Measure and Integration
Professor S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 26
Convergence in measure

So, we will continue with our study of properties of convergence in measures. So, now we look at some algebraic operations and the usual properties.

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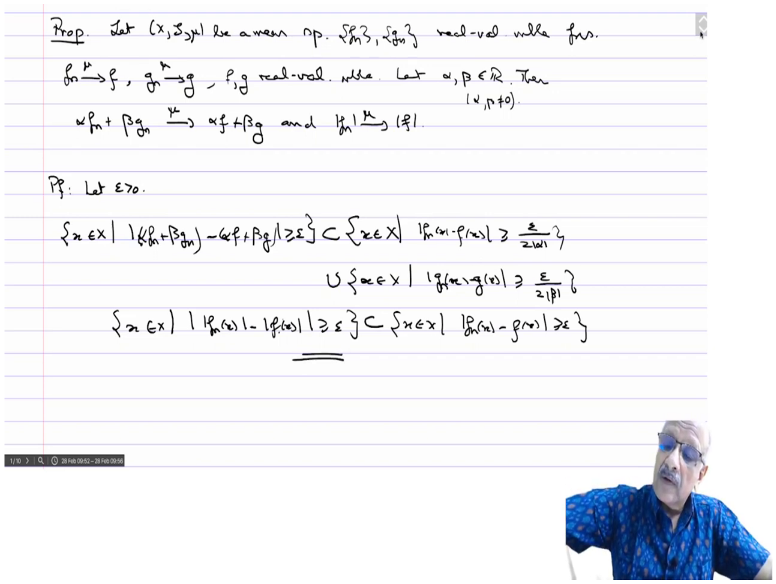

Proposition: (X, S, μ), be a measure space $\{f_n\}$, $\{g_n\}$ real valued measurable functions, $f_n \to f$, $g_n \to g$ in μ, f, g real valued measurable. Let $\alpha, \beta \in \mathbb{R}$, then $\alpha f_n + \beta g_n \to \alpha f + \beta g$ in measure and $|f_n| \to |f|$ in measure. So, these are the usual properties we expect from any reasonable notion of convergence.

Proof: so let $\varepsilon >$ so then

$$\left\{x \in X : |(\alpha f_n - \beta g_n) - (\alpha f - \beta g)| \geq \varepsilon\right\} \subset \left\{x \in X : |f_n(x) - f(x)| \geq \varepsilon/(2|\alpha|)\right\}$$

$$\cup \left\{x \in X : |g_n(x) - g(x)| \geq \varepsilon/(2|\beta|)\right\}$$

So, we can take of course alpha beta not equal to 0, otherwise there is nothing to prove. So, then the right hand side for n sufficiently large go the measure can be made as small as you like and therefore this measure also goes to 0. Similarly, you have

$$\{x \in X : ||f_n(x)| - |f(x)|| \geq \varepsilon\} \subset \{x \in X : |f_n(x) - f(x)| \geq \varepsilon\}$$

This is again by the triangle inequality mod of mod a minus mod b is equal to less than or equal to mod of a minus b. So, if this is greater equal to epsilon this will also be greater equal to epsilon. So, from these two the result follows immediately.

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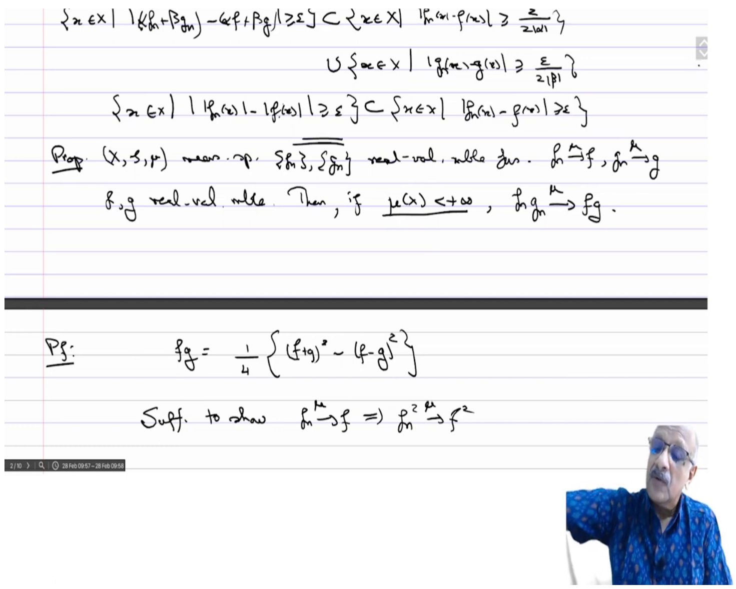

