Measure and Integration
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Lecture No- 3
1.3 – Measures on rings

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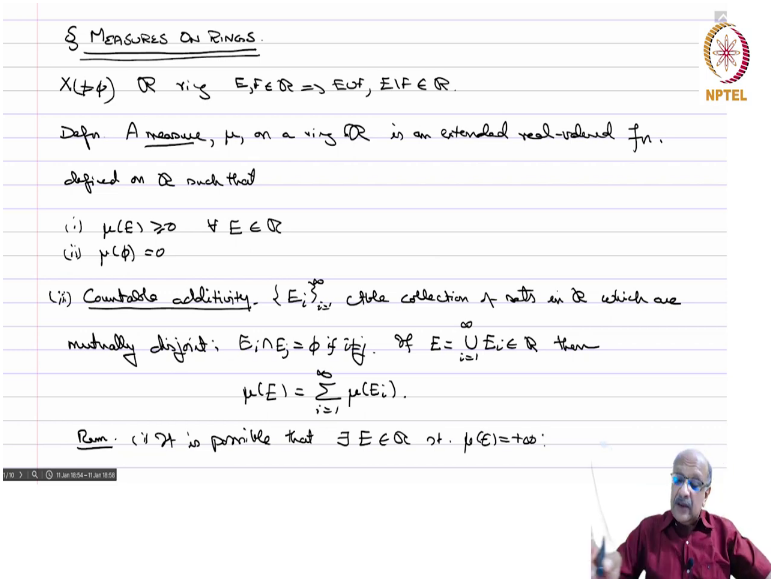

We will now study measures on rings. Recall X is a non empty set and you have R is a collection of subsets which is closed. So, this ring which means $E, F \in R \Rightarrow E\backslash F \in R$ and we also saw a sigma ring and such things.

So, let X is a set and R is a ring of subsets.

Definition: a measure μ on a ring R is an extended real valued function defined on R such that

(1) $\mu \geq 0, \forall E \in R,$

(2) $\mu(\phi) = 0,$

(3) Countable additivity: Let $\{E_i\}_{i=1}^{\infty}$ is a countable collection of sets in R which are mutually disjoint, that means $E_i \cap E_j = \phi \; if \; i \neq j$. If $E = \cap_{i=1}^{\infty} E_i \in R$, then

$$\mu(E) = \sum_{i=1}^{N} \mu(E_i).$$

So, before Lebesgue people did try to do things like this and they always stuck with finite additivity that means, if you have a finite collection of disjoint sets, then the measure of the union is the sum of the measures that is obvious because we have areas or lengths if you have disjoint sets and you put the union then you can define the size or length or the area to be the sum of the individual values.

But Lebesgue found that putting this condition of countable activity, which was not obvious intuitively, helped to make a very rich theory of integration and measure and that is the advantage of these things. So, we have as usual several remarks based on this definition.

Remark: (1) It is possible that there exists $E \in R$ such that $\mu(E) = \infty$, because it is an extended real valued function. So that can be sets whose measure is infinite.

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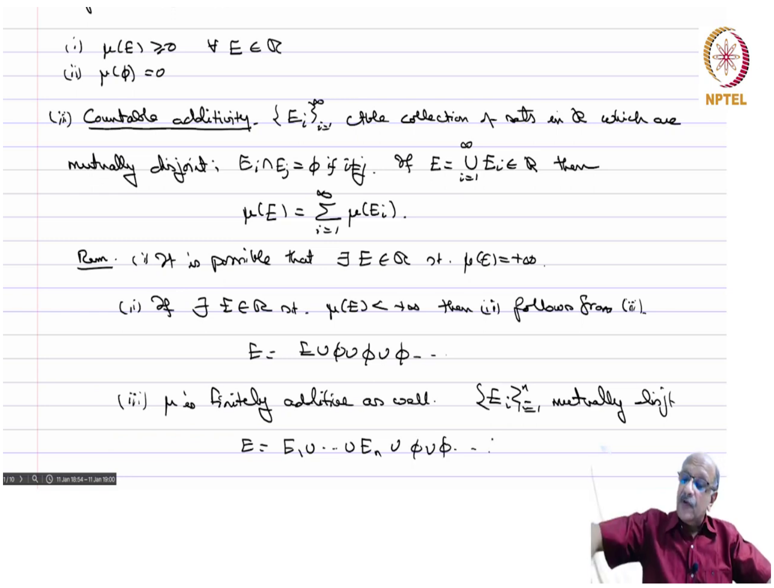

(2) If there exists any E in R, such that $\mu(E) < \infty$, then, (ii) follows from (iii) because you can write $E = E \cup \phi \cup \phi$.. and therefore, if you now write the countable additivity you get mu E equals mu phi plus mu phi plus mu phi plus mu phi etc.

This again I want to emphasize that when you should be able to cancel only when the quantities are finite in an equation if you want to do it if something is infinite, then canceling on both sides is absurd.

(3) μ is finitely additive. So, if $\{E\}_{i=1}^{n}$ mutually disjoint then you can write

$$E = E_1 \cup E_2 \cup .. \cup E_n \cup \phi \cup ....$$

and therefore, you get that.

