Measure and Integration
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Lecture 28
Integration: Simple Functions

We will start a new chapter and this is the main topic of this course. So, we will now start integration.

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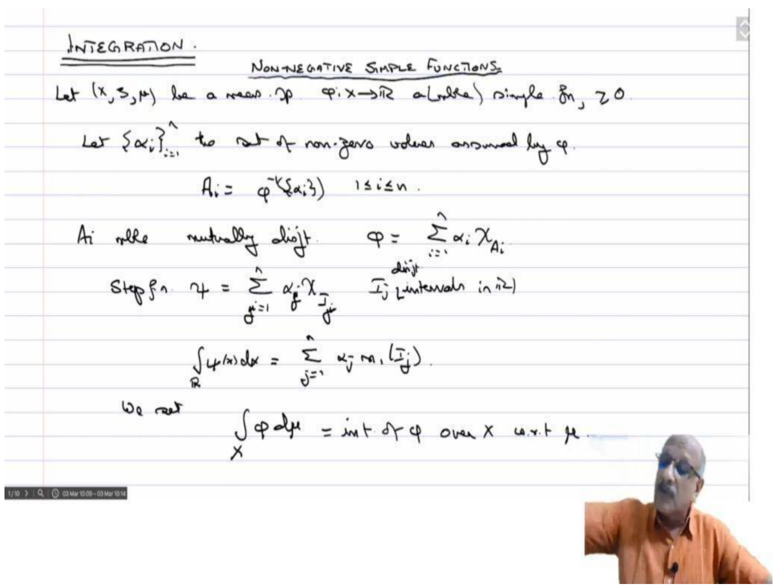

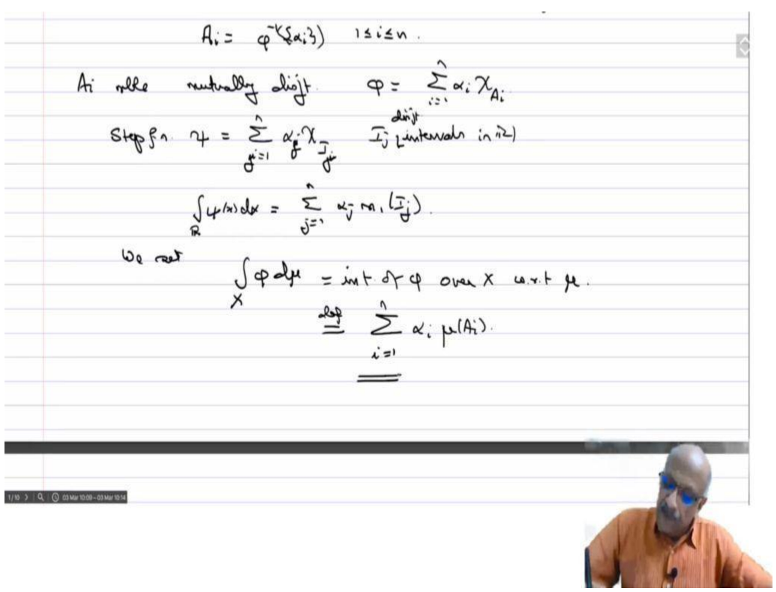

INTEGRATION:

So, we will study how to define the Lebesgue integral of measurable function over measure space; and then study the properties of the integral which we usually know for the Riemann integral. We will try to see what are the different kinds of theorems, which we can prove. So, let (X, S, μ) be a measure space and $f: X \to \mathbb{R}$, $f \geq 0$ a measurable function, which is non-negative.

