Measure and Integration
Professor S. Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 25
4.6 – Convergence in measure

So, we will continue with the study of Convergence in Measure. So, the previous session what we did was we define what is convergence and measure what is Cauchy in measure and then we studied its relationship with the point where is convergence. So, if it is convergence in measure, then you have a sub sequence which converges point wise and if it is finite measure space and point wise convergence always implies convergence in measure then not true for the infinite measure spaces.

Then we also saw that the limit was unique in the sense that if you have two functions to reach a certain sequence converges in measure then the two have to be equal almost everywhere and that is as good as uniqueness to this as far as you can go because of the very definition of convergence in measure. Today, we will look at the connection with almost uniform convergence and also what is Cauchy in measure, that convergence and measure that they imply each other. So, these are the things which we want to do today.

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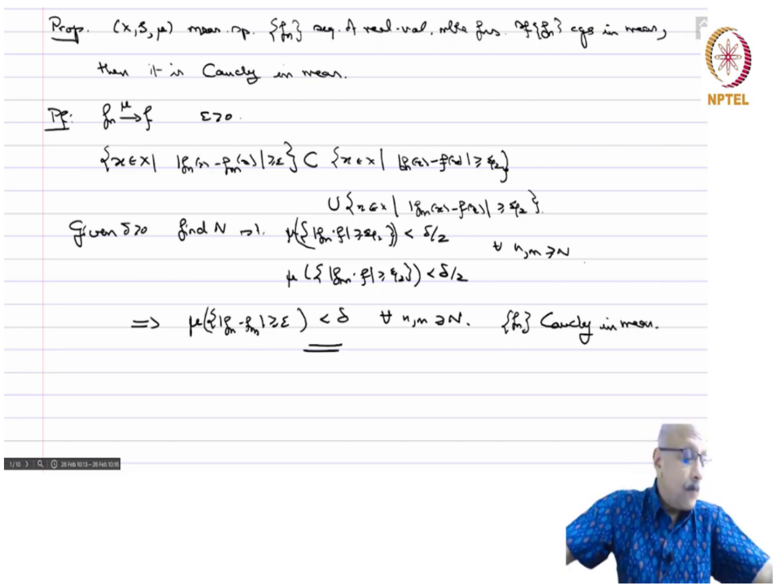

So, we will start with the following proposition So,

Proposition: (X, S, μ) measure space $\{f_n\}$ sequence of real-valued measurable functions if I do not say this please assume it always when we talk of real-valued measurable functions. Then, if fn converges in measure, then it is Cauchy in measure. So, this is always the easy part.

Proof: Let us take $f_n \to f\ in\ \mu$ and then if you if you take So, let us take $\varepsilon > 0$ then

$$\left\{x \in X: |f_n(x) - f_m(x)| \geq \varepsilon\right\} \subset \left\{x \in X: |f_n(x) - f(x)| \geq \varepsilon/2\right\}$$

$$\cup \left\{x \in X: |f_m(x) - f(x)| \geq \varepsilon/2\right\}$$

this is a usual argument we gave because this set is contained in because the triangle inequality $f_n - f_m$ is this an $f_n - f$ plus f minus f_m and then each of these is less than epsilon by 2. So, obviously, this will be less than epsilon.

So, if this has to be greater than equal to ε then it has to belong to one of these two sets. So, given $\delta > 0$ So, find N such that $\mu\left(\left\{x \in X: |f_n(x) - f(x)| \geq \varepsilon/2\right\}\right) < \delta/2$

$$\mu\left(\left\{x \in X: |f_m(x) - f(x)| \geq \varepsilon/2\right\}\right) < \delta/2$$

for all $m, n \geq N$. So, this will imply then $\mu\left(\left\{x \in X: |f_n(x) - f_m(x)| \geq \varepsilon\right\}\right) < \delta,\ for\ all\ n, m \geq N$

and that proves that $\{f_n\}$ is a Cauchy sequence in measure.