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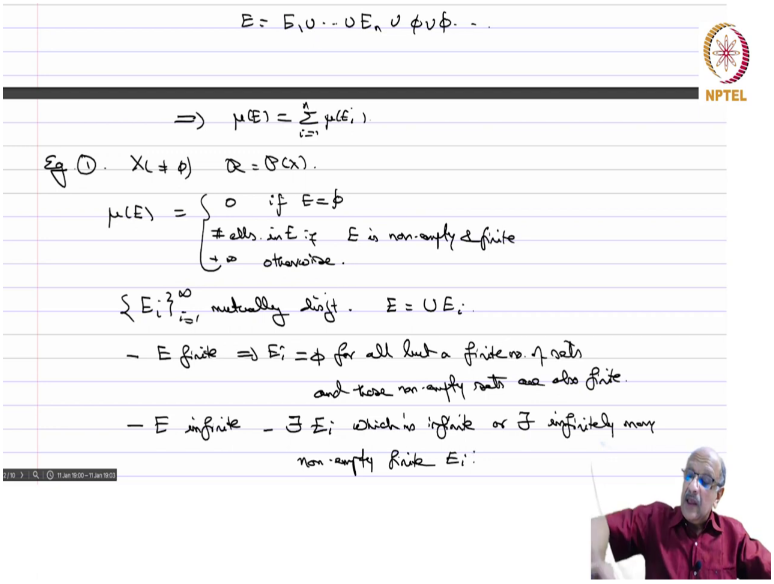

So, that will imply that $\mu(E) = \sum_{i=1}^{N} \mu(E_i)$.

So, now it is time to give some examples.

Example: (1) X any non empty set and $R = P(X)$. So, you then define

$$\mu(E) = 0, \; if\; E = \phi,$$

$$= \# \; of\; elements\; if\; E \neq \phi\; and\; finite,$$

$$= + \infty \; otherwise.$$

So, if you have an infinite number of elements in E then you put mu E equals infinity. So, we have to check. So, the first property is non negative and it is 0 for the empty set. So, we only have to check countable additivity. So, $\{E_i\}_{i=1}^{\infty}$ mutually disjoint of course, in this case we have the power sets so, $E = \cup_{i=1}^{\infty} E_i$.

First case E is finite. This implies E_i are empty for all but a finite number and those non empty sets are also finite. In this case additivity is obvious because the empty sets will all contribute 0 and you know the number of elements in the if you put the union of also a finite number of finite sets, then the number of elements in the union is the sum of the number of elements in the individual sets and that gives you precisely the additivity property.

Second case If E is infinite then you may either have there exists a E_i which is infinite or there exists infinitely many non empty finite E_i in either case the if you take the sum on 1 side and the other both sides will be infinity and therefore, the countable additivity is established

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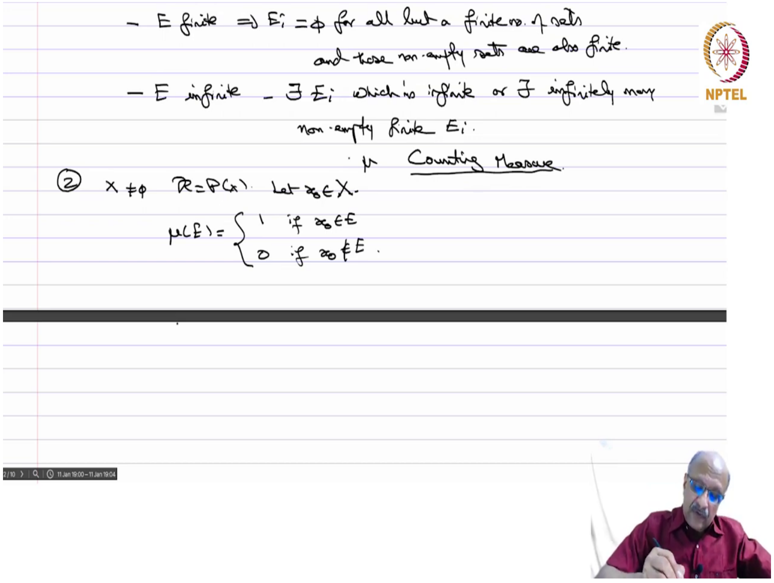

So, secondly, second example. So, this is called the counting measure so, μ is called the counting measure, because it counts the number of elements in the sets.

(2) X is any non-empty set and you have $R = P(X)$ and let $x_0 \in X$. Then you have

$$\mu(E) = 1, \; if \; x_0 \in E,$$

$$= 0, \; if \; x_0 \notin E.$$

So, now again this non-negative μ of the empty set is obviously 0. So, we have checked countable additivity.