So, first we will look at non-negative simple functions. Let $\{\alpha_i\}_{i=1}^{n}$, the set of non-zero values assumed by φ. So, we are taking a measurable simple function which is non-negative. So, it takes only a finite number of values being a simple function; and let us take

$$A_i = \varphi^{-1}(\{\alpha_i\}),\ 1 \leq i \leq n.$$

Then, A_i are all measurable and mutually disjoint; and you can write

$$\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}.$$

So, in order to define the integral, we are going to imitate what we did in the case of the Riemann integral. So, suppose you had a step function

$$\psi = \sum_{j=1}^{n} \alpha_j \chi_{I_j},\quad I_j \ \textit{are disjoint intervals}$$

So, let just put j so that, that not too many i is around, where I_j are the intervals in $\mathbb{R}$, and then disjoint. Then, how do you define the integral? Integral is nothing but the area under the curve. So, $\int_{\mathbb{R}} \psi(x)dx = \sum_{j=1}^{n} \alpha_j m(I_j)$.

This is what we said in the preamble, namely we look at go along the y axis, and take a aerial view of the function and then measure it.

And that is it is to find when instead of i, j if you had arbitrary sets, then what do you mean by measures of an arbitrary set. That is the true reason why we define measure theory up to now. So, imitating this, we define $\int_X \psi d\mu$. So,

$\int_X \psi d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i)$; so this is the meaning of the symbol. Now, before we proceed further, we have to, we can we note that you can express a function which is a characteristic simple function in more than one way as a sum of characteristics, linear combination of characteristic functions.

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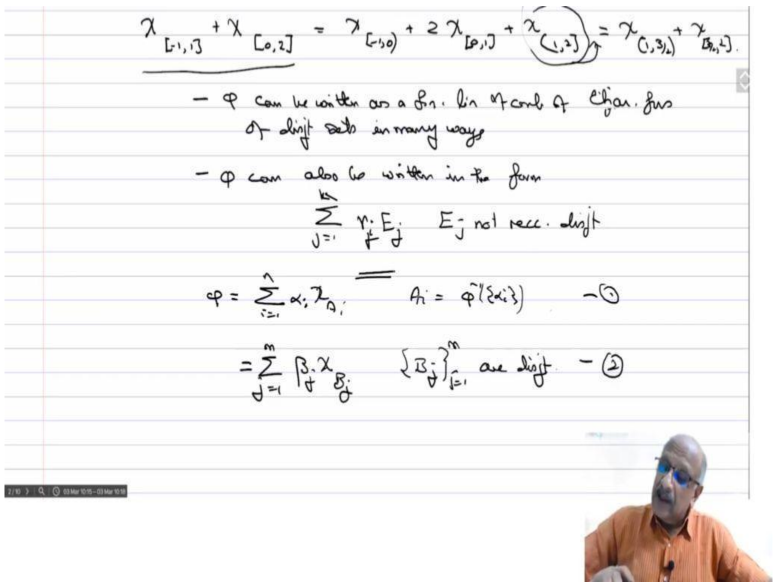

