Measure and Integration
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Lecture 27
Exercises

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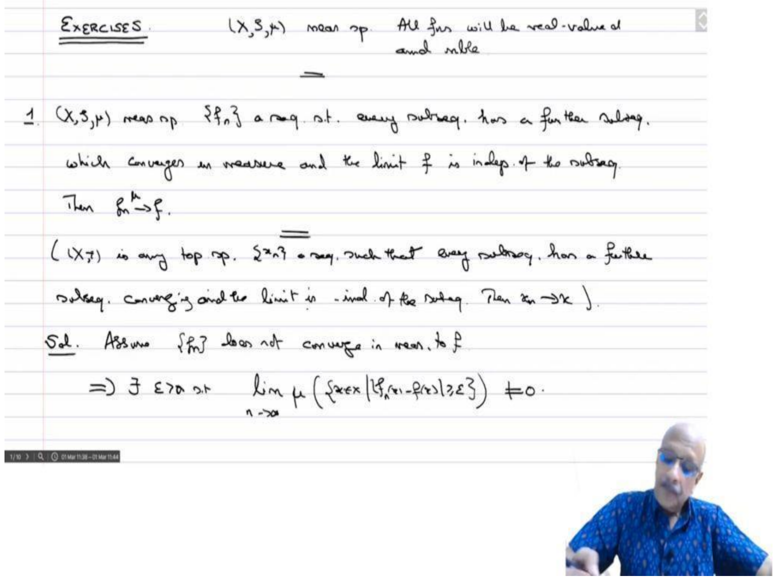

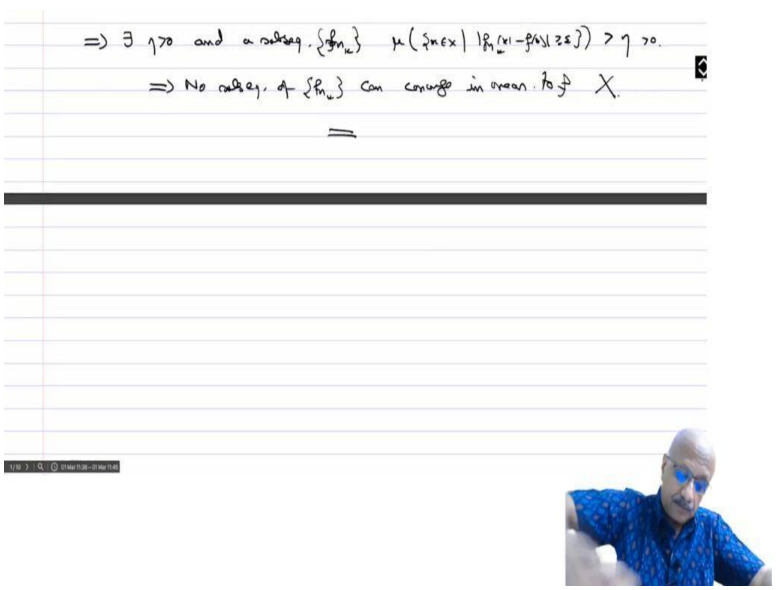

Now, it is time to do some exercises. So, in all that follows (X, S, μ) will be a measure space and if any all functions will be real valued and measurable; unless otherwise specified. But, this is what, so I want say it again and again so this running hypothesis which we will have all the time.

So, first exercise. So,

Exercise 1: (X, S, μ) measure space and $\{f_n\}$ a sequence such that every subsequence has a further subsequence, which converges in measure; and the limit f is independent of the subsequence. Then , $f_n \to f$ in measure. So, let us understand this hypothesis. So, do you have a sequence, take any subsequence. Then, you can extract to further subsequence which will be converging in measure; and the limit function which you get there is independent.

Whatever subsequence you are taking and further sub subsequence you are taking, the limit is always the same; it is independent of the subsequence chosen; then, $f_n \to f$ in measure. So, this is a very important property if you have. So, for instance, if (X, τ) is any topological space; this is a very useful, very trivial observation for limits. But, you should do this exercise; it take a couple of minutes to solve and it is extremely useful to know it.