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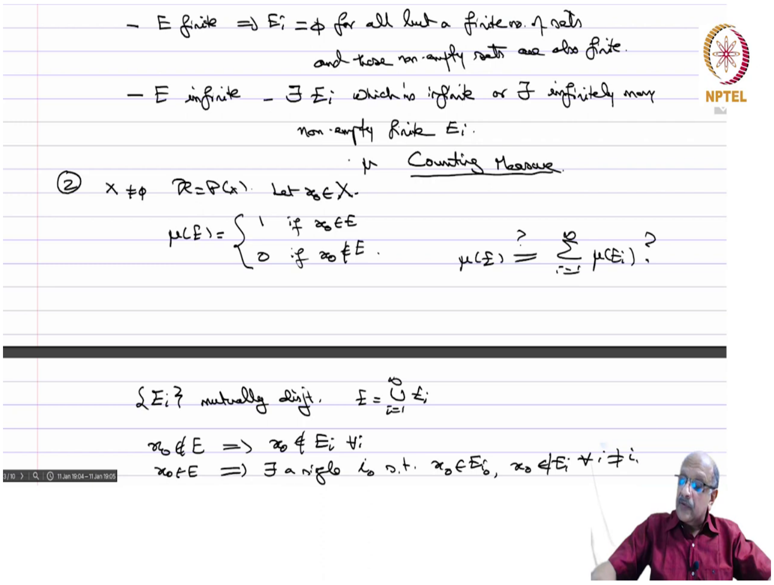

So, if $\{E_i\}_{i=1}^{\infty}$ is mutually disjoint and $E = \cup_{i=1}^{\infty} E_i$. So, if $x_0 \notin E \Rightarrow x_0 \notin E_i \; \forall \, i$.

So, we want to know if $\mu(E) = \sum_{i=1}^{\infty} \mu(E_i)$. So, in both sides of the equation everything is 0 and so, it is obviously true if $x_0 \in E$, this implies there exists a single i naught such that x naught belongs to Ei naught and x naught does not belong to Ei for all i naught equal to i naught because the sets are mutually disjoint.

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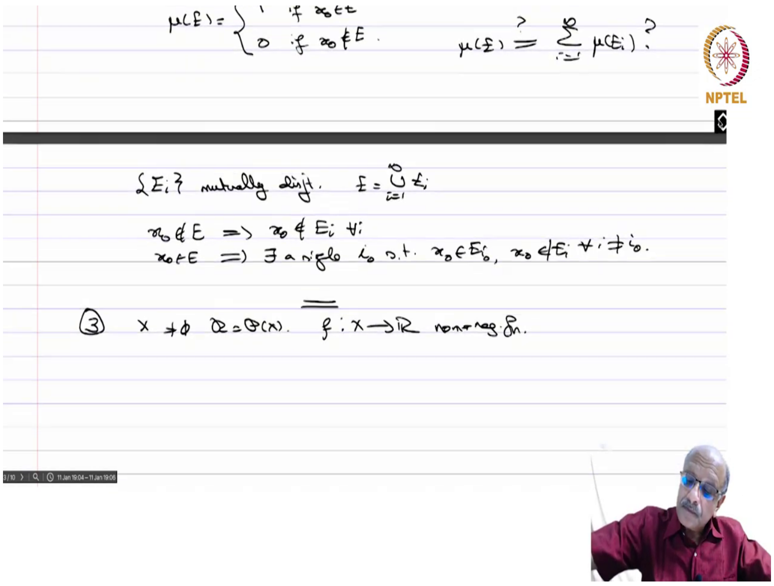

The third example is something like the first example.

(3) X is again any non empty set and $R = ring\ of\ finite\ sets$ and then $f: X \to \mathbb{R}$ non negative function.

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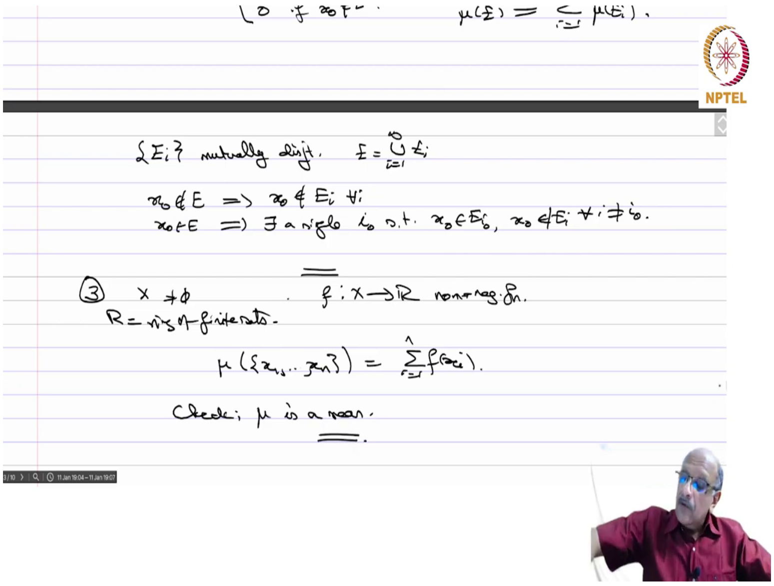