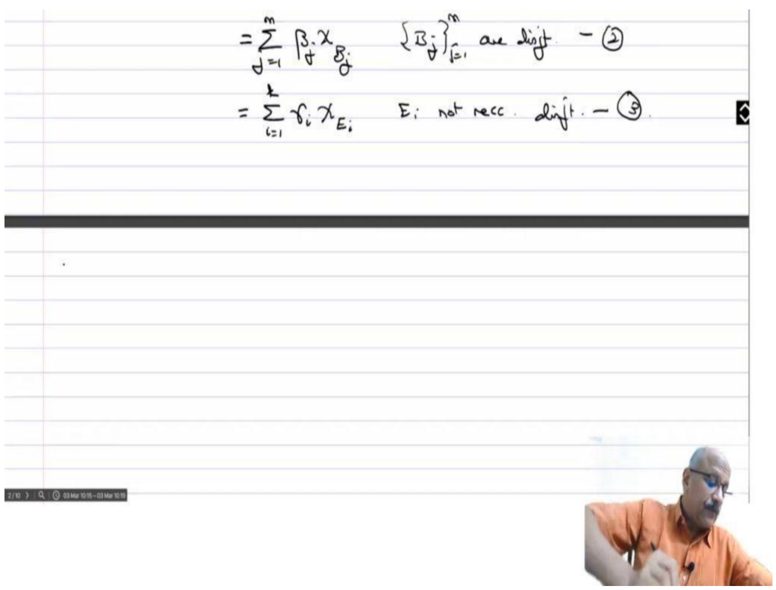

For instance, if I have chi of say,

$$\chi_{[-1,1]} + \chi_{[-1,1]} = \chi_{[-1,0)} + 2\chi_{[0,1]} + \chi_{(1,2]}$$

and these are disjoint. Of course, we can also write it in many other ways; so, we want to know which way are we going to. So, we can even write a function that is given in terms of disjoint sets, in terms of non-disjoint sets. So, we would like to define the integral in a way that we are not worried about how the function has been simple function has been written, whether it is.

So, if you like, for instance, so we can have many cases. One is phi can be written as a finite linear combination of disjoint sets; this of characteristic functions of disjoint sets in many ways. For instance, I can split this

$$= \chi_{(1,3/2)} + \chi_{[3/2,2]}$$

for instance. So, so there are so many ways in which you can write.

And even also $\varphi = \sum_{j=1}^{k} \gamma_j E_j$, E_j *is not necessarily disjoint.*

Example is here, this is not disjoint and so. So, we would like to define the integral in a way that is independent of the way in which we write; because it will be difficult for us to break up the function each time according to the original form in which we wrote.

And therefore, we would like to give more robust definition of the integral. So, let us assume, so start with $\varphi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ *where* $A_i = \varphi^{-}(\{\alpha_i\})$

so, this is the first form which you are taking. So, now let us assume that this is

$$\varphi = \sum_{j=1}^{m} \beta_j B_j, \quad \text{where } \{B_j\}_{j=1}^{m} \text{ are disjoint.}$$

So, this is the first form, this is a second form; they are all disjoint, but they are not necessarily the level sets. So, the third form is

$$\varphi = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad \text{where } \{E_i\} \text{ not necessarily disjoint.}$$

So, this is the third form in which we are going to write a given function.

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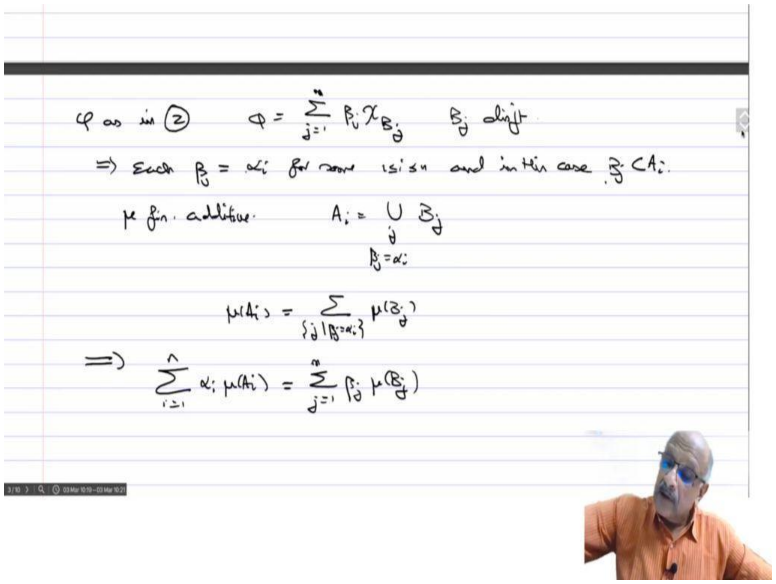

So, let us take so phi as in 2. So, this means $\varphi = \sum_{j=1}^{m} \beta_j \chi_{B_j}$, B_j *are disjoint*.

Then, each $B_j = \alpha_i$ for some $1 \le i \le n$; and in this case $B_j \subset A_i$. So, μ is finitely additive and you have $A_i = \bigcup_{j:\beta_j=\alpha_i} B_j$

j; because you can regroup according to the values.