So, (X, τ) is any topological space; then, a sequence such that every subsequence has a further subsequence converging to a fixed point independent of the subsequence. And the limit is independent of the subsequence; then then, $x_n \to x$. So, you see we can very often using compactness arguments, we can extract some subsequences which are convergent; but we would like to know if the entire sequence converges.

And if you can show that the limit is always unique in the sense that it does not depend whatever may be the subsequence you started off, the limit is always a certain fixed elements. Then, you can deduce that the entire subsequence will in fact converge to this point. So, this a very useful argument which we can, which can be can exploit in very very nice ways.

And the same is true also for convergence in measure; convergence in measure is not standard type of convergence as we saw; it depends on some measures of some sets going to zero. And what we have here is the same property. If you have a subsequence and the further subsequence which converges to in measure to a limit f, which is independent of the subsequence chosen; then the original sequence also converges to f.

So, this proof is I am going to do can be easily copied to do the general topological exercise which I have given here also.

Proof: So, assume $\{f_n\}$ does not converge in measure to f. So, what does it mean? That means, there exists an $\varepsilon > 0$ such that $\lim_{n\to\infty} \mu\left(\left\{x \in X : |f_n - f| \geq \varepsilon\right\}\right) \neq 0$.

What do you mean that limit of some positive numbers is not equal to zero? That means, you can so this implies there exists $\eta > 0$ and a subsequence such that $\{f_{n_k}\}$, such that $\mu\left(\left\{x \in X : |f_n - f| \geq \varepsilon\right\}\right) > \eta$. And this will be true for any subsequence of this subsequence also; that means no subsequence of $\{f_{n_k}\}$ can converge in measure to f.

This will always be bigger than eta and limit will not go to 0; and that is a contradiction to the hypothesis. Therefore, the original sequence converges in measure to f.

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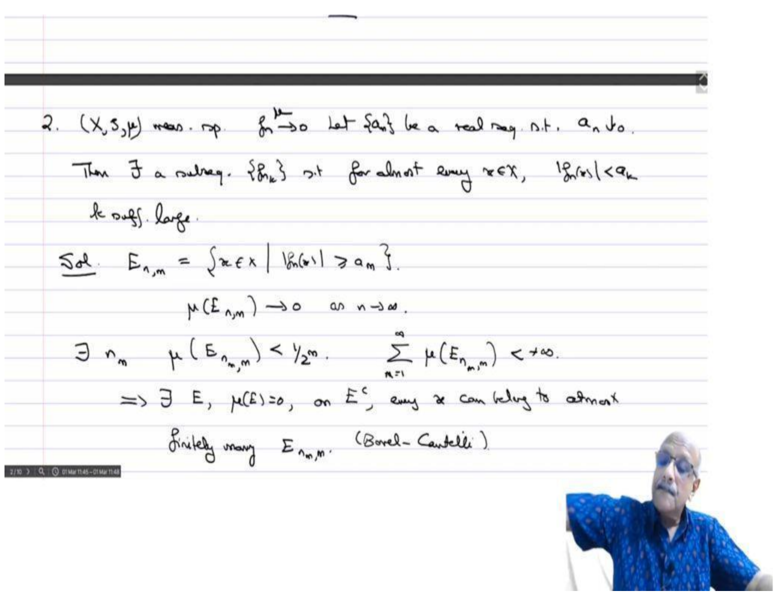

