Measure and Integration
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Lecture 29
Non-negative Functions

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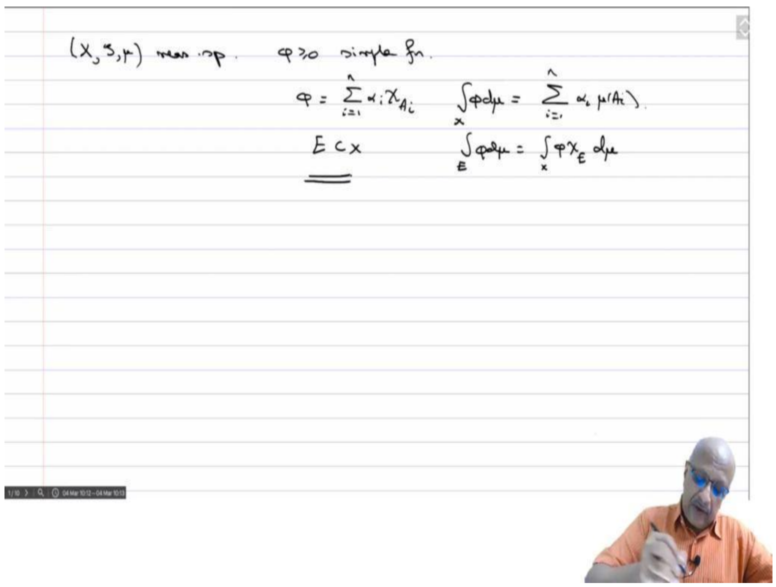

So, we are in the process of defining the Lebesgue integral. So, when we had φ, so (X, S, μ) is a measure space; and $\varphi \geq 0$ simple function. So, $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $\int_X \varphi \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i)$

$$E \subset X, \quad \int_E \varphi \, d\mu = \int_X \varphi \chi_E d\mu.$$

So, we now continue with this; so, now we are going to do non-negative functions.

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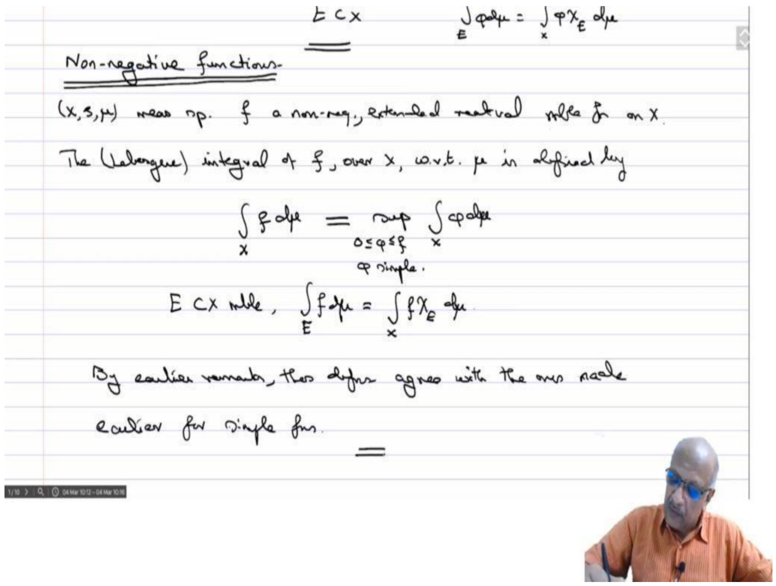

Non negative functions:

So, (X, S, μ) measure space; f non-negative, extended real valued function measurable function on X. Then, the Lebesgue integral of f over X with respect to the measure μ is defined by

$$\int_X f d\mu = sup_{\{0 \leq \varphi \leq f\ :\ \varphi\ simple\}} \int_X \varphi d\mu.$$

And if $E \subset X$ measurable $\int_E f d\mu = \int_X f \chi_E d\mu$..

So, recall that given we we proved a very important theorem earlier. If f is a non-negative measurable function, then it is the increasing limit of simple non-negative functions; and therefore, this definition makes sense. Now, by the remarks which you made at the end of the last session, we it is clear that since if you have two measurable simple functions, one less than the other than the integrals have the same inequality between them.