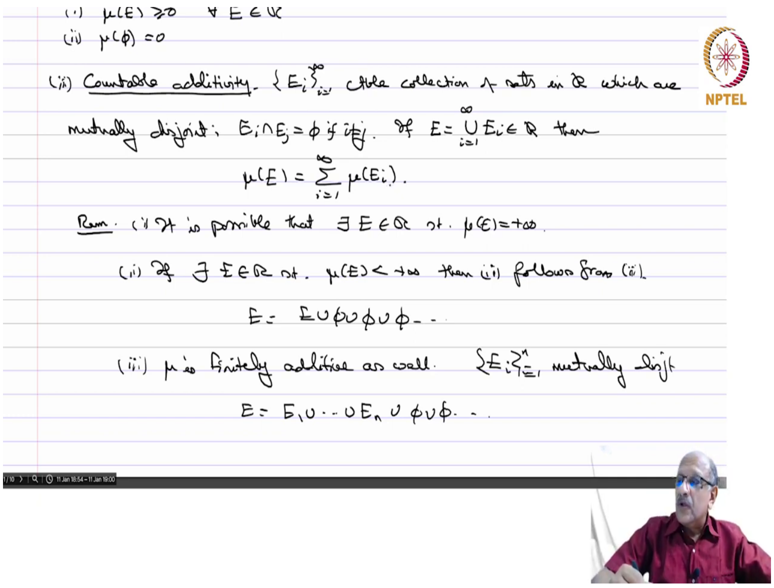

Then you define $\mu(\{x_1, ..., x_n\}) = \sum_{i=1}^{n} f(x_i)$.

So, check μ is a measure.

One remark I forgot to tell you in the countability additivity here. So, when we wrote the countable additivity equation relationship here, the order in which we write the Ei is not important because you have a series of positive terms and so, since all of this is now elements are positive you can write them in any order the answer will always be the same So, that is you do not have to worry about the convergence of the series

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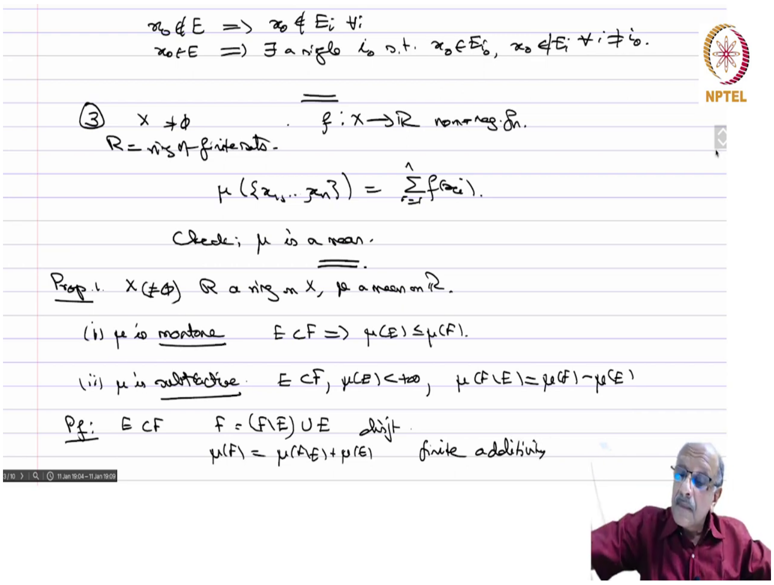

So, we have this. So, the most important and interesting example is the Lebesgue measure, which we will construct in detail a little later on. So, now, we will just prove some properties of measures.

Proposition 1. So, X non empty, R ring on X, μ measure on R. Then

(1) μ is monotone, i.e., $E \subset F \Rightarrow \mu(E) \leq \mu(F)$.

(2) μ is subadditivity, i.e., $E \subset F,\ \mu(E) < \infty,\ \mu(F \backslash E) = \mu(F) - \mu(E)$.

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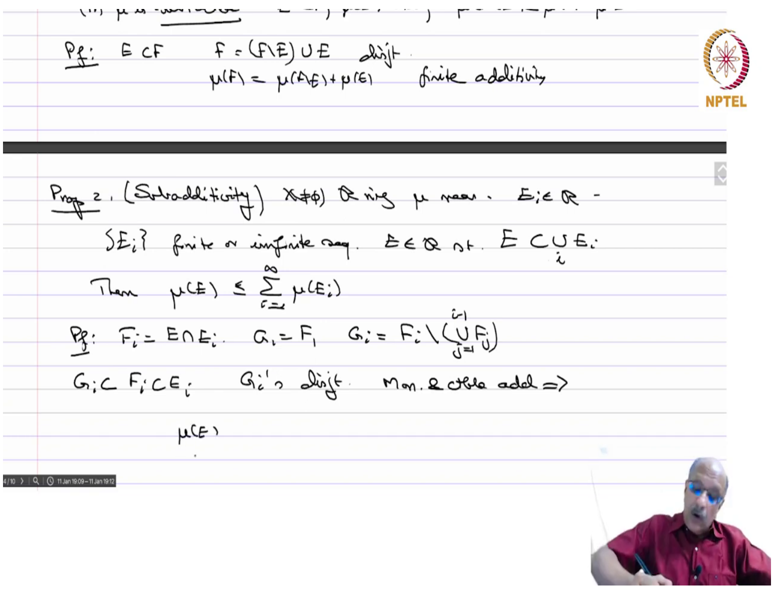
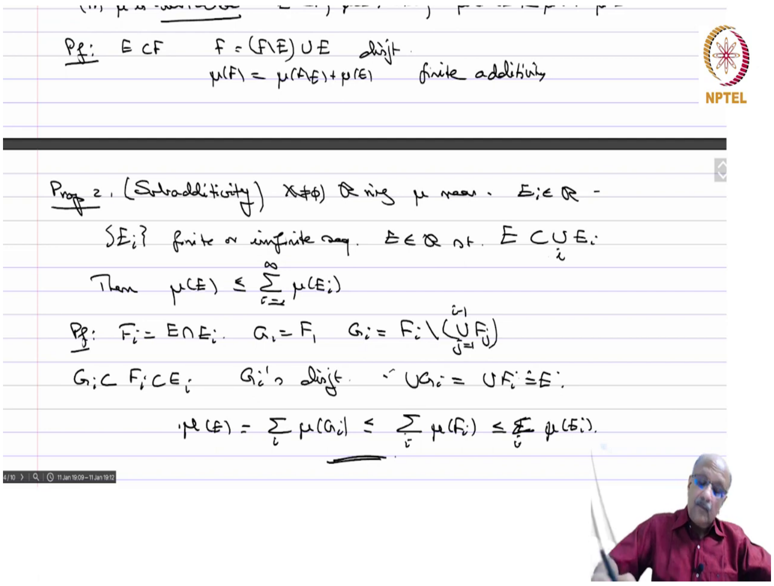