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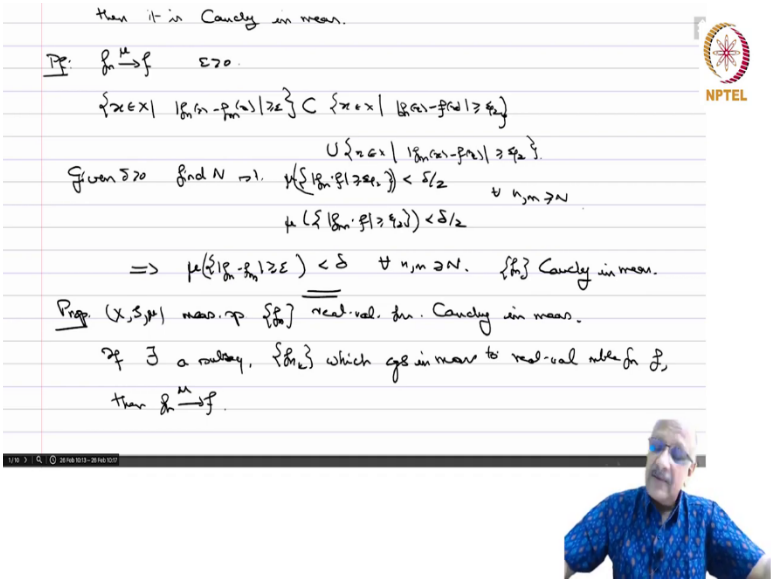

So, now we have to do the converse that takes some work and it also gets mixed up with the question about almost uniform convergence. So, we will say proposition we will do it in several stages.

Proposition: (X, S, μ) measure space $\{f_n\}$ sequence of real-valued functions Cauchy in measure so, the ultimate aim is to show that this converges in measure.

So, if there exists a subsequence $\{f_{n_k}\}$ which converges in measure to real-valued measurable for function f then $f_n \to f \; in \; \mu$. So, this is just like in real sequences you have a Cauchy sequence and if you have a convergent subsequence then the original sequence is also convergent and it has the same limit.

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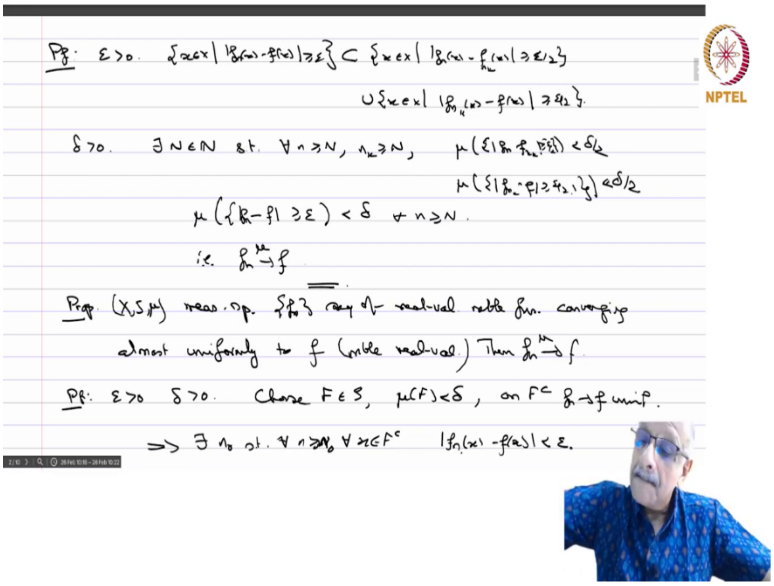

Proof: Let $\varepsilon > 0$, then

$$\{x \in X: |f_n(x) - f(x)| \geq \varepsilon\} \subset \left\{x \in X: |f_n(x) - f_{n_k}(x)| \geq \varepsilon/2\right\}$$

$$\cup \left\{x \in X: |f_{n_k}(x) - f(x)| \geq \varepsilon/2\right\}.$$

let $\delta > 0$ So, there exists a capital N such that for all $n, n_k \geq N$

$$\mu\left(\left\{x \in X: |f_n(x) - f_{n_k}(x)| \geq \varepsilon/2\right\}\right) < \delta/2$$

$$\mu\left(\left\{x \in X: |f_{n_k}(x) - f(x)| \geq \varepsilon/2\right\}\right) < \delta/2.$$

So, this will imply then $\mu\left(\left\{x \in X: |f_n(x) - f(x)| \geq \varepsilon\right\}\right) < \delta,\ for\ all\ n \geq N$.

that is a $f_n \to^{\mu} f\ in$ μ by the definition of convergence in measure.