So now next proposition about products.

Proposition: (X, S, μ), be a measure space $\{f_n\}$, $\{g_n\}$ real value measurable functions, $f_n \to f$, $g_n \to g$ in μ, f, g real valued measurable. Then if $\mu(X) < \infty$. So, now we are having condition here $f_n g_n \to fg$ in measure, so there is some extra restriction in this.

Proof, so we know that

$$fg = \frac{1}{4}\{(f + g)^2 - (f - g)^2\}$$

So, if f_n, g_n goes to f and g in measure then $f_n + g_n$ will go to $f + g$ in measure then $f_n - g_n$ will go to $f - g$ in measure, so sufficient to show that f_n converges to f in measure implies f_n square converges to f square in measure.

So, then this will go in measure $(f_n + g_n)$ will go to $(f + g)^2$ in measure, $(f_n - g_n)^2 \rightarrow (f - g)^2$ in measure and then by multiplication by 1 by 4 you can easily check is also no problem and therefore we will have completed the proof of this theorem.

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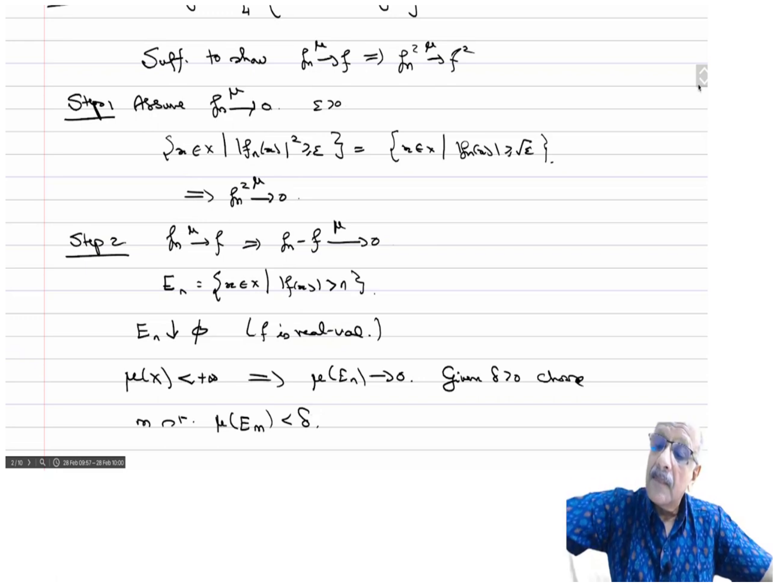

So, now this is done in a few steps. So, first step

Step 1: Assume $f_n \to 0$ in measure, so

$$\left\{x \in X : |f_n(x)|^2 \geq \varepsilon\right\} = \left\{x \in X : |f_n(x)| \geq \sqrt{\varepsilon}\right\}$$

so it is the same thing. So $\varepsilon > 0$ and therefore this implies $f_n^2 \to 0$ in measure.

Step 2 so, $f_n \to f \Rightarrow f_n - f \to 0$ in measure. Now, you let

$$E_n = \{x \in X : |f(x)| > n\}$$

Then E_n decreases because as n becomes larger the set becomes smaller and since f is real valued it decreases to so f is real value this decreases to the empty set. So, now you use the fact that $\mu(X) < \infty$ and this implies that $\mu(E_n) \to 0$.

