**Measure and Integration**

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**Lecture 24**

**4.5 – Convergence in measure**

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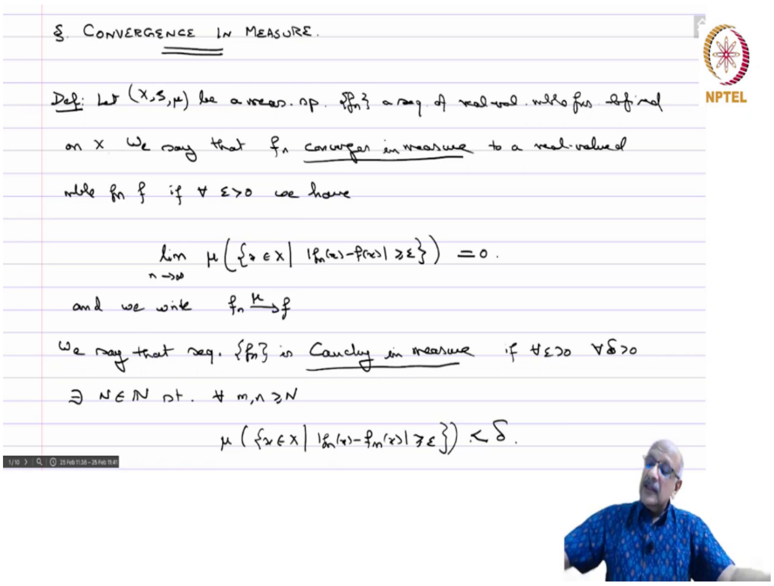


We will now study a new type of convergence. This is called Convergence in Measure. So, we will then study its properties and also its relationship to the other types of convergence which you have seen convergence point wise convergence almost everywhere point wise again and convergence almost uniformly so, all these interconnections we will see.

**Definition:** Let $(X, S, \mu)$ be a measure space and a $\{f_n\}$ a sequence of real-valued measurable functions defined on $X$, we say that a $f_n$ converges in measure to a function to a real-valued measurable function $f$ if $\forall\ \varepsilon > 0$ we have

$$lim_{n\to\infty}\mu(\{x\epsilon X|\ |f_n(x) - f\ (x)| \geq \varepsilon\}) = 0.$$

Think about it carefully. So, you take $\mu(\{x\epsilon X|\ |f_n(x) - f\ (x)| \geq \varepsilon\})$ should go to 0 as $n \to \infty$. So, this is a different type of convergence. We call this convergence in measure and we write

$$f_n \xrightarrow{\mu} f.$$

The symbol μ over the arrow which will mean that it converges in measure.

We say that sequence $\{f_n\}$ is Cauchy in measure if $\forall\ \varepsilon > 0$ and $\forall\ \delta > 0$ that $\exists\ N \in \mathbb{N}$ such that for all $m, n \geq N$

$$\mu(\{x \in X |\ |f_n(x) - f\ (x)| \geq \varepsilon\}) < \delta.$$

That means, $\mu(\{x \in X |\ |f_n(x) - f\ (x)| \geq \varepsilon\})$ can be made as small as you like for N sufficiently large. So, that is the corresponding idea of Cauchy.

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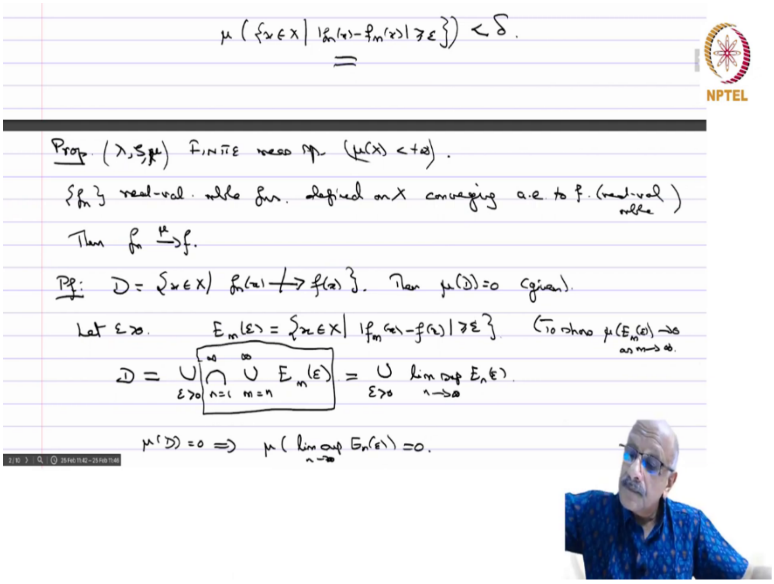