Proposition so, our aim will be if you have a Cauchy in measure we will try to construct a sub sequence which is convergent in measure and then those original sequence will also converge. So,

Proposition: (X, S, μ) measure space $\{f_n\}$ sequence of real-valued measurable functions converging almost uniformly to f measurable real value then $f_n \to f \ in\ \mu$.

So, if you have almost uniform convergence, then you have convergence in measure so, proof. So,

Proof: let $\varepsilon > 0$ then you and let $\delta > 0$. So, choose $F \in S$, $\mu(F) < \delta$ and on F^c $f_n \to f\ uniformly$. What does it imply there exists

$$\Rightarrow \exists\, n_0\ such\ that\ \forall n \geq n_0\ \ \forall x \in F^c, \quad |f_n(x) - f(x)| < \epsilon.$$

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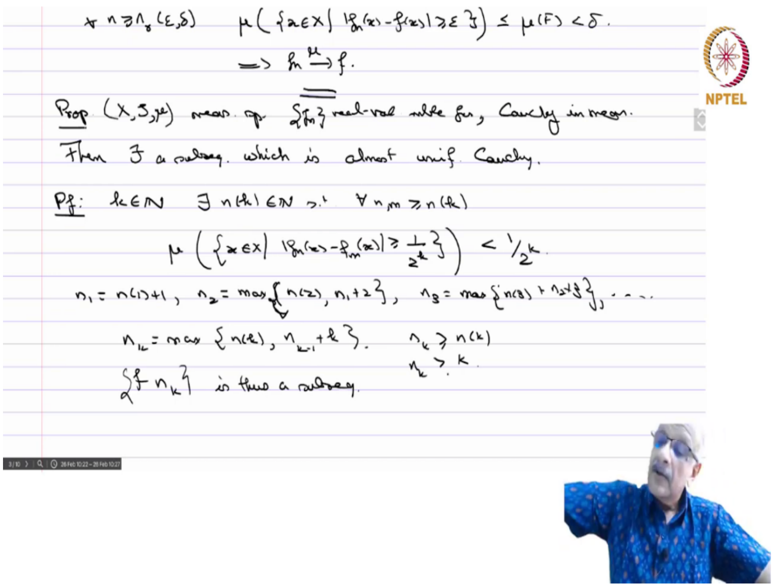

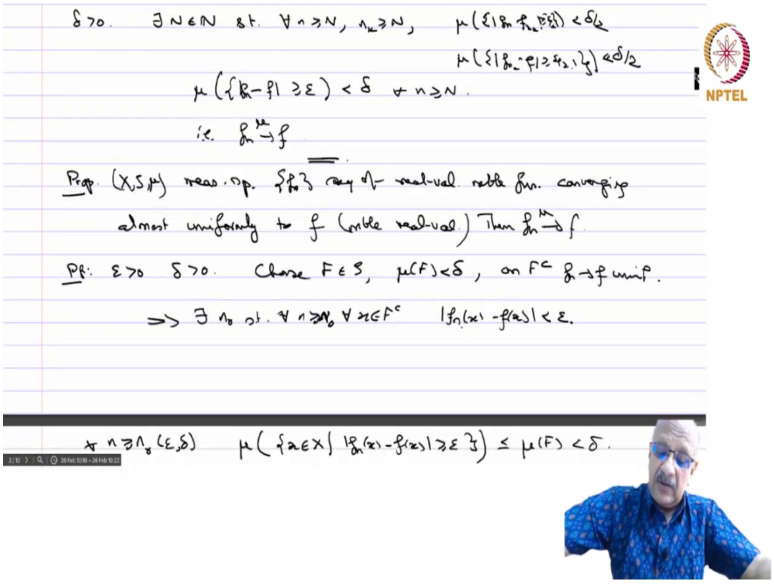

So, for all $n \geq n_0$ you will have, so, this depends on epsilon and really on delta also because you are using the δ comes because the choice of the set f and then therefore, F^c epsilon is because of the uniform convergence on f complement. So, we have this that

$$\mu\left(\left\{x \in X : |f_n(x) - f(x)| \geq \varepsilon\right\}\right) \leq \mu(F) < \delta,$$

and therefore, this proves that $f_n \rightarrow^{\mu} f$.