And therefore, by earlier remarks these definitions agree with the ones, one made earlier, ones made earlier for simple functions. So, if f is a simple function, then this definition is not agrees with the previous definition. So, that is the, that is obvious from the two remarks which I made at the end of the previous video; so you please go and check that those two.

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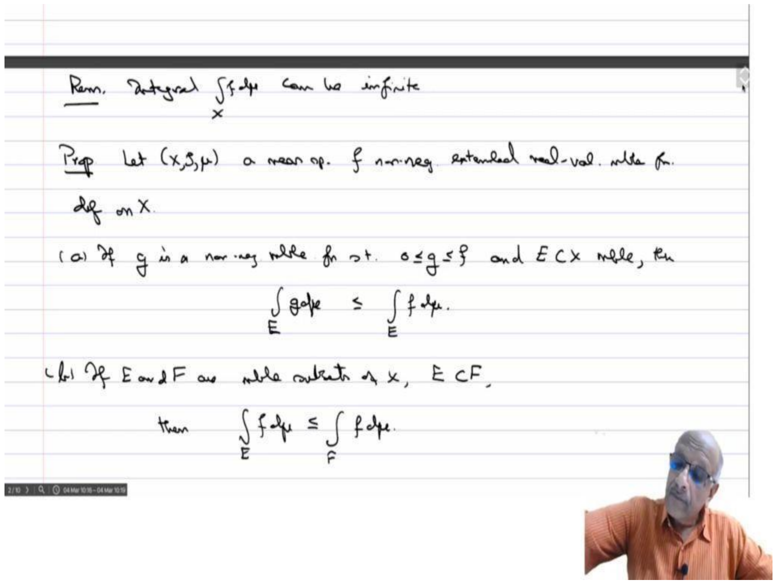

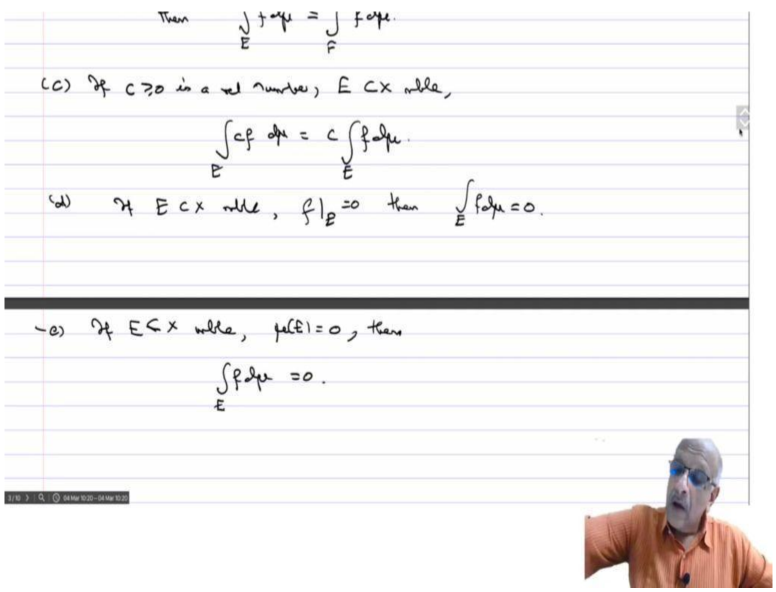

Remark. $\int_X f d\mu$ can be infinite. So, next proposition, very trivial one and immediate from the definitions.

Proposition: (X, S, μ) is a measure space and f non-negative extended real valued measurable function defined on X.

(a): If g is a non-negative measurable function such that $0 \leq g \leq f$; and $E \subset X$ measurable, then $\int_E g d\mu \leq \int_E f d\mu$.

(b) If E and F measurable subsets of X, and $E \subset F$; $\int_E f d\mu \leq \int_F f d\mu$

(c): If $C > 0$ is a real constant number, $E \subset X$ measurable; then

$$\int_E Cf d\mu = C \int_E f d\mu$$

(d), If $E \subset X$ measurable, $f|_E = 0$; then $\int_E f d\mu = 0$; and

(e). If $E \subset X$ measurable and $\mu(E) = 0$; then $\int_E f d\mu = 0$. All these have immediate one line deductions from the definitions.

