Measure and Integration
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Lecture no-23
4.4 - Egorov's theorem

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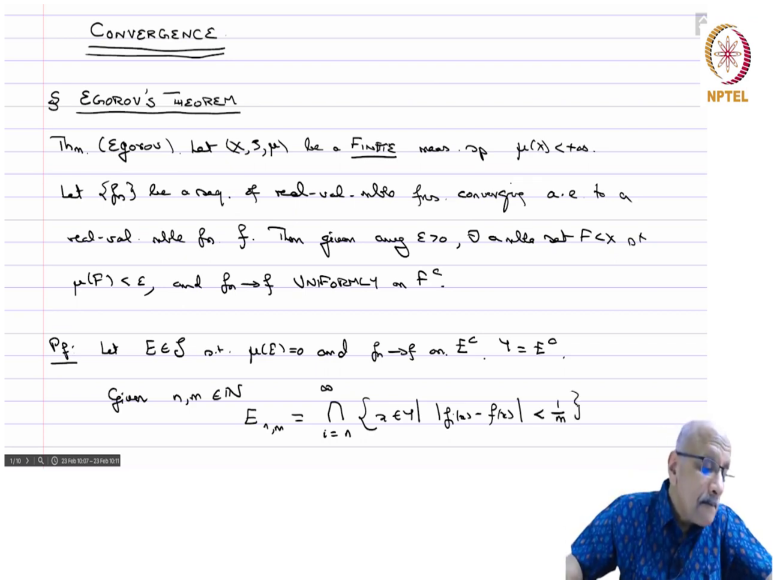