So, the next proposition is called subadditivity.

Proposition 2: So, again X non empty set, R ring, μ is measure and $E_i \in R$, and you have that so, $\{E_i\}$ is a finite or infinite sequence and $E \in R$ such that you have $E \subset \cup_{i=1} E_i$.

Then $\mu(E) \leq \sum_{i=1}^{\infty} \mu(E_i)$.

proof: so, you said $F_i = E \cap E_i$ and you define $G_1 = F_1$, $G_i = F_i \setminus \cup_{j=1}^{i-1} F_j$. So, then you have that $G_i \subset F_i \subset E_i$. So by monotonicity and countable additivity this implies that (Refer Slide Time: 18:01)



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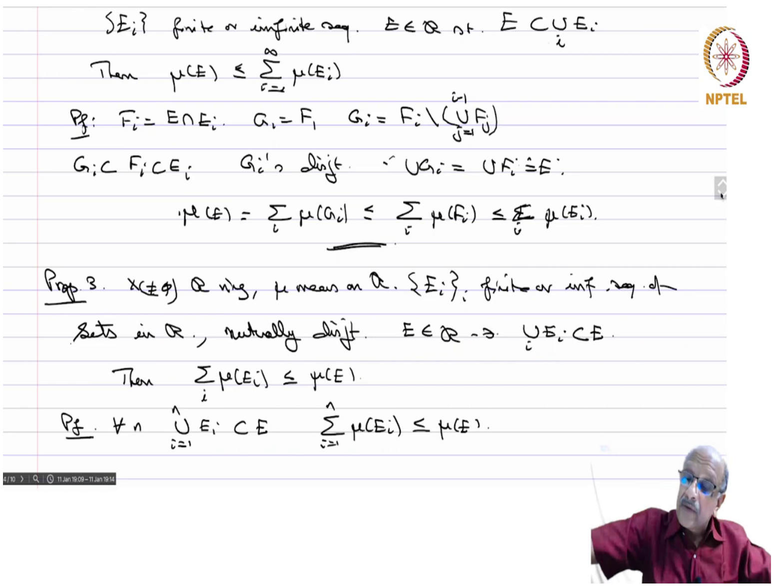

$\mu(E) = \sum_i \mu(G_i) \leq \sum_i \mu(F_i) \sum_i \mu(E_i)$.

Proposition 3: So, X non empty, R ring, μ measure on R, $\{E_i\}$ is a finite or infinite sequence and $E \in R$ such that you have $\cup_{i=1} E_i \subset E$. Then $\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(E)$.

proof: for every positive integer n, $\cup_{i=1}^{n} E_i \subset E$. Therefore you have $\sum_{i=1}^{n} \mu(E_i) \leq \mu(E)$.

Then from this you deduce the result immediately if n is finite and if the collection is infinite, you let $n \to \infty$.

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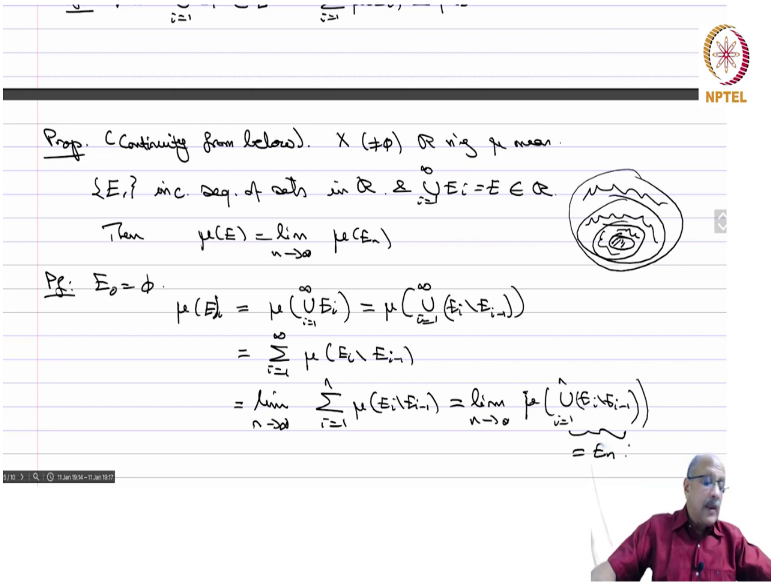

So, now we have a very important proposition, a very important property of measures.