And therefore, $\mu(A_i) = \bigcup_{j:\beta_j=\alpha_i} \mu(B_j)$, And now it immediately follows because of this that. So, this seem now implies $\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{j=1}^{m} \beta_j \mu(B_j)$

from this from this it follows immediately.

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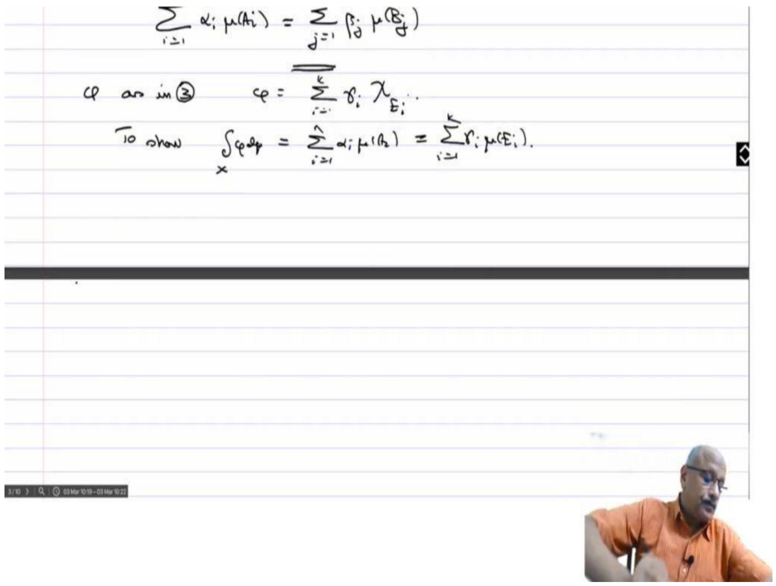

So, now let us look at the third form φ as in 3. That means

$$\varphi = \sum_{i=1}^{k} \gamma_i \chi_{E_i}$$

So, we want to show in this also, we can write the integral as sigma chi of Ei, mu of Ei, we want try to show that. So, to show

$$\int_X \varphi d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{i=1}^{k} \gamma_i \mu(E_i).$$

So, it does not matter which way you write the function; the answer is always, integral is immediate to the. Once you are given a simple function, we can simply write it as the replay in this fashion; so, this is the idea, so we have to prove this now.

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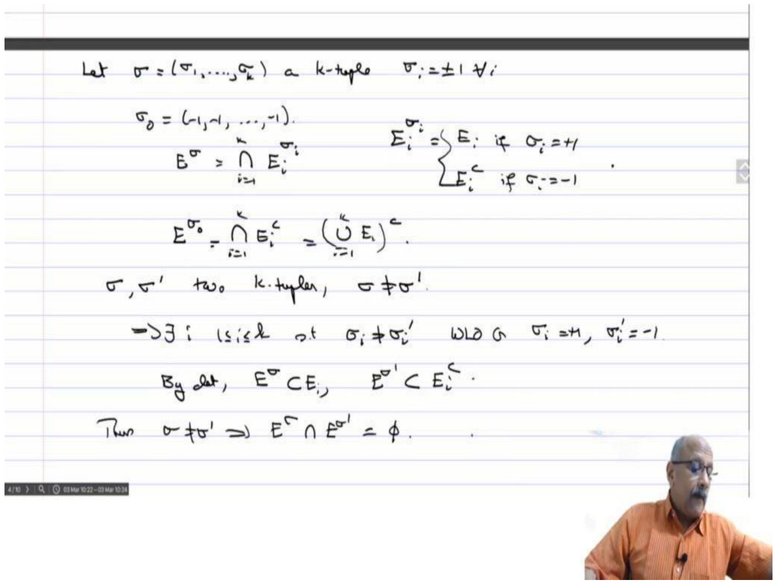

So, let us say $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_k)$; $\sigma_i = +- 1$ for each i. So, E1 to Ek are the sets, so k is the set index which comes here; and for each i you have. And you put

$$\sigma_0 = (-1, -1, \ldots, -1).$$

Now, we define $E^{\sigma} = \bigcap_{i=1}^{k} E_i^{\sigma_i}$ So, $E_i^{\sigma_i}$ equals E_i , if $\sigma_i = +1$; and E_i^c, if $\sigma_i = -1$.