So, we would like to know what are the various properties of this convergence, what is its relationship to convergence point wise etc. and then of course, the usual question convergence does it imply Cauchy in measure Cauchy in measure does not imply convergence in measures and so on and so forth. Let us start with the following proposition.

**Proposition**: Let $(X, S, \mu)$ finite measure space (that means $\mu(X) < \infty$). $\{f_n\}$ the real-valued measurable functions defined on $X$ convergence almost everywhere to $f$. Then of course, real-valued and measured then $f_n \xrightarrow{\mu} f$.

(So, if you are in a finite measure space then convergence almost everywhere the in place, convergence in measure.)

**Proof**: Let $D = \{x \epsilon X|\ f_n(x) \nrightarrow f(x)\ \}$(the set where the sequence diverges),

then we are given that $\mu(D) = 0$ (given). Let $\varepsilon > 0$,

$E_m(\varepsilon) = \{x \epsilon X|\ |f_n(x) - f\ (x)| \geq \varepsilon\}$ (To show mu of Em epsilon tends to 0 as m tends to infinity.)

(Now, you what is the set where the does not converge does not converge means there exists an $\varepsilon > 0$ set for every $n$ you have that exists in $m \geq n$ such that $|f_n(x) - f\ (x)| \geq \varepsilon$ this is the definition of not converging to $f\ (x)$, $f_n(x)$ does not convert $f(x)$ means, for every $n$ we can find an $m \geq n$ such that $|f_n(x) - f\ (x)| \geq \varepsilon$. So, this is the so, now, this is nothing but $\bigcup_{\varepsilon>0} limsup_{n\to\infty} E_m(\varepsilon)$. So, this quantity here is the definition of the limsup. So, we have already seen this. )

$$D = \bigcup_{\varepsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} E_m(\varepsilon) = \bigcup_{\varepsilon>0} limsup_{n\to\infty} E_m(\varepsilon).$$

Now you have that since $\mu(D) = 0$ and these are all subsets of that and therefore, this implies

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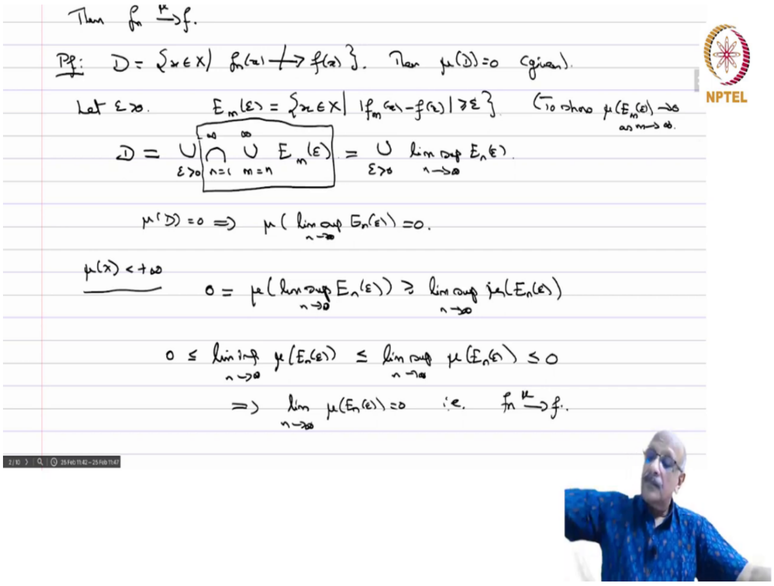

$$\mu(limsup_{n\to\infty} E_m(\varepsilon)) = 0.$$

Now, $\mu(X) < \infty$, (Therefore, everything is a finite measure in the set and therefore, we have given this either done it or we have done it in the exercises)

$$0 = \mu(limsup_{n\to\infty} E_m(\varepsilon)) \geq limsup_{n\to\infty} \mu(E_m(\varepsilon)).$$