Next proposition

Proposition: (X, S, μ) measure space $\{f_n\}$ real valued measurable functions Cauchy in measure then there exists a sub sequence which is almost uniformly Cauchy. So, you now see how we are going so, we have a sequence which is Cauchy in measure so, you have a sub sequence which is almost uniformly Cauchy.

Now, almost uniformly Cauchy means almost uniformly convergent, we have already seen that and almost uniformly convergent in place convergence in measures, so, what we have done this we have seen that there exists a subsequence of a Cauchy in measure which is almost uniformly Cauchy therefore, almost uniformly convergent therefore, convergent in measure and then by the previous proposition which we prove you have a subsequence which converges in measure and its Cauchy in measure means it converges in measure to that same function.

So, that is how we are going to prove and in that process we are proving also results connecting convergence in measure and almost uniform convergence. So,

Proof: so, let $k \in \mathbb{N}$ then there exists $n(k) \in \mathbb{N}$ such that for all $\forall\, n, m \geq n(k)$ you have the $\mu\left(\left\{x \in X : |f_n(x) - f_m(x)| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}$.

Now, you choose $n_1 = n(1) + 1,\ \ n_2 = max\left\{n(2), n_1 + 2\right\},\ \ n_3 = max\left\{n(3),\ n_2 + 3\right\}$, ... so on. So,

$$n_k = max\left\{n(k),\ n_{k-1} + k\right\}.$$

So, $n_k \geq n(k),\ \ n_k > k,$ also. So, therefore, we have n_k is a proper subsequence $\{f_{n_k}\}$ is thus sub sequence.

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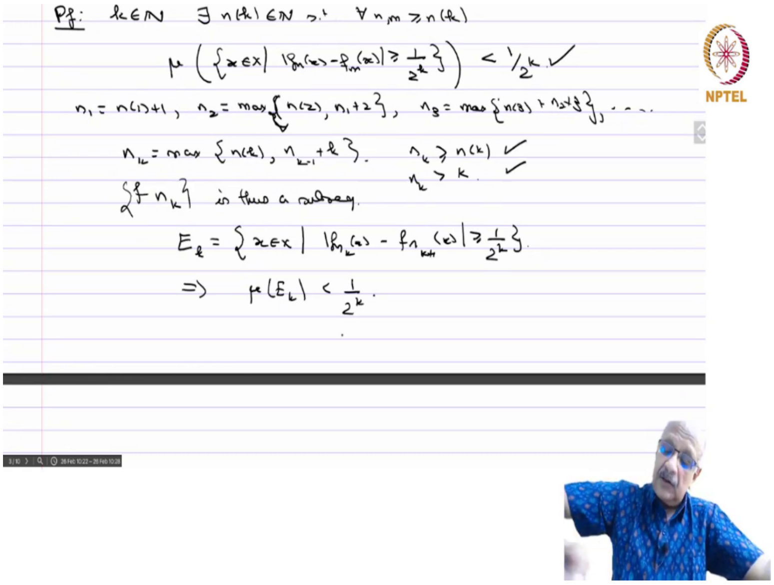

So, now, we will define $E_k = \left\{ x \in X \mid f_{n_k}(x) - f_{n_{k+1}}(x)| \geq \frac{1}{2^k} \right\} \Rightarrow \mu(E_k) < \frac{1}{2^k}$.

We connect with these conditions this condition so, that gives you this.