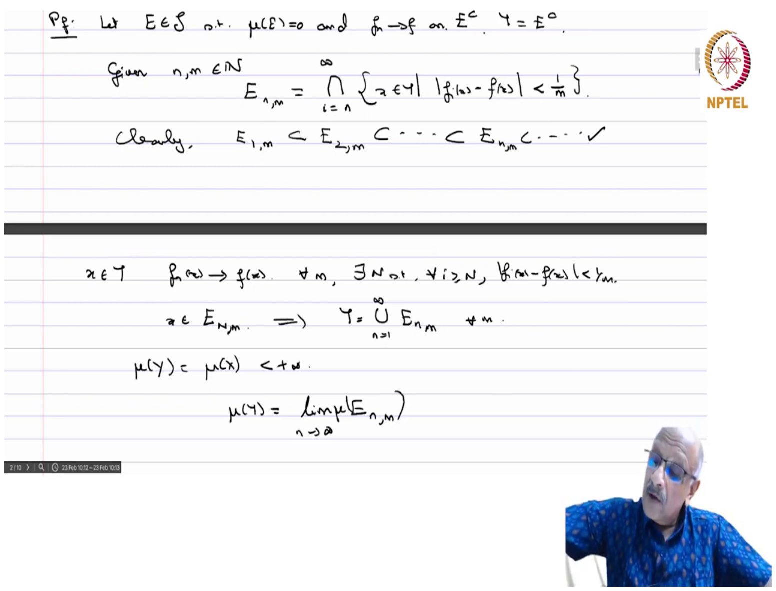

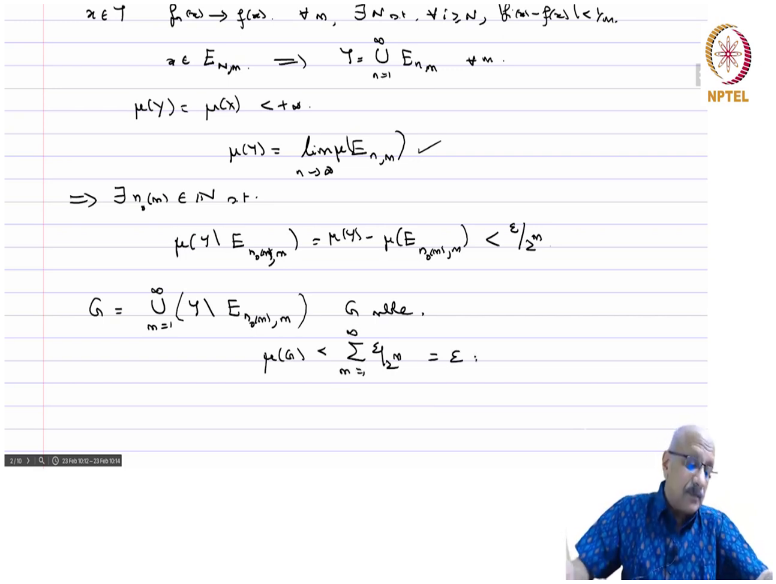

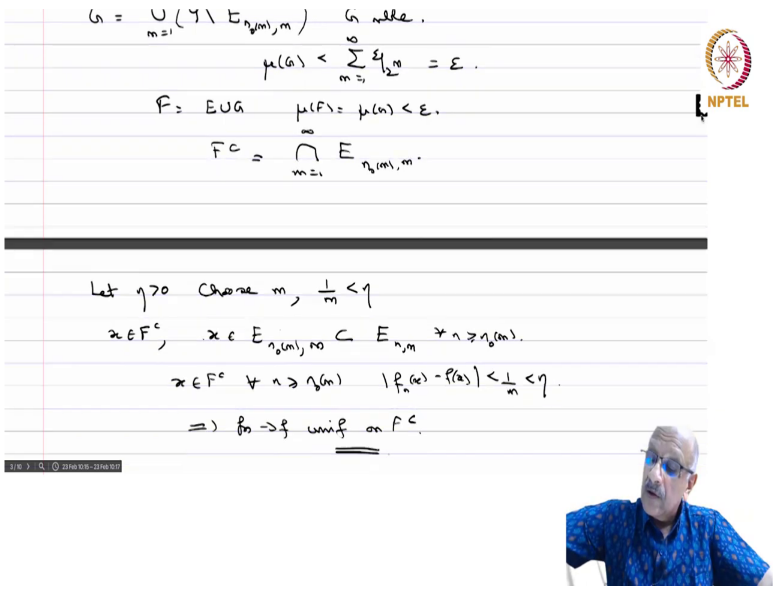

We now do a new chapter where we will study various types of convergence. So, the first section in this is Egorov's theorem:

<u>Egorov's Theorem:</u> Let (X, S, μ) be a finite measure space that means $\mu(X) < \infty$. Let f_n be a sequence of real-valued measurable functions converges almost everywhere to a real valued measurable function f. (So, we saw in the exercises that if μ is not complete, then it is not necessarily that f be measurable. So, we are now specifying that the limit is also a measurement function.)

Then given any $\varepsilon > 0$, $\exists$ a measurable set $F \subset X$ such that $\mu(F) < \varepsilon$ and

$f_n \to f$ uniformly on F^c. (So, we have convergence almost everywhere, then this convergence is practically like uniform convergence except, it may not be everywhere, so, then it is not obvious you cannot have an arbitrary convergence sequence to be uniformly convergent. But what is remarkable is if you are in the finite measure space, then you can choose the smallest set, measure with set with as small measures as you like. So, set on the complement the sequence is in fact uniformly convergent.)

<u>Proof</u>: Let $E\epsilon S$ such that $\mu(E) = 0$ $f_n \to f$ uniformly on E^c. So, you set $Y = E^c$.

So, then given $n, m\epsilon\mathbb{N}$. You define

$$E_{n,m} = \bigcap_{i=n}^{\infty} \{x\epsilon Y |\, |f_i(x) - f(x)| < 1/n\}.$$

Clearly, $E_{1,m} \subset E_{2,m} \subset ... \subset E_{n,m} \subset ......$ $\forall m$.

If $x\epsilon Y$, then $f_n(x) \to f(x)$ and $\forall\, m$, $\exists$ N such that for all $i \geq N$, we have

$$|f_i(x) - f(x)| < 1/m.$$

That means, $x\epsilon E_{n,m} \Rightarrow Y = \bigcup_{n=1}^{\infty} E_{n,m}$ $\forall m$.

So, $\mu(Y) = \mu(X) <+ \infty$.

(Because Y equals Y union E is the whole $f(X)$ and E has measure 0.)