So, you have to use the continuity from about to prove these results here and now, you know that

$$0 \leq limsup_{n\to\infty} \mu(E_n(\varepsilon)) \leq limsup_{n\to\infty} \mu(E_n(\varepsilon))) \leq 0$$

$$\Rightarrow \quad lim_{\ n\to\infty} \mu(E_n(\varepsilon)) = 0 \ i.e. f_n \to^{\mu} f.$$

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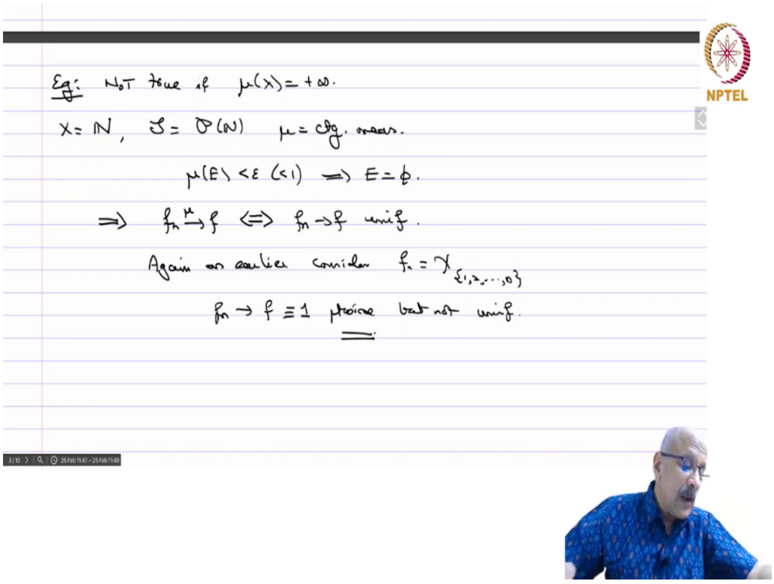


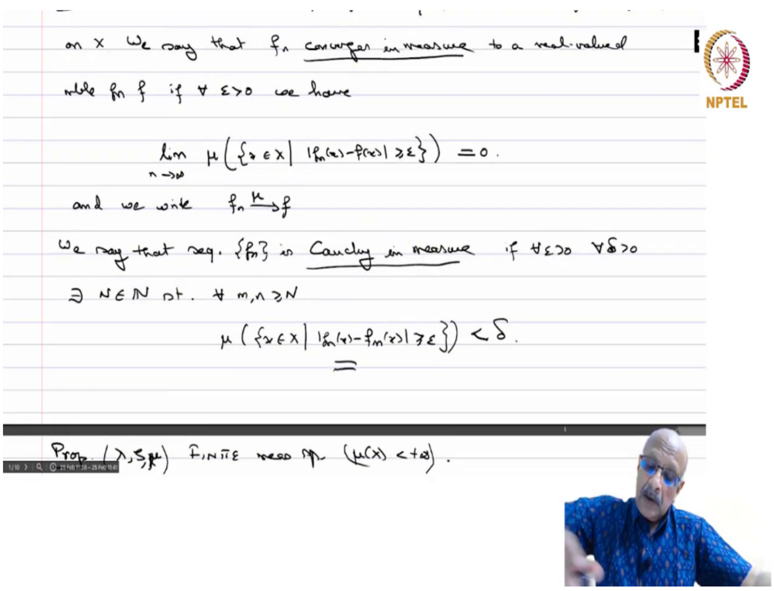


**Example:** Not true if $\mu(X) = +\infty$ so, again you take $X = \mathbb{N}$ , $S = P(\mathbb{N})$, and $\mu = counting\ measure.$

So, $\mu(E) < \varepsilon(< 1) = \phi$ .

(So, this implies convergence in measure when you want to say that this measure goes to 0 this should become empty after some time and that means, everything for every X for all N it should be less than or equal to less than epsilon.)

That means,

$$f_n \xrightarrow{\mu} f \Leftrightarrow f_n \rightarrow f \text{ uniformly.}$$

So, again as earlier, consider

$$f_n = \chi_{\{1,2.........n\}}.$$

Then $f_n \rightarrow f \equiv 1$ pointwise but not uniform so, you have point wise convergence, but that does not imply uniform convergence in the infinite measure space.