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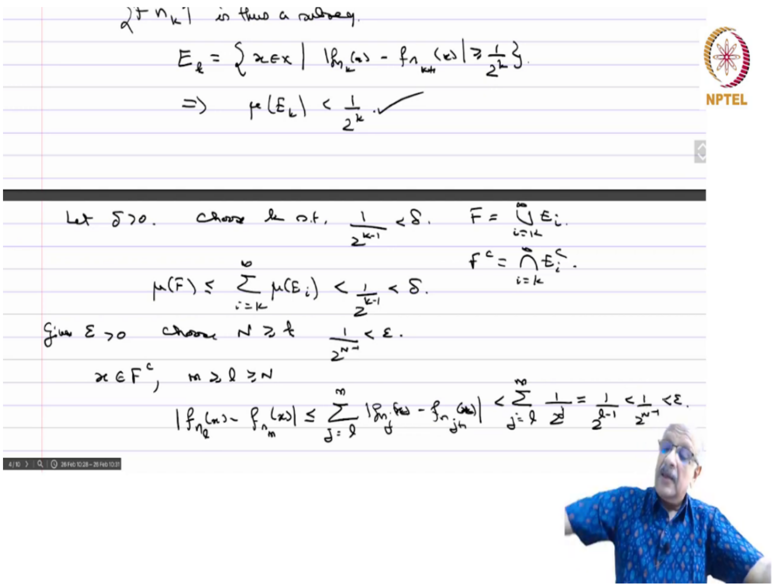

So, let $\delta > 0$. Choose k such that $\frac{1}{2^{k-1}} < \delta$. and you said. You know what is

$$F^c = \bigcap_{i=k}^{\infty} E_i^c.$$

So, what about $\mu(F) = \bigcup_{i=k}^{\infty} \mu(E_i) < \frac{1}{2^{k-1}} < \delta.$

And so that is a geometric sequence and therefore, you have that this is less than $\frac{1}{2^{k-1}}$ using the fact of this and that, of course, is less than δ by choice.

No also you chose epsilon. So, given $\varepsilon > 0,\ choose\ N \geq k\ \frac{1}{2^{N-1}} < \varepsilon$. So, now you take

$$x \in F^c,\ \ m \geq l \geq N,\ \ |f_{n_l}(x) - f_{n_m}(x)| \leq \sum_{j=l}^{m} |f_{n_j} - f_{n_{j+1}}(x)|$$

$$< \sum_{j=l}^{m} \frac{1}{2^j} = \frac{1}{2^{l-1}} < \frac{1}{2^{N-1}} < \varepsilon$$

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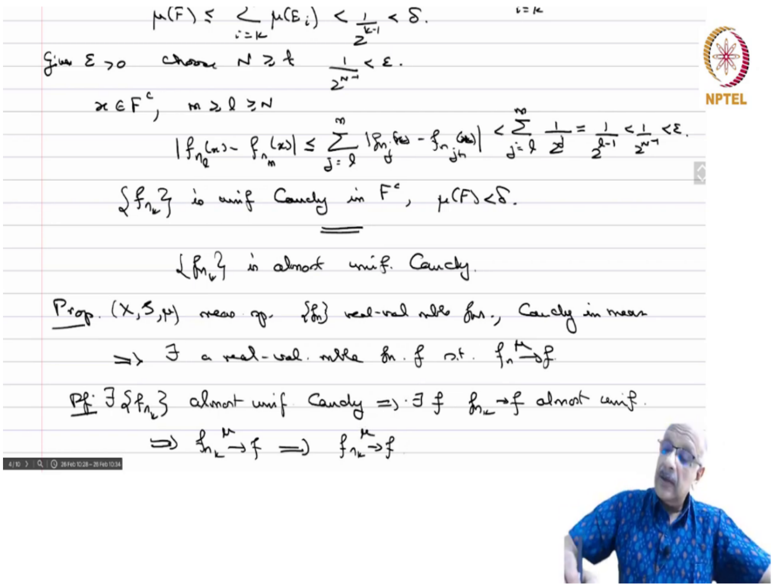

And therefore, this is true for all this and therefore, you have $\{f_{n_k}\}$ is uniformly Cauchy in F^c, and $\mu(F) < \delta$. So, this proves that you have subsequence which is unique almost. So, $\{f_{n_k}\}$ is almost uniformly Cauchy. Now, we have almost home. So, proposition

Proposition: (X, S, μ) measure space $\{f_n\}$ real valued measurable functions Cauchy in measure this implies ∃ a real valued measurable function f such that $f_n \to^{\mu} f$.