Now, therefore, you have

$$\mu(Y) = lim_{n\to\infty} \mu(E_{n,m}).$$

(And because of the finiteness you have)

$$\Rightarrow \exists\, n_0(m) \in \mathbb{N} \text{ such that}$$

$$\mu(Y\backslash E_{n_0(m),m}) = \mu(Y) - \mu(E_{n_0(m),m}) < \varepsilon/2^m.$$

Now, you say $G = \bigcup_{n=1}^{\infty} (Y\backslash E_{n_0(m),m})$, then G measurable.

And you have

$$\mu(G) < \sum_{m=1}^{\infty} \varepsilon/2^m = \varepsilon.$$

Now, you say $F = E \cup G,\ \ \mu(F) = \mu(G) < \varepsilon$.

And what is F^c? F^c is precisely

$$F^c = \bigcap_{m=1}^{\infty} E_{n_0(m),m}.$$

Let $\eta > 0$, choose m, says that $1/m < \eta$.

So, now, if $x \epsilon F^c$, therefore $x \epsilon E_{n_0(m),m} \subset E_{n,m}\ \forall\, n \geq n_0(m)$.

So, if $x \epsilon F^c$, $\forall\, n \geq n_0(m)$, you have

$$|f_i(x) - f(x)| < 1/m < \eta.$$

So, this is true $\forall\ x \epsilon F^c$, and therefore, you have

$$f_n \rightarrow f \text{ uniformly on } F^c \text{ and } \mu(F) < \varepsilon,$$

and therefore, that completes the proof of Egorov's theorem.

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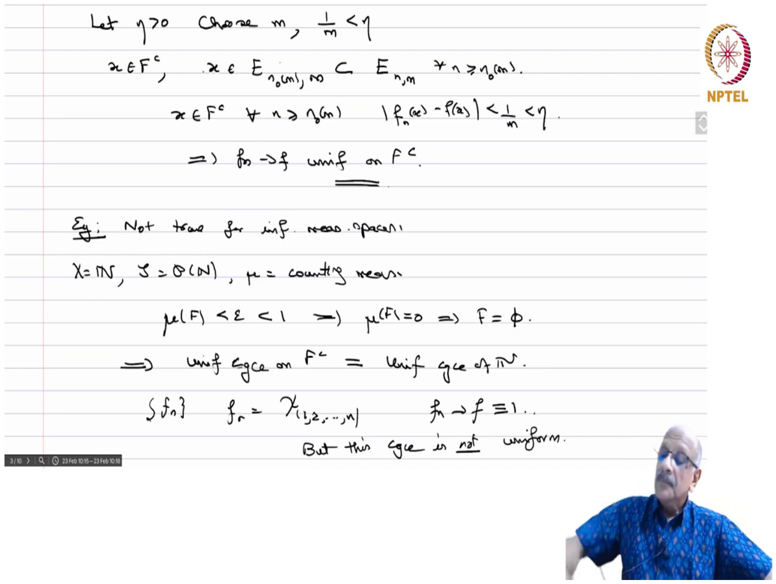

Example: Not true for infinite measure spaces. So, let us take

$X = \mathbb{N}, X = P(\mathbb{N})$ and $\mu =$ counting measures.

$$\mu(F) < \varepsilon < 1 \Rightarrow \mu(F) = 0 \Rightarrow F = \phi$$

Therefore, uniform convergence on F^c is the same as uniform convergence on $\mathbb{N}$. So, now, we will look at $\{f_n\}$, $f_n = \chi_{\{1,2,\dots,n\}}$.

Then $f_n \to f \equiv 1$. But this convergence is not uniform. So, Egorov's theorem fails in the case of infinite measures.