Proposition:(continuity from below) X non empty, R ring, μ measure. Now, $\{E_i\}$ is a increasing sequence of sets in R and $\cup_{i=1}^{\infty} E_i = E \in R$. Then $\mu(E) = \lim_{n \to \infty} \mu(E_n)$.

proof: We said $E_0 \neq \phi$, and then you can write

$$\mu(E) = \mu(\cup_{i=1}^{\infty} E_i) = \mu(\cup_{i=1}^{\infty} [E_i \backslash E_{i-1}]) = \sum_{i=1}^{\infty} \mu([E_i \backslash E_{i-1})$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} \mu([E_i \backslash E_{i-1}) = \lim_{n \to \infty} \mu(\cup_{i=1}^{n} [E_i \backslash E_{i-1}])$$

$$= \lim_{n \to \infty} \mu(E_n).$$

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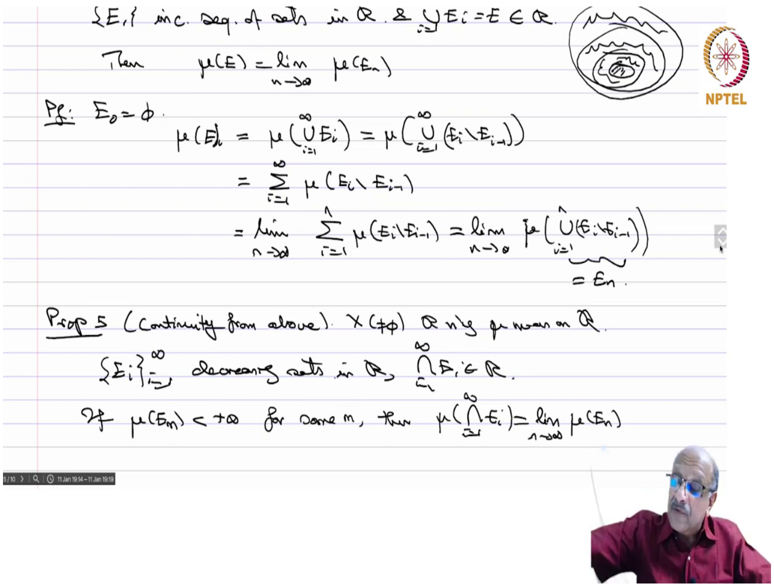

Proposition 5: (continuity from above) So, X non empty, R ring, μ measure on R, and then you have $\{E_i\}$ is a increasing sequence of sets in R and $\cap_{i=1}^{\infty} E_i = E \in R$. If $\mu(E_n) < \infty$,for some n, then $\mu(E) = \lim_{n \to \infty} \mu(E_n)$.

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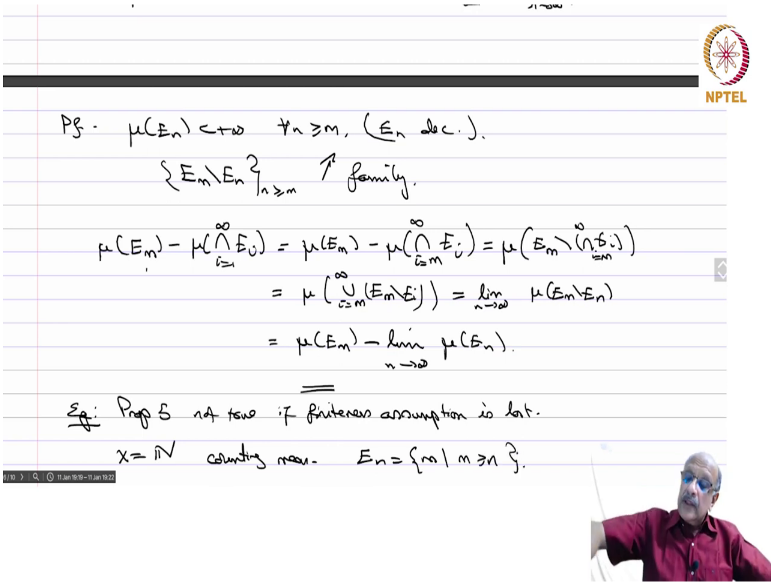

proof: We have $\mu(E_n) < \infty \, \forall \, n \geq m$. So, you have $\{E_n \backslash E_m\}_{n\geq m}$ is an increasing family. So now you write

$$\mu(E_m) - \mu(\cap_{i=1}^{\infty} E_i) = \mu(E_m) - \mu(\cap_{i=m}^{\infty} E_i) = \mu(E_m \backslash (\cap_{i=m}^{\infty} E_i))$$

$$= \mu(\cap_{i=m}^{\infty} (E_m \backslash E_i))$$

$$= \lim_{n\to\infty} \mu(E_m \backslash E_n)$$

$$= \mu(E_m) - \lim_{n\to\infty} \mu(E_n).$$

Since $\mu(E_m)$ is finite, I can cancel it on both sides and therefore I get the result which I need.