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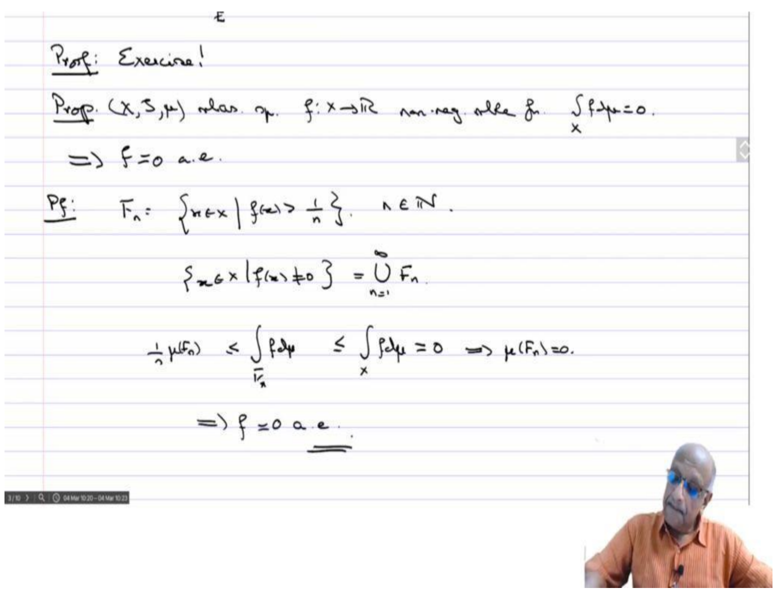

So proof, exercise. Next proposition:

Proposition: (X, S, μ) measurable measure space $f: X \to \mathbb{R}$ non-negative. So, now non-negative measurable function and integral over X, $\int_E f d\mu = 0$; then $f = 0 \quad a.e.$

Proof: So, if you have a no-negative function whose integral is 0, then the function has to be essentially 0. So, let

$F_n = \{x \in X : f(x) \geq 1/n\}$

I do not know how to put a modulus, because we are dealing with non-negative functions. And this is true for all $n \in \mathbb{N}$.

Then, $\{x \in X : f(x) \neq 1\} = \bigcup_{n=1}^{\infty} F_n$. Now, if you have if you take $\int_E f d\mu = 0$. But, that is

$$\frac{1}{n}\mu(F_n) \leq \int_{F_n} f d\mu \leq \int_X f d\mu = 0 \Rightarrow \mu(F_n) = 0, \ \forall n.$$

And therefore, since it is the union of sets of measure 0, so this implies $f = 0 \quad a.e.$

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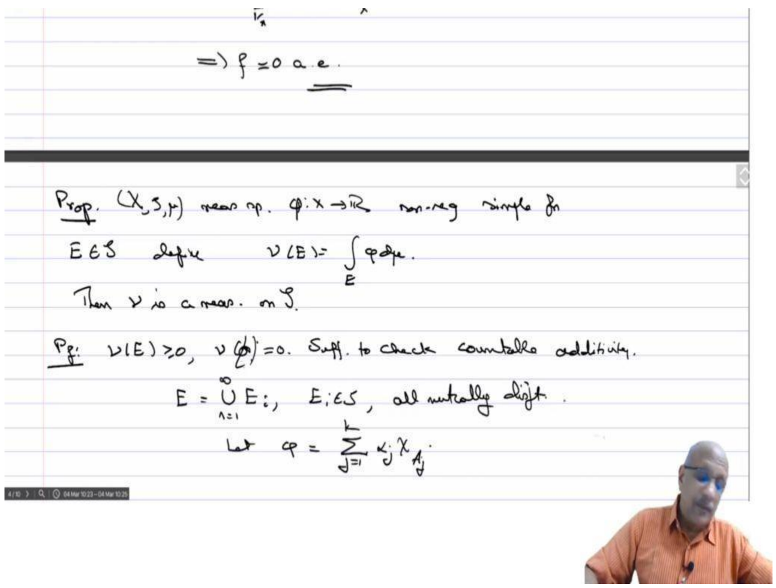