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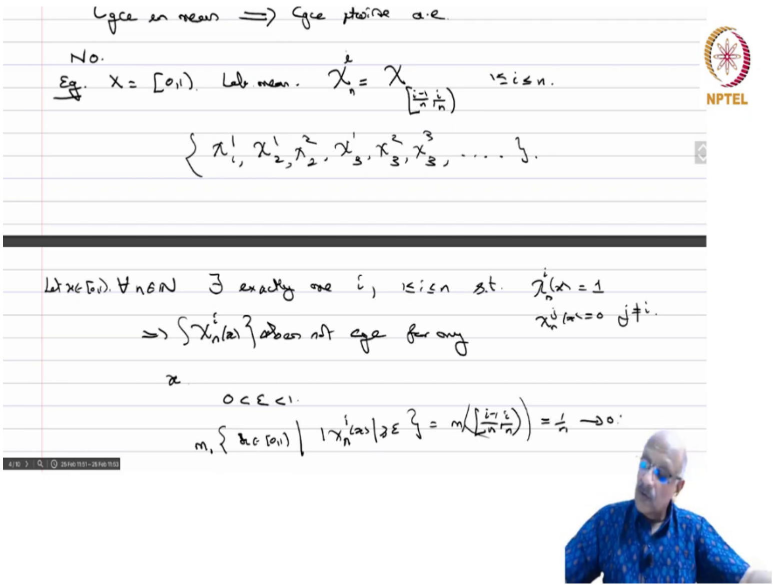


Now, what about convergence in measure? So, convergence in measure does it imply convergence point wise almost everywhere so, again the answer is no. So, we have the following example.

**Example**: Take $X = [0, 1)$ equipped with the Lebesgue measure and then you define

$$\chi_n^{\,i} = \chi_{[\frac{i-1}{n}, \frac{i}{n})}, 1 \leq i \leq n.$$

So, now, you look at the sequence

$$\{\chi_1^{\,1},\ \chi_2^{\,1}, \chi_2^{\,2}, \chi_3^{\,1},\ \ \chi_3^{\,2}\ \ \chi_3^{\,3},..........\}.$$

Let $x \in [0, 1)$. $\forall\ n \in \mathbb{N}$ $\exists$ exactly i, $1 \leq i \leq n$ such that $\chi_n^i(x) = 1$ and $\chi_n^j(x) = 0$ for all $j \neq i$.

(So, that means if you look at this equation you will have a 1 for the first one then one of these will be 1 the other will be 0, then one of these three will be 1 and the rest will be 0 and so on. So, there will be 1 which is popping up all the time.)

So, this implies,

$$\{\chi_n^i(x)\} \text{ does not converge for any } x.$$

So therefore this is you do not have pointwise convergence, but what about convergence in measures?

So, if you take $0 < \varepsilon < 1$ then

$$m_1(\{x \in [0, 1) \mid |\chi_n^i(x)| \geq \varepsilon\}) = m_1([\tfrac{i-1}{n}, \tfrac{i}{n})) = 1/n \to 0.$$

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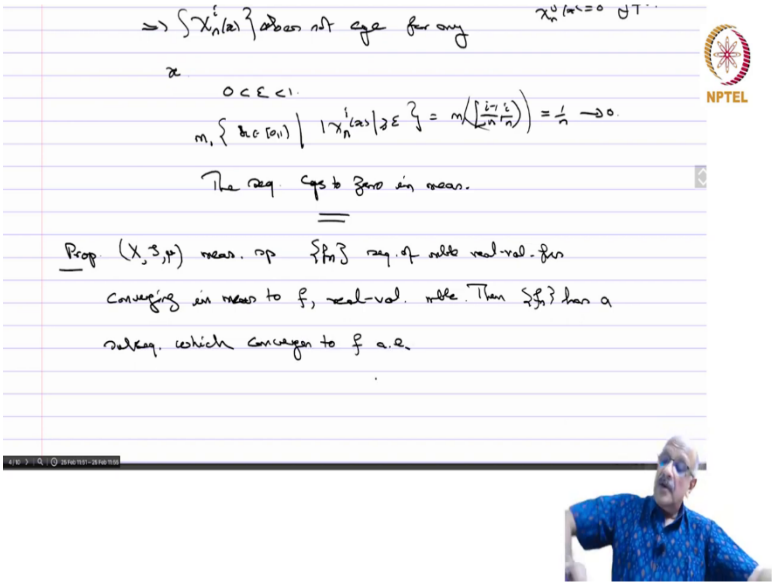