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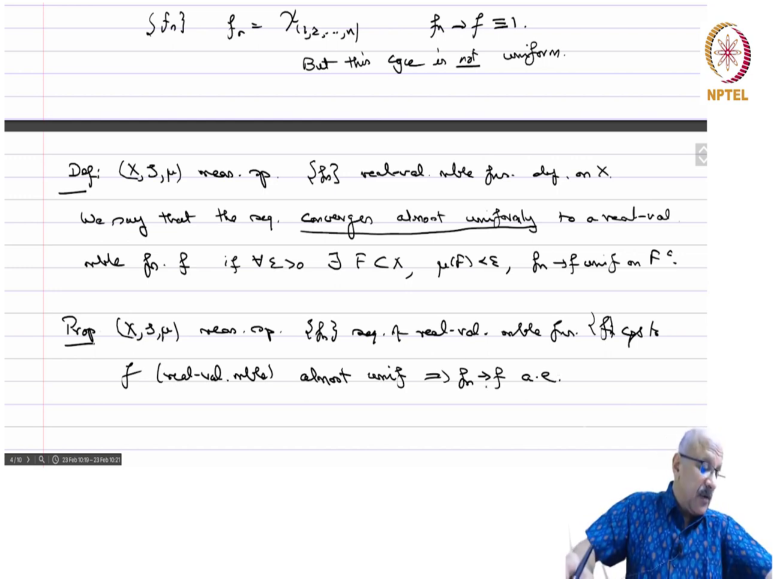

So, inspired by Egorov's theorem we make the following:

Definition: Let (X, S, μ) measure space $\{f_n\}$ real-valued measurable functions defined on X. We say that the sequence **converges almost uniformly** (here you must, this almost is not in the sense of what we have been saying before about almost everywhere) to real-valued measurable function f if $\varepsilon > 0\ \exists\ F \subset X,\ \mu(F) < \varepsilon$ and $f_n \to f$ uniformly on F^c.

So, what does Egorov's theorem say that if you have a finite measure space convergence almost everywhere is the same as convergence almost uniformly.

Converse of Egorov's theorem is true in any measure space. So, if you have almost uniform convergence then you have convergence almost everywhere.

Proposition: Let (X, S, μ) measure space $\{f_n\}$ real-valued measurable functions defined on X. $f_n \to f$ (real valued measurable function) almost uniformly $\Rightarrow$ $f_n \to f$ almost everywhere.

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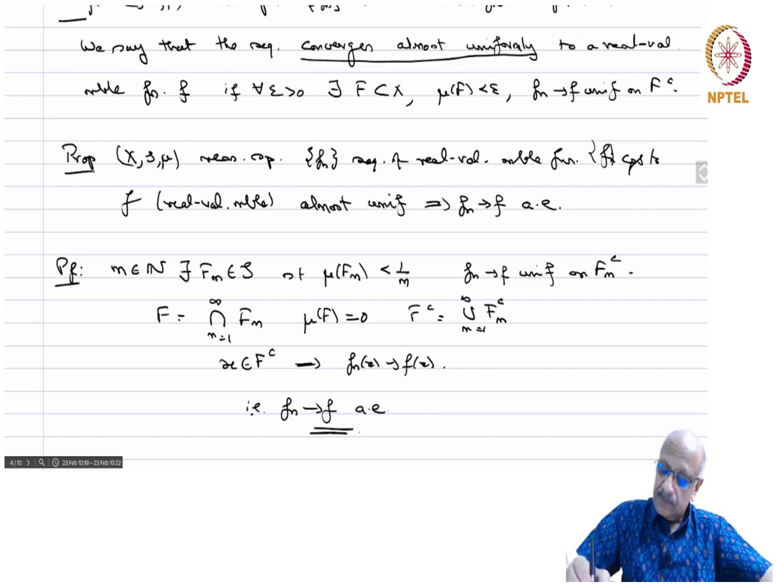

Proof: Let $m \epsilon \mathbb{N}$. You chose $F_m \epsilon S$ such that $\mu(F_m) < 1/m$ and $f_n \to f$ uniformly on $F_m^{\ c}$.

$$\text{So, } F = \bigcap_{n=1}^{\infty} F_m,\ \mu(F) = 0,$$

(because it is less than $\mu(F_m) < 1/m$. So, $\mu(F_m) < 1/m$ for each m. So, $\mu(F) = 0$.)

$$\text{and } F^{\ c} = \bigcup_{m=1}^{\infty} F_m^{\ c}.$$

And in any of these sets, so if $x \epsilon F^{\ c}$ It belongs to some $F_m^{\ c}$ and this implies

$$f_n(x) \to f(x)$$

that is, $f_n \to f$ a. e.