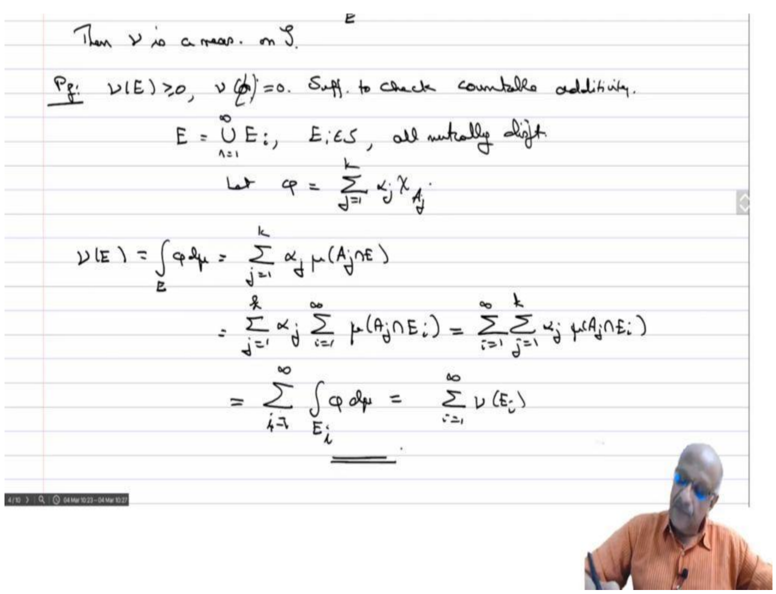

Proposition: (X, S, μ) measurable measure space $\varphi: X \to \mathbb{R}$ non-negative simple functions. Let $E \in S$, define $\nu(E) = \int_X \varphi d\mu$; then, μ is a measure on S. So, the indefinite integral gives you a measure, so

Proof. Indefinite integral, because I am saying the integral taken on over arbitrary sets of the sigma algebra.

So, $\nu(E) \geq 0$ and ν of the empty set is obviously 0; so, sufficient to check countable additivity.

So, let $E = \bigcup_{n=1}^{\infty} E_i$, $E_i \in S$ all mutually disjoint. So, Let $\varphi = \sum_{j=1}^{k} \alpha_i \chi_{A_j}$.

So, what is $\nu(E)$?

$$\nu(E) = \int_E \varphi d\mu = \sum_{j=1}^{k} \alpha_j \mu(A_j \cap E) = \sum_{j=1}^{k} \alpha_j \sum_{i=1}^{\infty} \mu(A_j \cap E_i) = \sum_{i=1}^{\infty} \sum_{j=1}^{k} \alpha_j \mu(A_j \cap E_i).$$

$$= \sum_{i=1}^{\infty} \int_{E_i} \varphi d\mu = \sum_{i=1}^{\infty} \nu(E_i)$$

So, that completes the proof.

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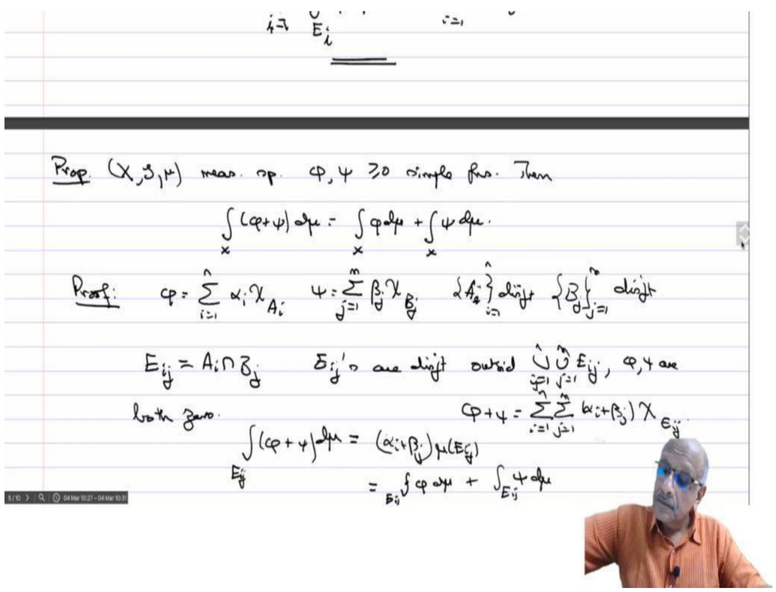