Example: Proposition 5 is not true if finiteness assumption is lost. So, for instance, you take $\mathbb{N}$ and the counting measure and you take $E_n = \{m: m \geq n\}$.

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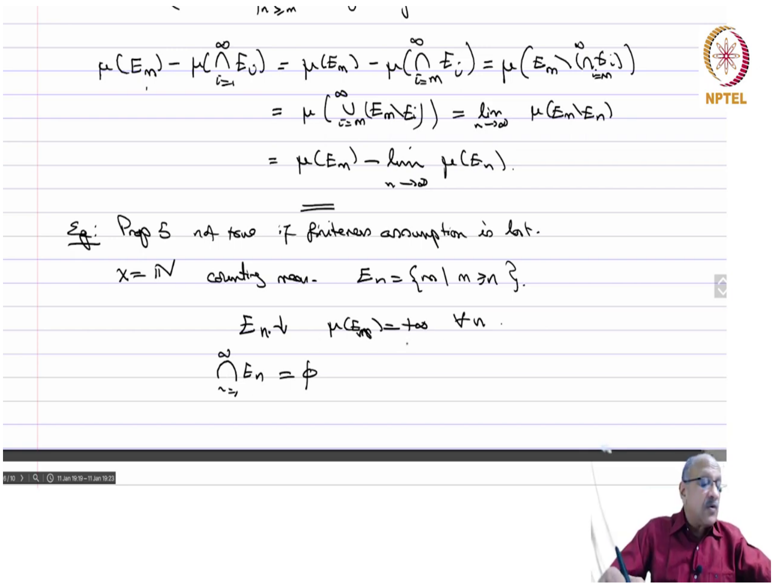

So, then E_n are decreasing and $\mu(E_n) = \infty, \forall n$ and you have $\cap_{i=1}^{\infty} E_i = \phi$.

So, the measure of the intersection is 0, but the limit of this is still always infinity. So, the equation is that that is why it is not true. And you have here.

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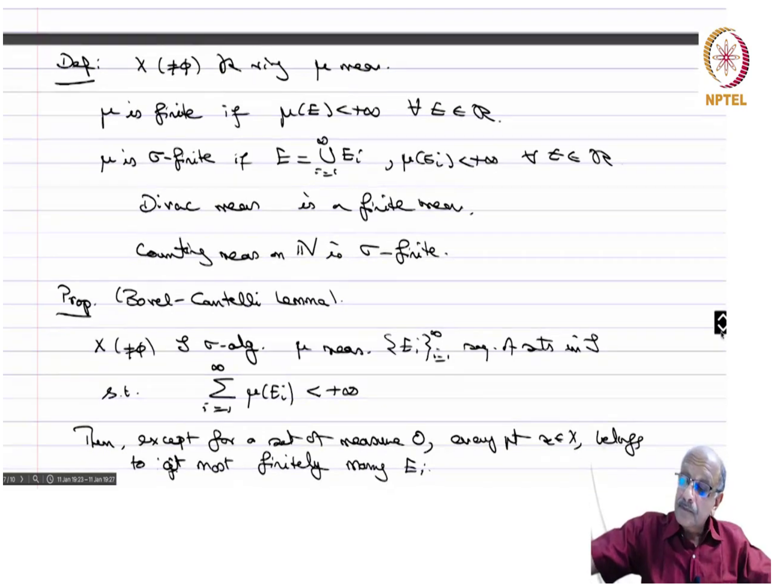

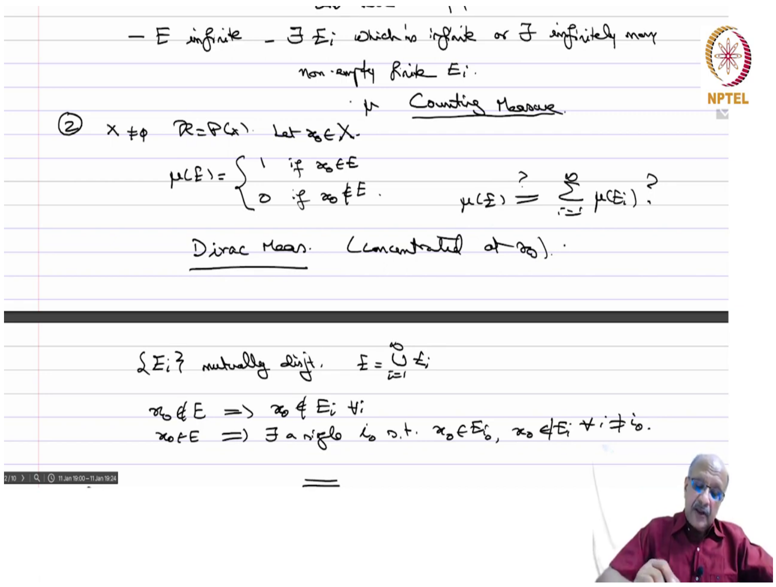

Now, we have one more definition again.

Definition: So, X non empty, R ring and μ a measure, then we say that μ is finite if

$\mu(E) < \infty, \forall E \in R$. We call μ is sigma finite if E can be written as $E = \cup_{i=1}^{\infty} E_i$ and $\mu(E_i) < \infty \, \forall \, i$.

