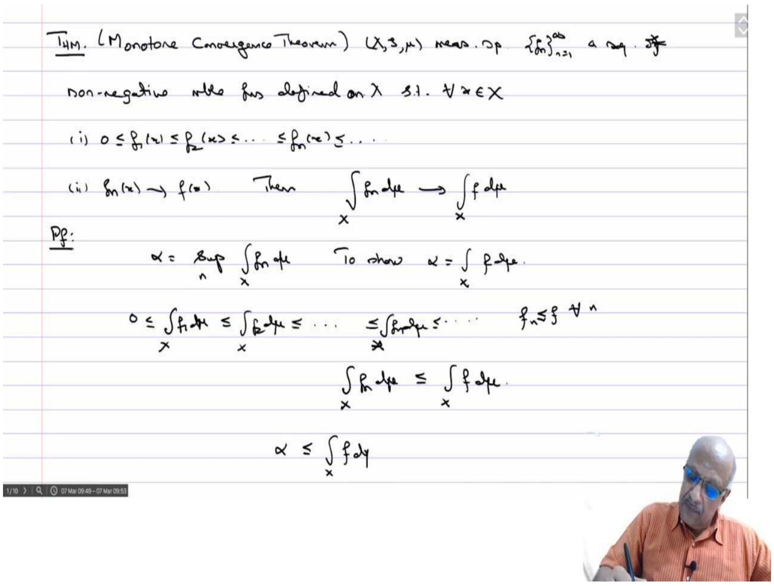

So, we will now prove the first important limit theorem. So, this is theorem called the **monotonic convergence theorem**. So,

Theorem: (X, S, μ) is a measure space and $\{f_n\}_{n=1}^{\infty}$ a sequence of non-negative measurable functions defined on X such that for every $x \in X$ we have

(i) $0 \leq f_1(x) \leq f_2(x) \leq \ .\ .\ .\ .\ \leq f_n(x) \leq \ .\ .\ .$

(ii) $f_n(x) \to f(x)$,

Then $\int_X f_n \, d\mu \to \int_X f \, d\mu$

So, we are having the integral limit of the integrals is the integral of the limit. So, this is the you can interchange the integration is a limiting process by itself we also have pointwise limits and integral of the, so you can interchange the two limit process the limit of the integrals is the same as the integral of the limit.

In other words a limit can go inside the integral side. So,

Proof, so you let $\alpha = sup_n \int_X f_n \, d\mu$

So, we need to show in fact $\alpha = \int_X f \, d\mu$. So, since you have a increasing sequence you have

$$0 \le \int_X f_1 \, d\mu \le \int_X f_2 \, d\mu \le \ . \ . \ . \ \le \int_X f_n \, d\mu . . .$$

and you also have $\{f_n\}_{n=1}^{\infty}$ is increasing sequence and so, $f_n \le f$ for all n.

So, $\int_X f_n \, d\mu \le \int_X f \, d\mu$. So, this is an upper bound for all these things and therefore, you have that $\alpha \le \int_X f \, d\mu$. So, we have to show the converse.

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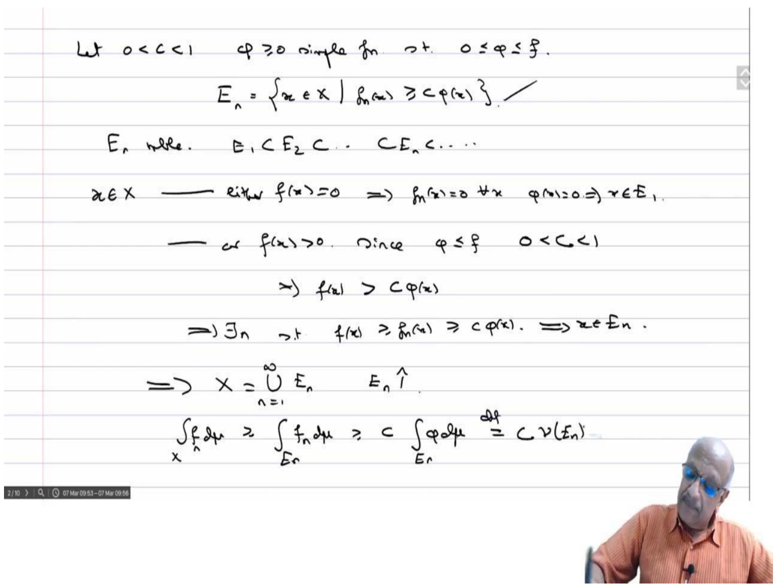