So, you take all these intersections and you define this. So, you have E^{σ_0} for instance, is

$$E^{\sigma_0} = \bigcap_{i=1}^{k} E_i^c = \left(\bigcup_{i=1}^{k} E_i\right)^c$$

So, if σ and σ' are two k tuples, such that $\sigma \neq \sigma'$; that means, there exists an i, 1 less than equal to i less than equal to k, such that $\sigma_i \neq \sigma_i'$.

So, without loss of generality, let us assume $\sigma_i = +1$ and $\sigma_i' = -1$. Then, by definition $E^{\sigma} \subset E_i$ $and,$ $E^{\sigma'} \subset E_i^c$. So, thus$\sigma \neq \sigma'$ implies $E^{\sigma} \cap E^{\sigma'} = \Phi$; so, they are always disjoint.

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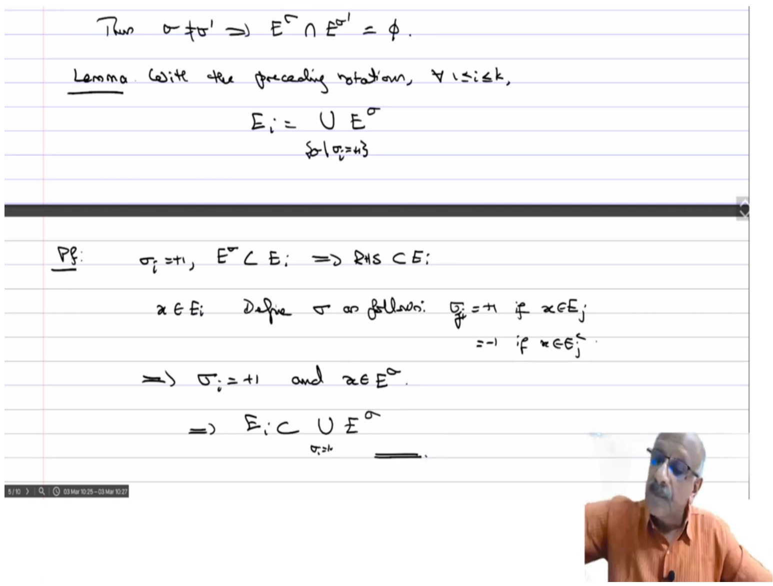

So Lemma,

Lemma: With the preceding notations for each $1 \leq i \leq k$,

$$E_i = \bigcup_{\{\sigma:\, \sigma_i = +1\}} E^{\sigma}.$$

So, you take all k tuples such that the i-th coordinate equal to plus 1. Take the corresponding E sigma, take the union; then E_i will be equal to that union; so

Proof. If $\sigma_i = +1$, then $E^{\sigma} \subset E_i$; so implies RHS is contained in E_i.

So, now, for the reverse part, we have to show E_i is contained in this union. So, let us take $x \in E_i$; define σ as follows: $\sigma_j = +1 \ if \ x \in E_j$

$$= -1 \ if \ x \in E_j^c$$

And this implies that $\sigma_i = +1$, because $x \in E_i$; and $x \in E^{\sigma}$. So, this implies that

$$E_i \subset \bigcup_{\sigma=1} E^{\sigma}$$

; and therefore you have the Lemma is proved.