This is the continuity from above which we have proved, if you have the finite measure space then you have the intersection if E_n decreases to empty set then $\mu(E_n) \to 0$. So, then choose so given $\delta > 0$ choose m such that $\mu(E_m) < \delta$.

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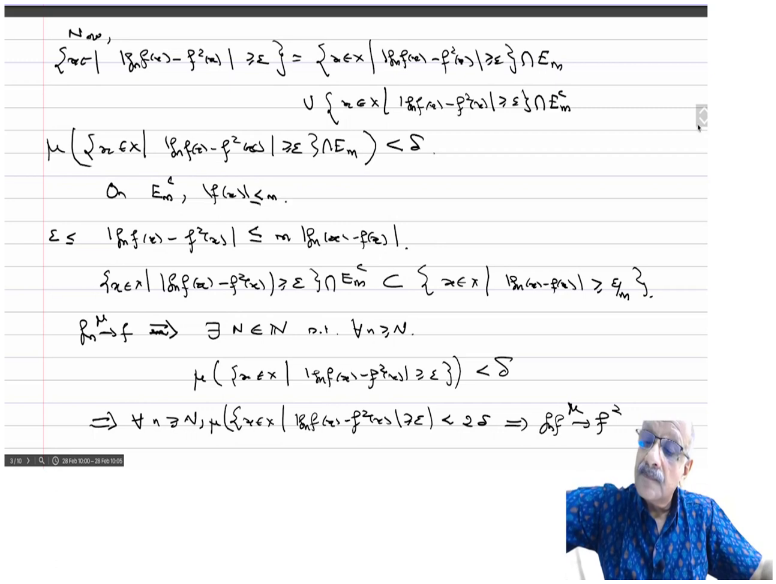

Now

$$\left\{x \in X: |f_n f(x) - f^2(x)| \geq \varepsilon\right\} = \left\{x \in X: |f_n f(x) - f^2(x)| \geq \varepsilon\right\} \cap E_m$$

$$\cup \left\{x \in X: |f_n f(x) - f^2(x)| \geq \varepsilon\right\} \cap E_m^{\ c}$$

so it is, that is straight forward thing, so now.

So if this set, so

$$\mu\left(\left\{x \in X: |f_n f(x) - f^2(x)| \geq \varepsilon\right\} \cap E_m\right) < \delta$$

the measure of the set is obviously less than the measure of E m and therefore that is less than delta. Now, on $E_m^{\ c}$, $|f(x)| \leq m$

therefore you have $\varepsilon \leq |f_n f(x) - f\ (x)| \leq m|f_n(x) - f(x)|$

Therefore, $\left\{x \in X: |f_n f(x) - f^2(x)| \geq \varepsilon\right\} \cap E_m^{\ c} \subset \left\{x \in X: |f_n(x) - f\ (x)| \geq \varepsilon/m\right\}$

And this, measure of this set goes to 0 because f_n converges to f in measure, so f_n converges to f in measure implies there exists a capital $N \in \mathbb{N}$ such that for all $n \geq N$ you have the measure

$$\mu\left(\left\{x \in X : |f_n f(x) - f^2(x)| \geq \varepsilon\right\} \cap E_n^{\,c}\right) \leq \delta.$$

So, consequently for all $n \geq N$ you have

$$\mu\left(\left\{x \in X : |f_n f(x) - f^2(x)| \geq \varepsilon\right\}\right) \leq 2\delta$$

and so that implies that $f_n f \to f^2$ in measure.