$$\int_X f\,d\mu \geq \int_{E_n} f_n\,d\mu \geq c\int_{E_n}\varphi\,d\mu = c\,\nu(E_n) \checkmark$$

$$\nu(E_n) = \int_E \varphi\,d\mu$$

$\nu(E) = \int_E \varphi\,d\mu$ defines a measure

$$\alpha \geq c\lim_{n\to\infty}\nu(E_n) = c\,\nu(X) = c\int_X\varphi\,d\mu.$$

True $\forall\ 0\le\varphi\le f$ $\qquad \alpha \geq c\int_X f\,d\mu$

" $\forall\ 0<c<1$ $\qquad \Rightarrow \alpha \geq \int_X f\,d$

$$\int_X f_n\,d\mu \le \int_X f\,d\mu.$$

$$\alpha \le \int_X f\,d\mu. \checkmark$$

Let $0<c<1$ $\quad \varphi\geq 0$ simple fn. s.t. $0\le\varphi\le f$.

$$E_n = \{x\in X \mid f_n(x) \geq c\,\varphi(x)\}$$

E_n mble. $\quad E_1\subset E_2\subset\cdots\subset E_n\subset\cdots$

$x\in X$ — either $f(x)=0 \Rightarrow f_n(x)=0\ \forall n \quad \varphi(x)=0 \Rightarrow x\in E_1$.

— or $f(x)>0$. since $\varphi\le f \quad 0<c<1$

$\nu(E) = \int_E \varphi\,d\mu$ defines a measure

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True $\forall\ 0\le\varphi\le f$ $\qquad \alpha \geq c\int_X f\,d\mu$

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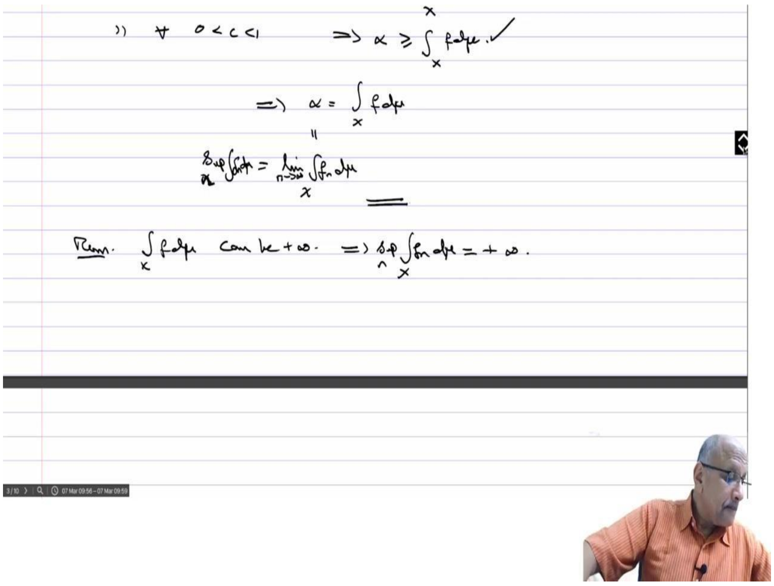
$$\Rightarrow \alpha = \int_X f\,d\mu$$

$$\sup_n \int f_n = \lim_{n\to\infty}\int_X f_n\,d\mu$$



So, now, $0 < c < 1$, be a fixed constant and φ non-negative simple function such that $0 \leq \varphi \leq f$. So, define $E_n = \left\{x \in X : f_n(x) \geq c\varphi(x)\right\}$.