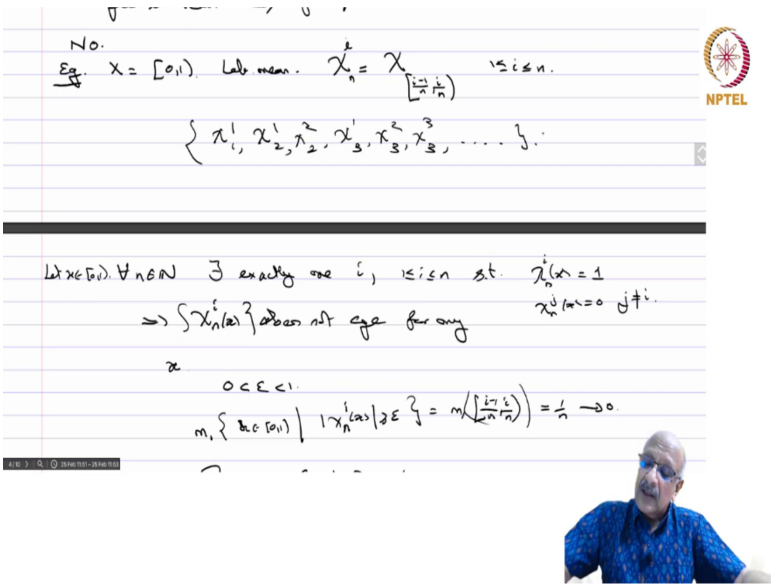


Therefore, the sequence converges to 0 in measure.

( So, it converges in measure, but it does not converge point wise, but if you look at this carefully, you can look at the sequence once again and then there exists a subsequence which you can pick out namely just take 1 from each n 1 the i where it is 1 or if you take the 1 from where it is 0 for each n, then also you will get the subsequence which converges. So, you have a sub sequence which will converge point wise. )

**Proposition:** Let $(X, S, \mu)$ measure spaces $\{f_n\}$ sequence of measurable real-valued function converging in measure to $f$ real valued measurable. Then $f_n$ has a subsequence which converges to $f$ almost everywhere.

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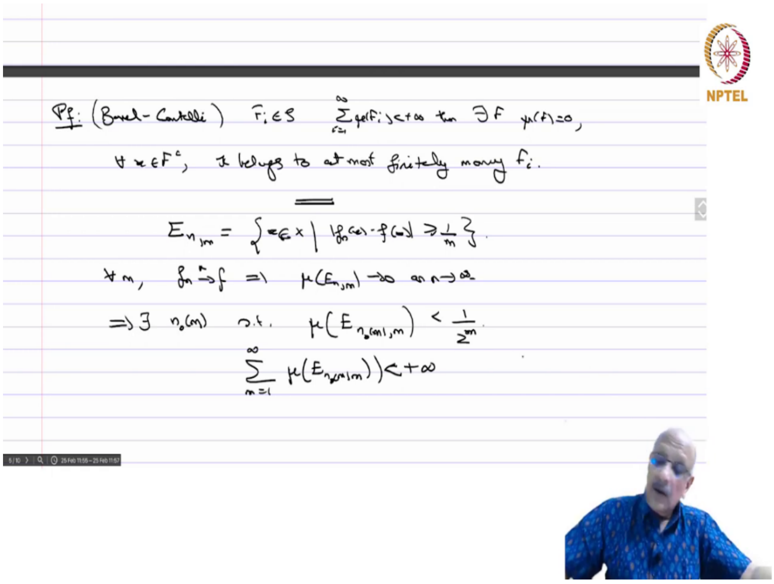


(So, prove we can do this in several ways, and I am going to use now to indicate a proof on the proof later on. So, we are going to use the Borel Catelli Lemma)

**Proof:** $F_i \epsilon S$, $\sum_{i=1}^{\infty} \mu(F_i) < \infty$ then there exists an F, $\mu(F\ ) = 0$ and for all $x \epsilon F^c$, x belongs to at most finitely many $F_i$. This is the Borel Cantelli Lemma.