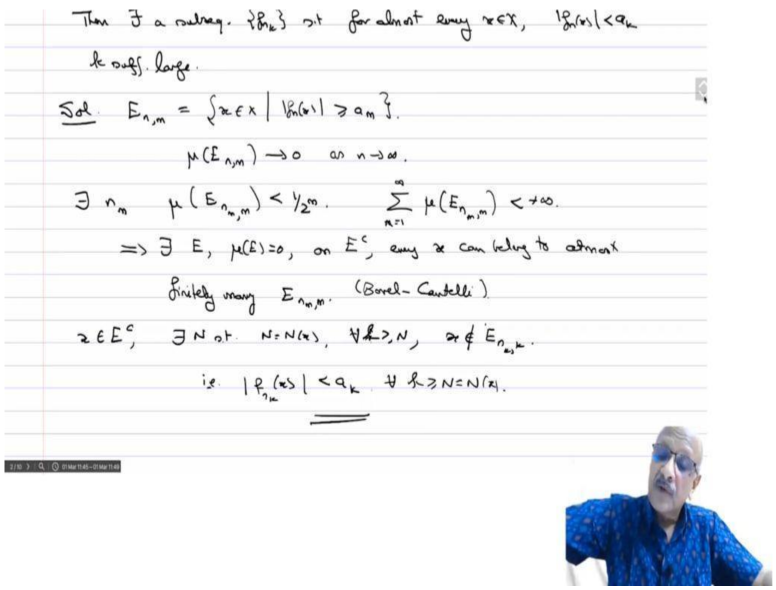

Exercise 2: (X, S, μ) measure space and $f_n \to 0$ in measure μ. Let $\{a_n\}$ be a real sequence such that an decreases to 0; so, it is a monotonically decreasing sequence whose infimum is 0. Then, there exists a subsequence $\left\{f_{n_k}\right\}$ such that for almost every $x \in X$, $|f_{n_k}(x)| < a_k$, k sufficiently large; so it depends on x of course, the k how large may depend on x.

Solution; this again application of the Borel-Cantelli lemma; so, we are going to use the same argument. So,

$$E_{n,m} = \left\{x \in X : |f_n(x)| \geq a_m\right\}.$$

So, $\mu\left(E_{n,m}\right) \to 0$ as n tends to infinity, m is fixed. So, now since this goes to 0, So $\exists n_m$, such that $\mu\left(E_{n,m}\right) < \frac{1}{2m}$. $\quad \sum_{m=1}^{\infty} \mu\left(E_{n,m}\right) < \infty$

because the geometric series, so this is finite. Then, there $\exists\, E$ with $\mu\left(E\right) = 0$, such that on E^c every x can belong to at most finitely many $E_{n,m}$, this is starting but Borel-Cantelli. So, if

$$x \in E^c,\ \exists N = N(x),\ \forall\ l \geq N,\ x \notin E_{n_k,k}.$$

That is $|f_{n_k}(x)| < a_k, \ \forall k \geq N$. So, that proves, so it is a very nice application again of the Borel-Cantelli lemma.

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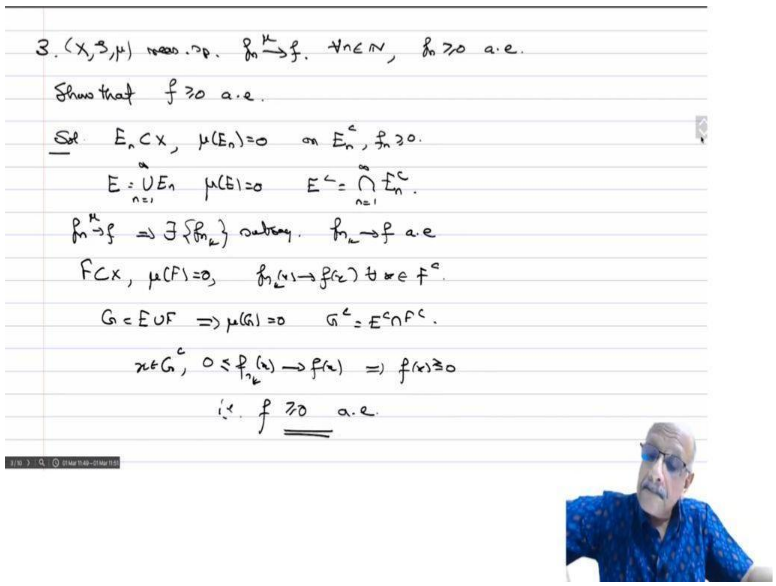