Now, E_n is of course measurable and you have that

$$E_1 \subset E_2 \subset \ . \ . \ . \ \subset E_n \subset \ . \ . \ . \ ,$$

because f_n is an increasing sequence, so if $x \in E_n$ then f and x is bigger than $c\varphi(x)$ that means, $f_{n+1}(x) \geq c\varphi(x)$ that means $x \in E_{n+1}$, so E_n is an increasing sequence. So, there are two possibilities for $x \in X$.

So, either $f(x) = 0$ this implies $f_n(x)$ for all x and $\varphi(x) = 0$ and you have in this case all this implies

$x \in E_1$ or $f(x) > 0$ and since $\varphi \leq f$, $0 < c < 1$, this implies that $f(x) \geq c\varphi(x)$.

Because this is the supremum there exists an n such that $f(x)$ is greater or equal to $f_n(x)$ greater or equal to $c\varphi(x)$, which implies that $x \in E_n$. So, in other words, this implies that

$$X = \bigcup_{n=1}^{\infty} E_n, \quad E_n \uparrow$$

Now, $\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq C \int_{E_n} \varphi \, d\mu = C\, \nu(E_n)$.

So, $\nu(E_n) = \int_{E_n} \varphi \, d\mu$. But what do you know φ is a simple non-negative simple function.

So, $\nu(E\ ) = \int_E \varphi \, d\mu$ defines a measure and therefore, if you pass to the limit in this relationship here, so, if you then you will get

$\alpha \geq C \lim_{n \to \infty} \nu(E_n) = C\nu(X) = C \int_X \varphi \, d\mu$.

So, this is true for all phi through $0 \leq \varphi \leq f$. Therefore, by definition $\alpha \geq C \int_X \varphi \, d\mu$, by definition. Now, $0 < C < 1$ this is true for all C. So, this implies that $\alpha \geq \int_X \varphi \, d\mu$ and that is what we wanted to prove.

So, we already have α less than equal to $f\, d\mu$ and now we have α greater than equal to $f\, d\mu$ this implies $\alpha = \int_X f \, d\mu$ and therefore, this is nothing but limit integral fn d mu over x, this equal to also Sup integral fn d mu over n. So, this proves the monotone convergence theorem very useful, very simple to apply, you need a sequence of non-negative measurable functions which is increasing so, monotonic increasing then the limit of the integrals is the integral of the limit.

So,

Remark: $\int_X f \, d\mu = +\infty$. So, this just implies that the $sup \int_X f_n \, d\mu = +\infty$. So, both sides say infinity and that is it.

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Prop $(X, \mathcal{S}, \mu)$ measure sp. $\{f_n\}$ non-neg. mble fns.

$$\text{Define } f(x) = \sum_{n=1}^{\infty} f_n(x) \quad x \in X$$

Then $f \geq 0$, mble and

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu.$$

Pf: $g_n = f_1 + \cdots + f_n \Rightarrow g_n$ mble. $g_n \uparrow f \Rightarrow f$ is mble.

$0 \leq \varphi_n \leq f_1$, $\quad 0 \leq \psi_n \leq f_2 \quad \varphi_n, \psi_n$ simple $\quad \varphi_n \uparrow f_1 \quad \psi_n \uparrow f_2$.

$0 \leq \varphi_n + \psi_n$ simple $\uparrow f_1 + f_2$.