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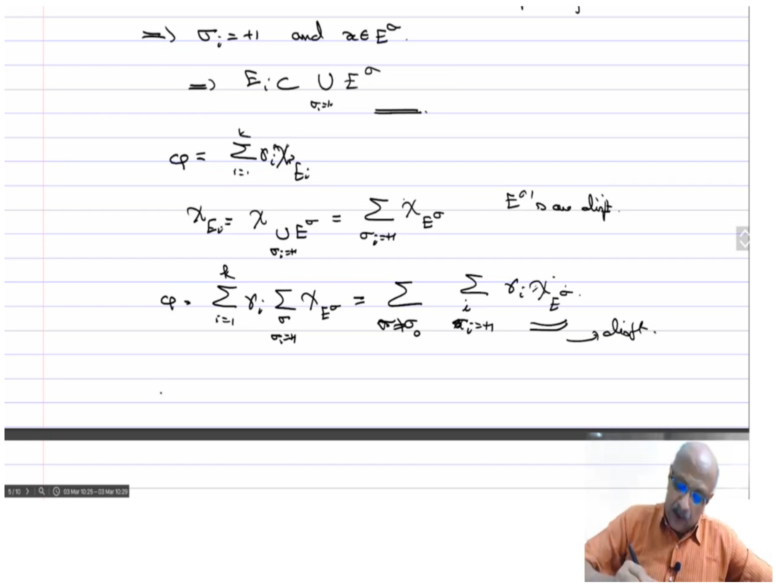

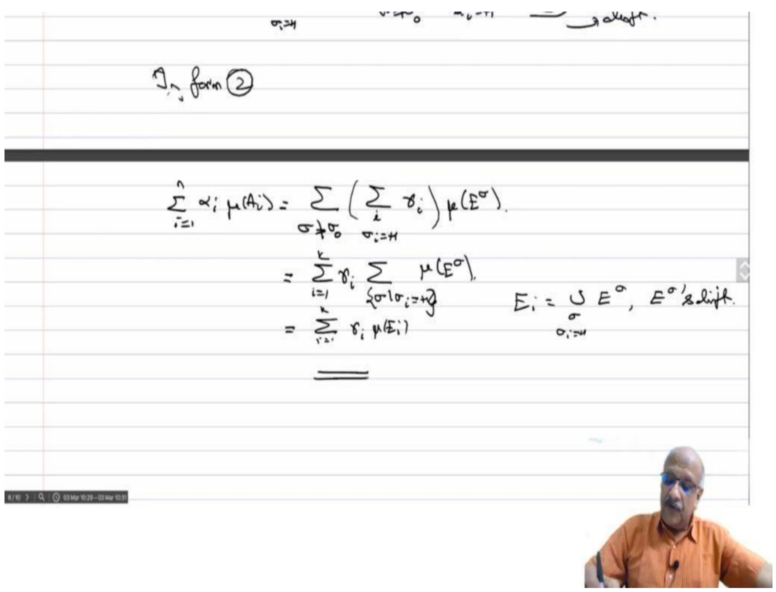

So, now let us assume $\varphi = \sum_{i=1}^{k} \sigma_i \chi_{E_i}$.

Now, $\chi_{E_i} = \chi_{\bigcup_{\sigma_i=1} E^{\sigma}} = \sum_{\sigma_i=1} \chi_{E^{\sigma}}$; because all the E^{σ} are disjoint. Consequently,

$$\varphi = \sum_{i=1}^{k} \gamma_i \sum_{\sigma_i=1} \chi_{E^\sigma} = \sum_{\sigma_i \neq \sigma_0} \sum_{i:i=1} \gamma_i \chi_{E^\sigma}$$

So, this is the, I am interchanging the order of the i; so, $\gamma_i \chi_{E^\sigma}$. But, so this is another way of writing the function phi in terms of disjoint sets.

So, by 2, so this is in form **(2)**, the

$$\sum_{i=1}^{n} \alpha_i \mu(A_i) = \sum_{\sigma \neq \sigma_0} \left(\sum_{\{i:\sigma_i=+1\}} \gamma_i \right) \mu(E^\sigma)$$

$$= \sum_{i=1}^{k} \gamma_i \sum_{\{\sigma:\sigma_i=+1\}} \mu(E^\sigma)$$

$$\sum_{i=1}^{k} \gamma_i \mu(E_i), \qquad since \quad E_i = \bigcup_{\sigma:\sigma_i=+1} E^\sigma, \; E^{\sigma}\text{'} \; disjoint.$$

So, whatever form you write the definition.