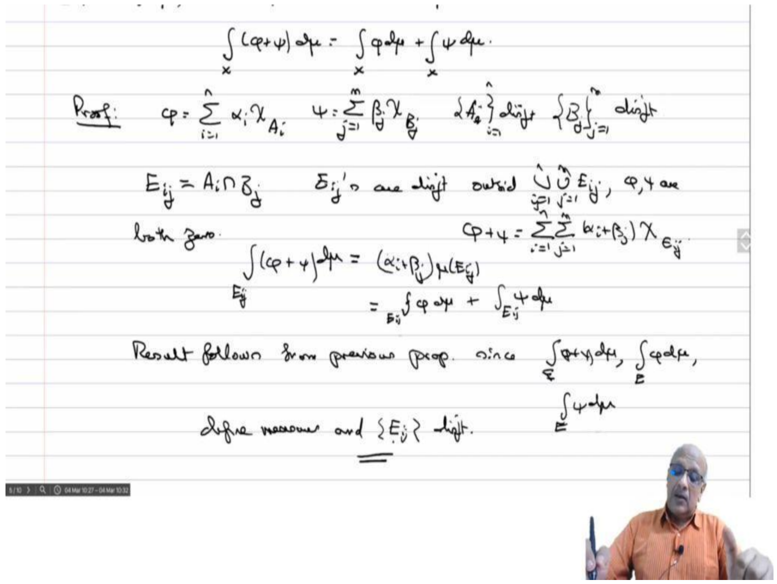

So, now we are going to prove in first step of an important property. When you know, you have integration that is a linear operation. If you take sum of functions and integrate, it is the sum of the integrals and so on. But, these things have to be proved now with the definition which we have given. And therefore as a first step towards this, we have the following result.

Proposition: (X, S, μ) measurable measure space φ, ψ non-negative simple functions; then,

$$\int_X (\varphi + \psi) d\mu = \int_X \varphi d\mu + \int_X \psi d\mu$$

Proof: Take

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \quad \psi = \sum_{j=1}^{m} \beta_j \chi_{B_j}, \quad \{A_i\}_{i=1}^{n} \textit{ are disjoint and } \{B_i\}_{j=1}^{m} \textit{ are disjoint}$$

You can write in many forms as I said, and you can always write it in terms of disjoint sets.

So, now, you set

$E_{i,j} = A_i \cap B_j$, $E_{i,j}$ are disjoint

outside $\bigcup_{i=1}^{n} \bigcup_{j=1}^{m} E_{i,j}$; φ, ψ are both 0; so, now,

$$\int_{E_{i,j}} (\varphi + \psi)d\mu = (\alpha_i + \beta_j)\mu(E_{i,j}) = \int_{E_{i,j}} \varphi d\mu + \int_{E_{i,j}} \psi d\mu$$

So, $\varphi + \psi$, so now you have again a disjoint thing here; and therefore,

So, now over union of $E_{i,j}$, $E_{i,j}$'s are disjoint; each of these is a measure. This is a measure, this is a measure, this is a measure; and therefore, you know by countable additivity, you can write over the union. So, result follows from previous proposition, since

$\int_E (\varphi + \psi)d\mu$, $\int_E \varphi d\mu$, $\int_E \psi d\mu$ define measures; and $E_{i,j}$ are all disjoint.

So, therefore by countable additivity, the result will (follow). So, now, we will of course, expand this result to a much better result later on; next time, we will there now look at limits. So, one of the drawbacks we saw in the Riemann integral was the limit of integrable function is not integrable; so, those are the things. So, we will now try to study something about limits of sequence integrals of sequences of functions. And, of course, we are still restricted to the case of non-negative functions. And we will do that next time.