By MCT, $\lim_{n\to\infty} \int_X (\varphi_n + \psi_n) \, d\mu = \int_X (f_1 + f_2) \, d\mu$

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$$\int_X \varphi_n \, d\mu \to \int_X f_1 \, d\mu$$

$$\int_X \psi_n \, d\mu \to \int_X f_2 \, d\mu.$$

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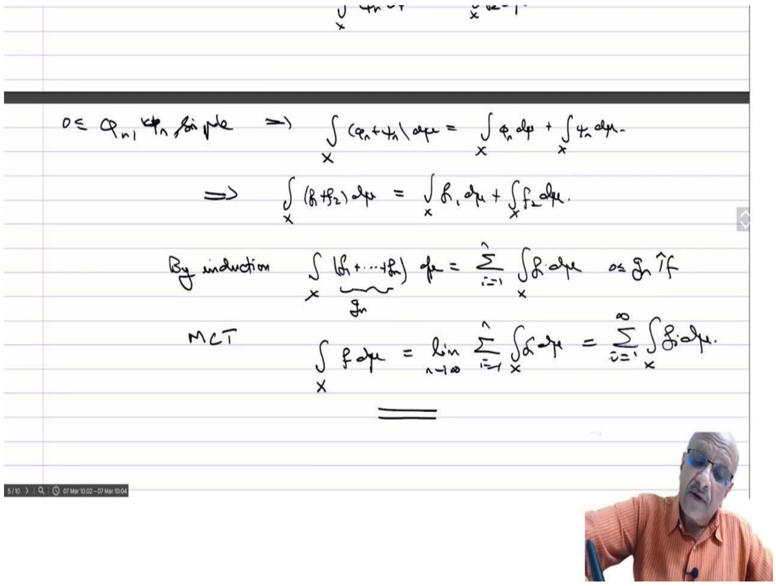
$$\int_X \psi_n \, d\mu \to \int_X f_2 \, d\mu.$$

$0 \leq \varphi_n, \psi_n$ simple $\Rightarrow \int_X (\varphi_n + \psi_n) \, d\mu = \int_X \varphi_n \, d\mu + \int_X \psi_n \, d\mu.$

$$\Rightarrow \int_X (f_1 + f_2) \, d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d$$



So, now we can give an application of this. So,

Proposition: (X, S, μ) measure space $\{f_n\}$ non-negative measurable functions. Define

$$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

, then $f \geq 0$, measurable and

$$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu \ .$$

So, it is a point vise sum and the integral is also the you can then take the if you take the integral of f, the integral and the summation sign can be interchange. So, this is again a non-trivial observation works in working for non-negative functions.

Proof. So, you take $g_n = f_1 + f_2 + \ . \ . \ . \ + f_n$, implies g_n is measurable and then g_n increases to f implies g, f is measurable, we already saw the supremum of measurable functions is measurable. So, that makes it measurable. So, now, let us take φ_n and ψ_n simple, $0 \le \varphi_n \le f_1$, $0 \le \psi_n \le f_2$, and $\varphi_n \uparrow f_1$, $\psi_n \uparrow f_2$

we can always do this because of that theorem. Then we also have the $\varphi_n + \psi_n$ is simple and increases to $f_1 + f_2$.

So, we have by the monotone convergence theorem, you have a

$$\lim_{n \to \infty} \int_X (\varphi_n + \psi_n)\, d\mu = \int_X (f_1 + f_2)\, d\mu$$

$$\int_X \varphi_n \, d\mu \to \int_X f_1 \, d\mu \text{ and } \int_X \psi_n \, d\mu \to \int_X f_2 \, d\mu$$

But we also saw φ_n *and* ψ_n are simple, soφ_n *and* ψ_n simple, this implies a non-negative, so, this implies that

$$\int_X (\varphi_n + \psi_n)\, d\mu = \int_X \varphi_n \, d\mu + \int_X \psi_n \, d\mu$$

And therefore, passing to the limit you get

$$\int_X (f_1 + f_2)\, d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu$$

So, by induction

$$\int_X (f_1 + \ldots + f_n)\, d\mu = \sum_{i=1}^{n} f_i \, d\mu.$$

Now, this is the function $g_n = f_1 + \ldots + f_n$ and g_n increases to f all non-negative, therefore, again by the monotone convergence theorem,

$$\int_X f \, d\mu = \lim_{n \to \infty} \sum_{i=1}^{n} \int_X f_i \, d\mu = \sum_{i=1}^{n} \int_X f_i \, d\mu.$$

and that completes the proof. So, we will now do some examples in the next session.