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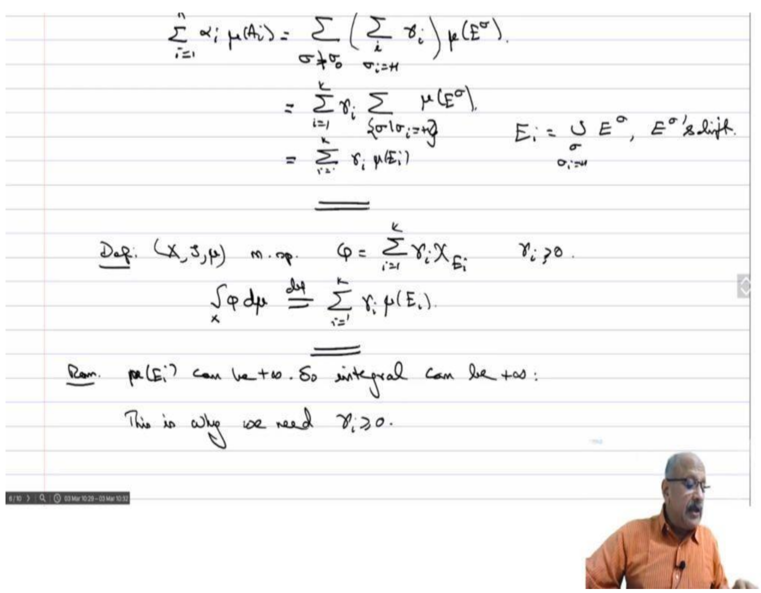

Definition, (X, S, μ) measure space and $\varphi = \sum_{i=1}^{k} \gamma_i \chi_{E_i}$; $\gamma_i \geq 0$. Then, the Lebesgue

$$\int_X \varphi d\mu \stackrel{def}{=} \sum_{i=1}^{k} \gamma_i \mu(E_i).$$

So, this is the definitions of the integral when you have a simple function, which is non-negative; so remark.

Remark: Measure of $\mu(E_i)$ can be infinite; so, integral can be infinity. And this is why we need $\gamma_i \geq 0$. If $\gamma_i \geq 0$ change sign and you have $\gamma_i \geq 0$ equals with $\mu(E_i) > 0$ and $\gamma_i < 0$, I mean infinite; and γ_j is also $\mu(E_j)$ is negative infinite and $\gamma_j < 0$, then you cannot add these numbers meaningfully. When you have two infinities with opposite signs, then you cannot add. And therefore, so that we do not have any ambiguity in the thing; we use only non-negative functions to start with.

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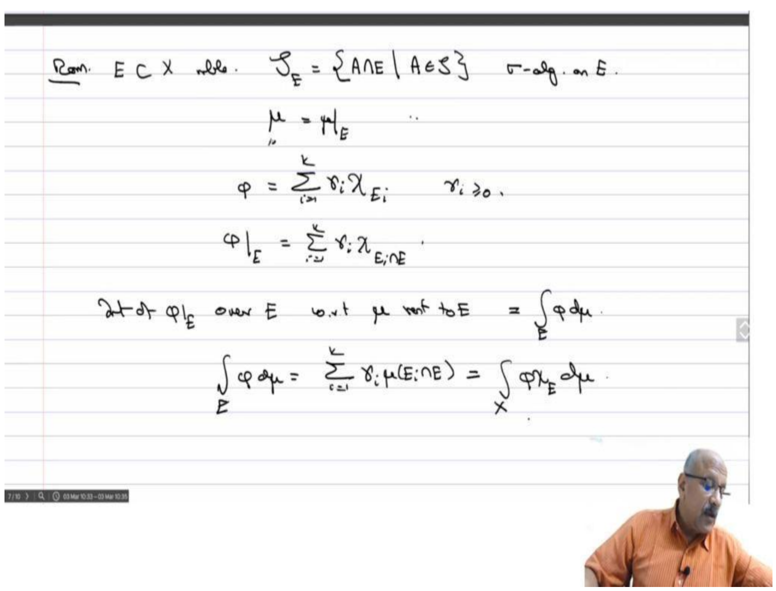