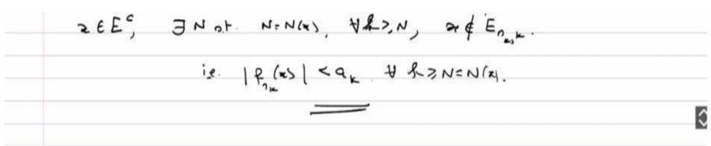

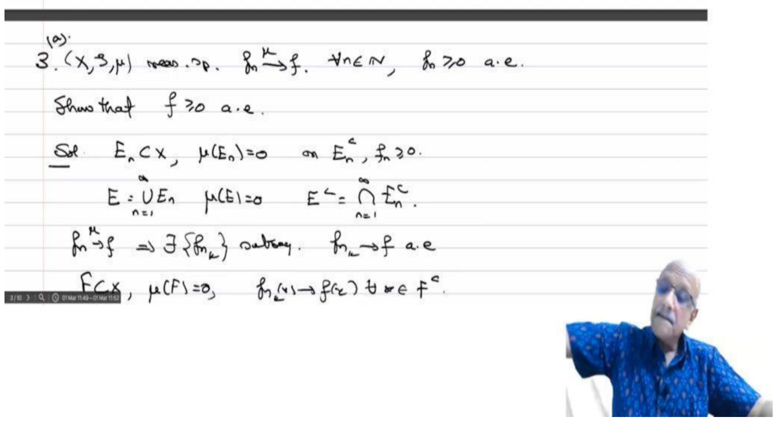

Exercise 3: (a) (X, S, μ) measure space, $f_n \to f$ in measure μ, $\forall n \in \mathbb{N}$. You have $f_n \geq 0$ almost everywhere. Show that $f \geq 0$ almost everywhere. So, it is behaving very much like ordinary convergence. Namely, if you have non-negative functions, the limit in measure must also be non-negative almost everywhere.

Solution, so let $E_n \subset X$, $\mu(E_n) = 0$; and on E_n^c, $f_n \geq 0$. So, you take $E = \bigcup_{n=1}^{\infty} E_n$, $\mu(E) = 0$, and $E^c = \bigcap_{n=1}^{\infty} E_n^c$. Now, $f_n \to f$ *in* μ implies there exists some sequence $\{f_{n_k}\}$, such that $f_{n_k} \to f$ point wise almost everywhere.

So, let $F \subset X$, $\mu(F) = 0$ and $f_{n_k}(x) \to f(x)$, for every x in F^c. So, now you said

$$G = E \cup F \Rightarrow \mu(G) = 0, \quad G^c = E^c \cap F^c.$$

So, on F^c means everything goes to 0.

So, if x belongs to G complement, you have $f_{n_k}(x) \to f(x)$, and $f_{n_k}(x) \geq 0$; and this implies $f(x) \geq 0$. That is except on G which just measures 0; that is f is greater than equal to 0 almost everywhere. Let me call this a.

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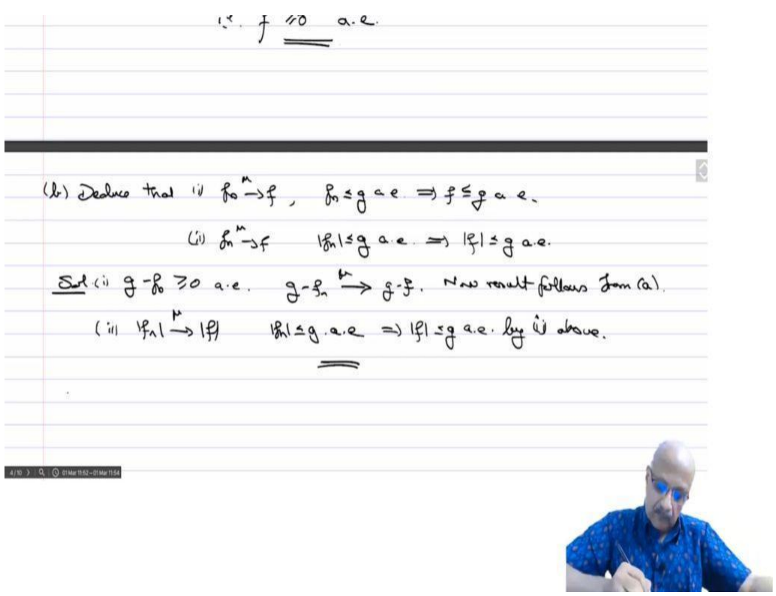