So, we are going to use this. We have proved this earlier and this is with nice application of that sector of the lemma. There are several nice applications of Boreal Catelli especially in probability theory, which you can say so, therefore, in measure theory also and therefore, this is one such.

So, we are going to define

$$E_{n,m} = \{x \epsilon X |\, |f_n(x) - f\ (x)| \geq 1/m\}.$$

$$\forall\, n,\ f_n \to^{\mu} f \Rightarrow \mu(E_{n,m}) \to \infty\ as\ n \to \infty.$$

$$\Rightarrow \exists\ n_0(m) \text{ such that } \mu(E_{n,m}) < \frac{1}{2^m}.$$

Since $m$ is fixed so I am going to choose in this fashion. So, now if you take

$$\sum_{n=1}^{\infty} \mu(E_{n_0(m),m}) <+ \infty.$$

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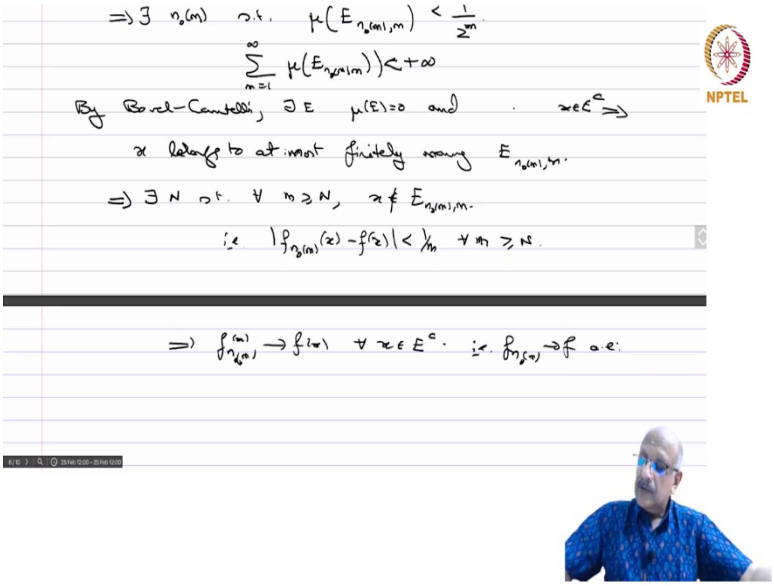

By the Borel Catelli $\exists\, E$ such that $\mu(E\ ) = 0$ and $x \epsilon\, E^c$

$\Rightarrow$ x belongs to utmost finitely many $E_{n_0(m),m}$.

$\Rightarrow\ \exists\, N$ such that $\forall\, m \geq N,\ x \notin E_{n_0(m),m}$.

That is, $|f_{n_0(m)}(x) - f\ (x)| < 1/m \ \ \forall\, m \geq N$

This means that

$$f_{n_0(m)}(x) \to f\ (x)\ \ \forall\ x \epsilon\, E^c.$$

That is $f_{n_0(m)} \to f$ a. e.

So, this completes the proof so, we have a very nice application. So, given any kind of sequence converging in measure you can always find a subsequence which converges pointwise almost everywhere.

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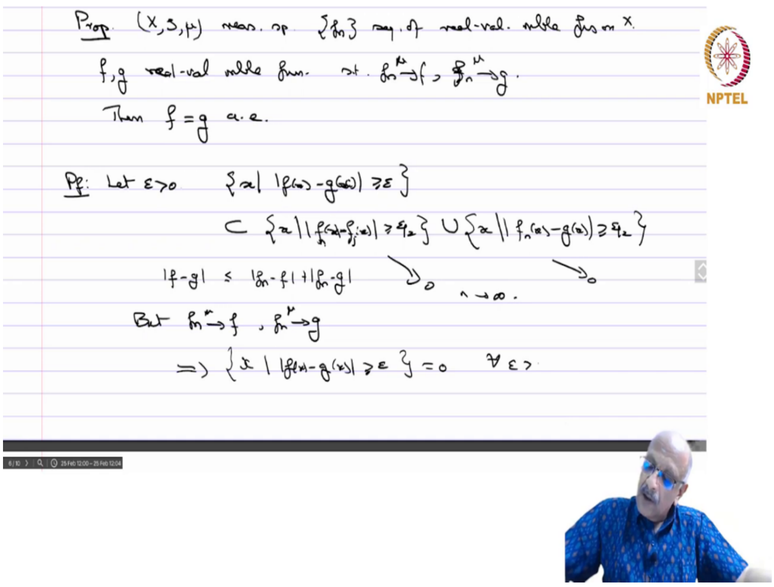