Proof: So, $\{f_n\}$ is Cauchy in measure so, $\{f_{n_k}\}$ almost uniformly Cauchy implies there exists an f.

So, say $\{f_{n_k}\}$ almost uniformly and that implies that $f_{n_k} \to^{\mu} f$ in measure because we have proved that and because you have some sequence which converges this measure no general is in Cauchy So, this in place $f_{n_k} \to^{\mu} f$.

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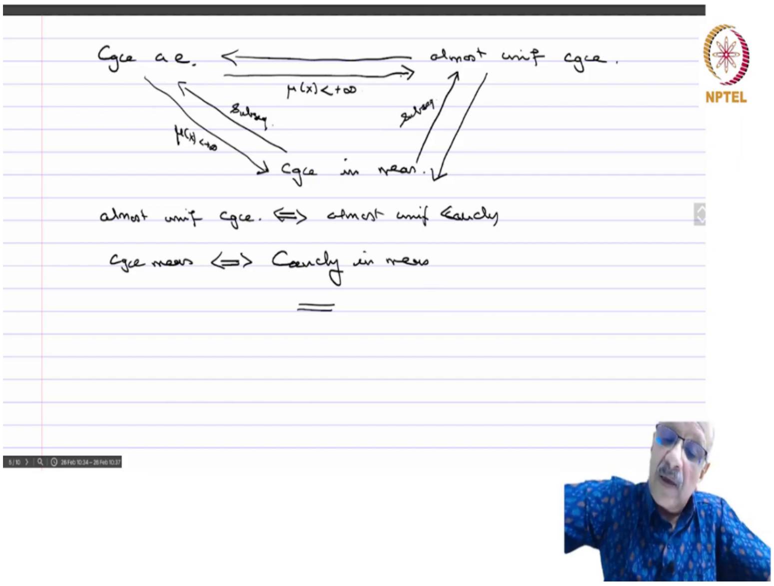

So, now we have proved a lot of things, so, it is better to have an idea what we have proved. So, we have convergence almost everywhere then you have almost uniform convergence and then we have convergence in measure and then we have that almost uniform convergence is the same as almost uniform Cauchy. And then convergence in measure is the same as Cauchy in measure.

By this I mean if you have a sequence that is almost uniformly convergent, then it is almost uniformly Cauchy, it is almost uniformly Cauchy, then it is almost uniformly convergent. Similarly, convergence in measure in place of Cauchy in measure implies convergence measure. So, these we have proved.

Now, convergence almost everywhere and convergence almost uniform convergence implies convergence almost everywhere we have seen, almost uniform convergence also implies convergence and measure we have seen. Now, convergence almost everywhere implies almost uniform convergence provided the measure is finite.

Similarly, convergence almost everywhere in place convergence in measure provided the measure is finite. Now, convergence in measure implies almost uniform convergence for the subsequence because, convergence in measure means Cauchy in measure the Cauchy in measure means is a subsequence which is almost uniformly Cauchy and therefore, almost uniformly convergent.

Similarly, convergence in measure also means convergence everywhere for a subsequence. So, these are the various inter relationships which we have. Now, if you have a sequence which is convergence in measure the subsequence which converges almost everywhere we assured using the Borel Cantelli Lemma, but you can also know use do it this way, convergence in measure means is almost uniformly Cauchy subsequence, which means almost uniformly convergent subsequence and that subsequence will imply first point wise everywhere and therefore, this another proof using these two arrows, which you can do for that particular theorem.

So, now we want to do what is the relationship of convergence in measure to the Algebraic operations alpha fn plus beta gn where fn, gn convergence and measure what is fn and gn and then what about mod fn and such things and therefore we will see those things next time.