Remark. So, let $E \subset X$ measurable. So, you will define

$$S_E = \{A \cap E : A \in S\}, \ \sigma - \ algebra \ on \ E.$$

set of all sets of the form A intersection E, A belongs to S. So, this will give you aσ algebra on E; and then $\mu = \mu|_E$. So, $\mu(E)$ if you like; so, $\mu(E)$ is nothing but the same measure; so, you just $\mu(E) = \mu|_E(E)$.

So, now if $\varphi = \sum_{i=1}^{k} \gamma_i \chi_{E_i}, \quad \gamma_i \geq 0$.

Then, $\varphi|_E = \sum_{i=1}^{k} \gamma_i \chi_{E_i \cap E}$.

So, $\int_E \varphi d\mu = \sum_{i=1}^{k} \gamma_i \mu(E_i \cap E)$.

so, we call this as integral over E $\varphi d\mu$.

And this is nothing but $\int_E \varphi d\mu = \int_X \varphi \chi_E d\mu$

So, you can think of this as another definition of the integral; so, it is just this this thing here.

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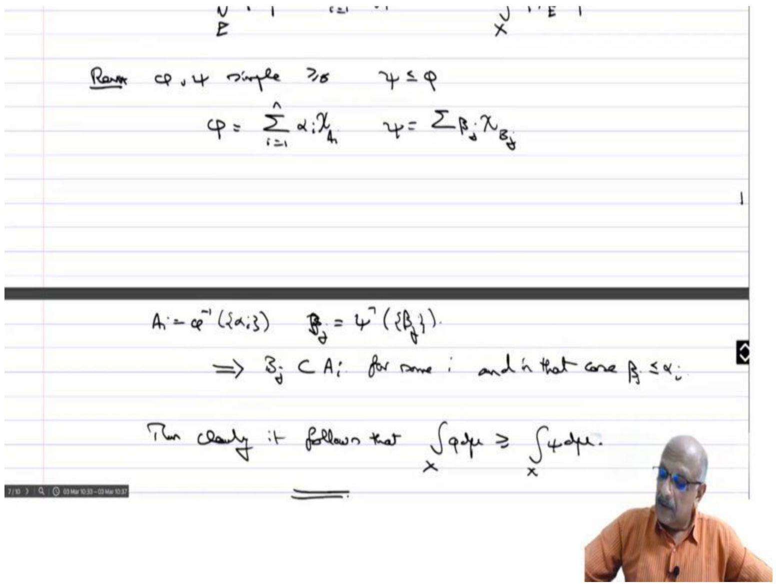

Remark: Finally, φ, ψ simple non-negative and $\psi \leq \varphi$. So, then we write

$$\varphi = \sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad \psi = \sum_{j=1}^{k} \beta_j \chi_{B_j}.$$

So, $A_i = \varphi^{-1}(\{\alpha_i\})$ $B_j = \psi^{-1}(\{\beta_j\})$

This will imply that $B_j \subset A_i$ *for some i*. And in that case $\beta_j \leq \alpha_i$; so, this is how it can happen. So, when you are written in terms of disjoint sets this, then clearly it follows that

$$\int_X \varphi d\mu \geq \int_X \psi d\mu.$$

So, you have the integral for the bigger function is bigger; just as you expects in the case of area under the curve, the bigger function will have more area under the curve and so on. So, this is these are some of the important things. So, next we will have to extend from simple functions to other functions. So, we will do that next.