So, finally, before closing this section we will of course, return to this instead we will study much more but for the moment we will the last reason they want to prove in this session is that the limit you see whenever you have a sequence which converges in some sense you want the limit to be unique.

So, this is what we are going to say here now.

**Proposition:** Let $(X, S, \mu)$ measure spaces $\{f_n\}$ sequence of measurable real-valued function on $X$. $f$, g real-valued measurable functions such that $f_n \to^{\mu} f$ , $f_n \to^{\mu} g$. Then

$f = g$ almost everywhere.

(which is as good as unique because when you know what you mean by convergence in measure it means that the limit of the set where the disagrees does not converge has to be 0. So, except on the set of measures 0 these two are equal and therefore, this is as good as saying that the limit is unique. So, you have uniqueness up to equality almost everywhere.)

**Proof**: Let $\varepsilon > 0$. So, you then take a set

$$\{x|\ |f(x) - g(x)| \geq \varepsilon\} \subset \{x|\ |f_n(x) - f(x)| \geq \varepsilon/2\} \cup \{x|\ |f_n(x) - g(x)| \geq \varepsilon/2\}$$

because you have

$$|f - g| \leq |f_n - f| + |f_n - g|.$$

So, if both of these are less than $\varepsilon/2$ then $|f - g|$ will be less than ε and therefore, if you have $|f(x) - g(x)| \geq \varepsilon$ at least one of these two must be true and therefore, you have this contained in this.

But, $f_n \to^{\mu} f$ , $f_n \to^{\mu} g$ and therefore, $|f_n(x) - f(x)|$ to 0 and $|f_n(x) - g(x)|$ also goes to 0 as $n \to \infty$ and this implies that

$$\mu(\{ |f(x) - g(x)| \geq \varepsilon\}) = 0, \ \forall \varepsilon > 0.$$

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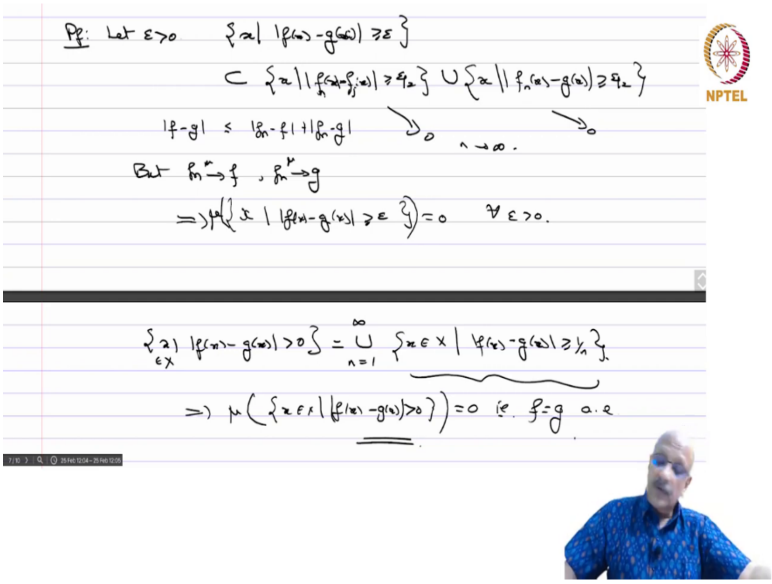


So, the set

$$\{x \epsilon X |\, |f(x) - g(x)| > 0\} = \bigcup_{n=1}^{\infty} \{x \epsilon X |\, |f(x) - g(x)| \geq 1/n\}.$$

$$\Rightarrow \mu(\{x \epsilon X |\, |f(x) - g(x)| > 0\}) = 0$$

i.e. $f = g$ a.e.

So, that proves so, we will continue with convergence and measure next time.