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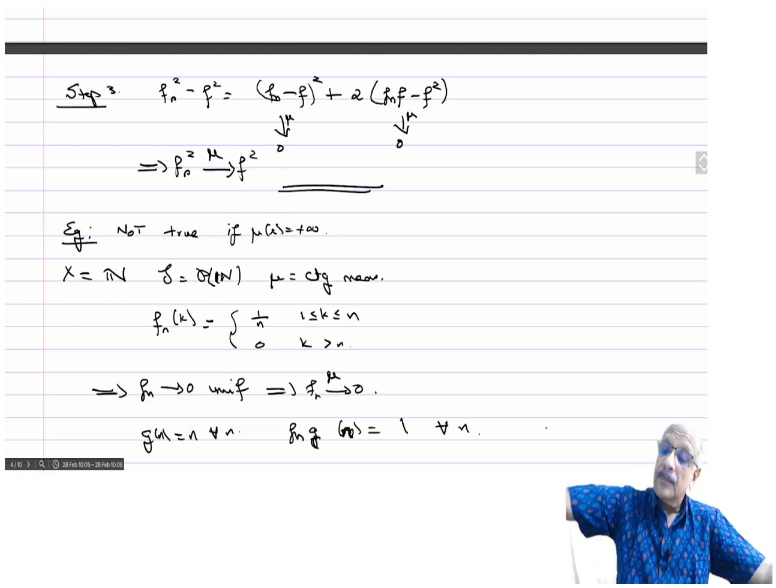

Step 3, you have

$$f_n^{\,2} - f^2 = (f_n - f)^2 + 2(f_n f - f^2)$$

Now, this goes to 0 in measure because f_n goes to f in measure the square when it goes to $f_n - f$ goes to 0 in measure so the square goes to 0 in measure and just now we saw that this goes to 0 in measure and therefore you have $f_n \to f^2$ in measure. So, this completes the proof of this proposition.

So, example

Example: Not true if $\mu(X) = \infty$, so again you take the standard set $X = N$, $S = P(N)$ and μ equals counting measure. Now, you define

$$f_n(k) = \frac{1}{n} \quad if \;\; 1 \leq k \leq n$$

$$= 0 \quad if \quad n > k.$$

Then this implies that $f_n \to 0$ uniformly implies $f_n \to 0$ in measure, we have seen that. Now, you take

$$g\ (n) = n \quad \forall\, n.$$

$f_n g\ (n) = 1$ you have a fixed sequence you do not even have to need the sequence.

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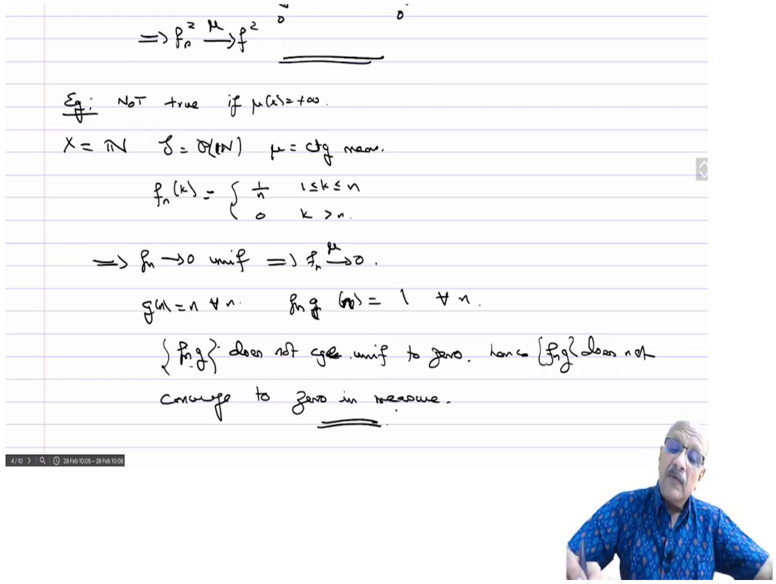

So, this will mean that $\{f_n g\}$ does not go to 0 uniformly, so $\{f_n g\}$ does not converge uniformly to 0. Hence, $\{f_n g\}$ does not converge to 0 in measure. So, this is not true for infinite measure spaces. So, we will conclude our study of convergence issues with this and before proceeding further we will do some exercises.