So, if you take Dirac measures I do not know if I mentioned the name. This measure is called the Dirac measure concentrated at x naught. This is called Dirac measure concentrated at x naught, and that is of course a finite measure.

So, Dirac measure is a finite measure. Counting measure on N is sigma finite because if you write a take any infinite collection of natural numbers you can write it as a disjoint union of singleton. And each singleton has measure once so it is finite and therefore you can write it as a union of a sum, union of sets of finite measure, countable union of sets of finite measure and therefore this becomes a sigma finite measure.

So, now we conclude with a very, very useful result if we will use it later. It is a very pretty results proposition. This is called the Borel-Cantelli Lemma, a very nice result.

Lemma: (Borel-Cantelli) So, X non empty, you have S sigma algebra, μ measure, $\{E_i\}_{i=1}^{\infty}$ sequence of sets in S such that $\sum_{i=1}^{\infty} \mu(E_i) < \infty$. Then except for a set of measures 0 every point x in X belongs to at most finitely many E_i.

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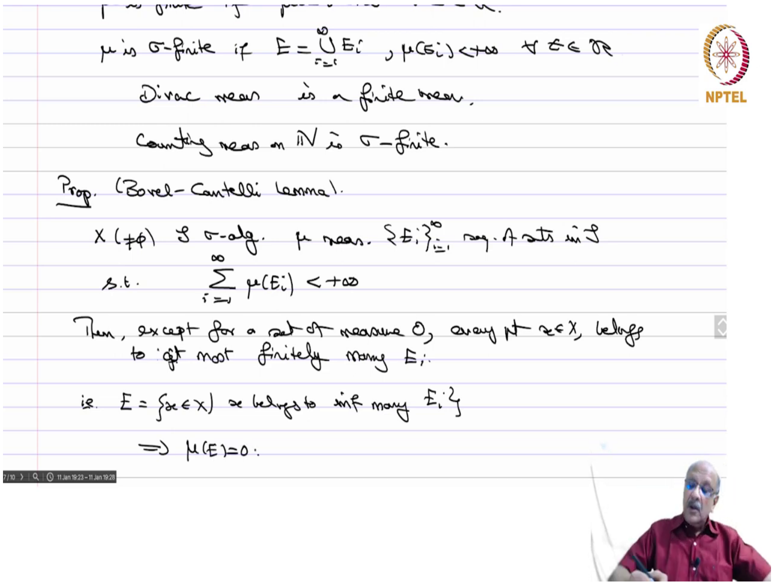

So, that means one what so you take $E = \{x \in X\}$. So set x belongs to infinitely many $\{E_i\}$. Then Lemma says that $\mu(E) = 0$.

So, this is a very, very beautiful result and very useful in many situations.

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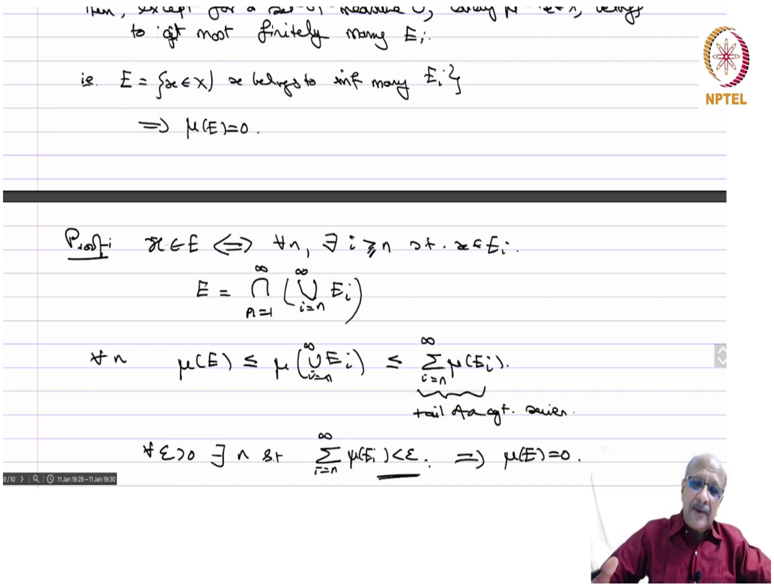

proof: So, let us see what is E? $x \in E$ if and only if $\forall\, n,\ \exists\, i \geq n\ s.t.\ x \in E_i$ and

$$E = \cap_{n=1}^{\infty}(\cup_{i=n}^{\infty} E_i).$$

So, you have $\mu(E) \leq \mu(\cup_{i=n}^{\infty} E_i) \leq \sum_{i=n}^{\infty} \mu(E_i)$. Therefore, for every $\epsilon > 0$, that exists n such that $\sum_{i=n}^{\infty} \mu(E_i) < \epsilon \Rightarrow \mu(E) = 0$.

So, this is the Borel-Cantelli lemma.

So, our next objective is to give a measure on a ring we would like to see if we can extend it to something. So, for instance, given a ring I can think of the smallest sigma ring containing this ring we have already seen such things given any arbitrary collection of sets: what is the smallest ring containing the collection or what is the smallest sigma ring containing a collection. So, if I am given a measure on a ring, I would like to extend it to the smallest sigma ring containing it and so on. So, such extensions we would like to study and we will do it next time.