(b): Deduce that (i) $f_n \to f$ *in* μ, $f_n \leq g$ almost everywhere; implies $f \leq g$, a.e. (ii), $f_n \to f$ *in* μ, $|f_n| \leq g$ a.e, implies $|f| \leq g$ *a.e*

Solution, $g - f_n \geq 0 \ a.e$; and $g - f_n \to g - f$ and then the result follows. Similarly, mod fn converges to $|f|$ almost everywhere. And so, so this is one; now result follows from a. Then, $|f_n| \to |f|$; and you have mod fn less than equal to g almost everywhere, implies $|f| \leq g \ a.e$ by one above.

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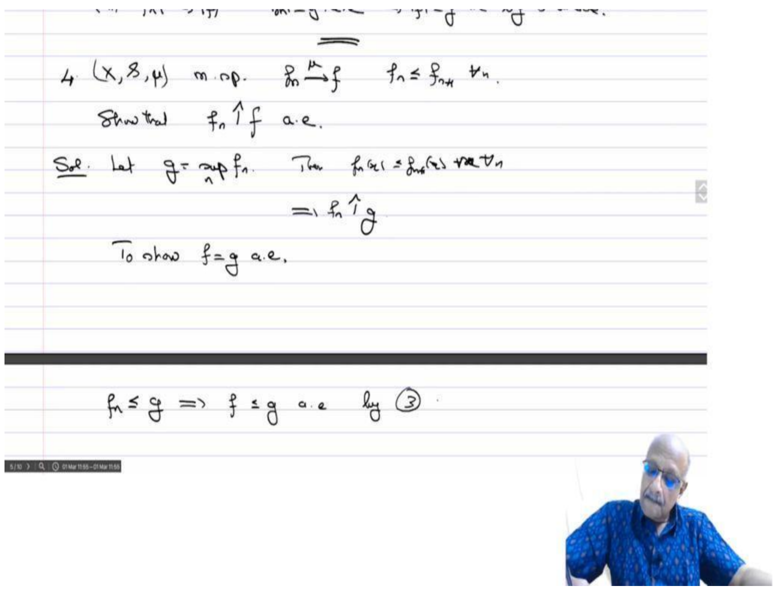

Exercise 4: (X, S, μ) measure space, $f_n \to f$ in measure μ, $\forall n \in \mathbb{N}$. You have $f_n \leq f_{n+1}$. Show that $f_n \uparrow f \ \ a.e$.

Solution. Let $g = sup_n f_n$, then $f_n(x) \leq f_{n+1}(x)$. for all x, for all n; implies $f_n \uparrow g$. So, to show $f = g \ a.e$; so let would do the trick.

But, $f_n \leq g \Rightarrow f \leq g \ \ a.e$, by (3) (b) (i). Now, fix $n \in \mathbb{N}$. Consider the sequence $\{f_{n+k}\}_{k=1}^{\infty}$. Then, that is a subsequence and therefore, $f_{n+k} \to f \ in \ \mu$. And you have $f_n \leq f_{n+k}$, for all k; and this implies that $f_n \leq f$ for all n.

So this is true. Now, true for n; this implies f_n is (())(18:06) implies g which is the supremum is less than equal to f. And this implies that $g \leq f, f \leq g \Rightarrow f = g \quad a.e.$ So, we will stop with this and next time we will start integration.