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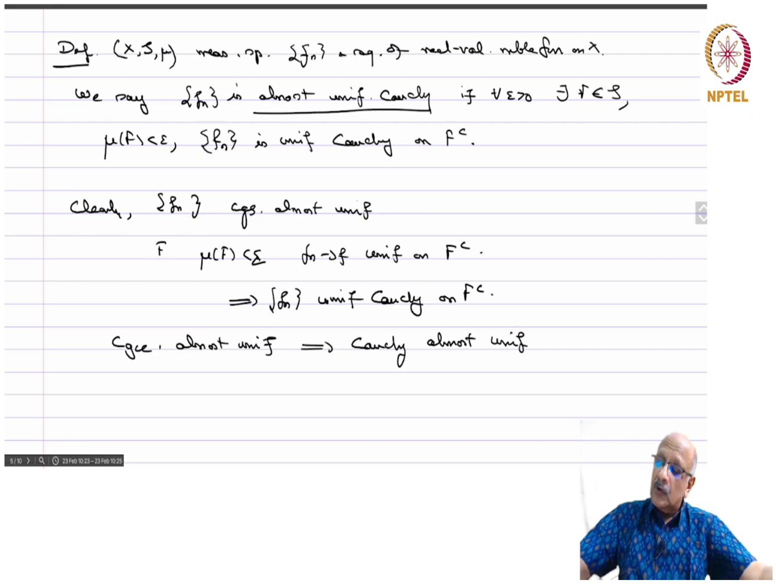

So, whenever you have a notion of convergence you also have associated notion of cauchy and then usually convergence will always implies cauchy and one is always interested in knowing whether cauchy implies convergence that is like important thing which we had the completeness of the reals in the complex numbers namely if the cauchy sequence has a limit.

So, the same idea we will always see whenever we introduce a notion of convergence, we will have an associated notion of cauchyness and then convergence should imply cauchy and cauchy should imply convergence.

Definition: Let (X, S, μ) measure space $\{f_n\}$ real-valued measurable functions defined on X.

We say $\{f_n\}$ is almost uniformly cauchy if $\forall\ \varepsilon > 0, \exists\ F \in S, \mu(F) < \varepsilon$, $\{f_n\}$ is uniformly cauchy on F^c.

Clearly if $\{f_n\}$ converges almost uniformly then there exists an F such that $\mu(F) < \varepsilon$, f_n converges to f uniformly on F^c.

$\Rightarrow$ $\{f_n\}$ uniformly cauchy on F^c.

So, convergence almost uniformly $\Leftrightarrow$ Cauchy is almost uniform.

So, we want to know about the converse. Suppose you have a sequence which is cauchy almost uniformly, is it going to be convergent almost uniformly?

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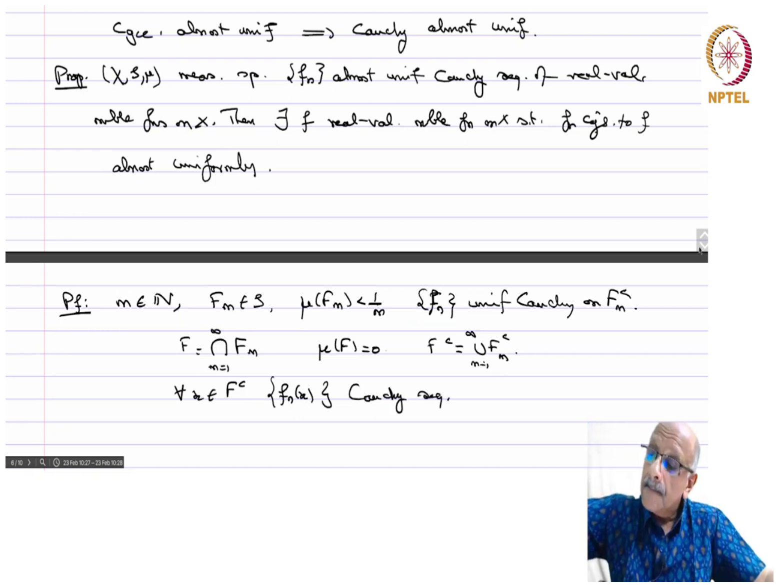

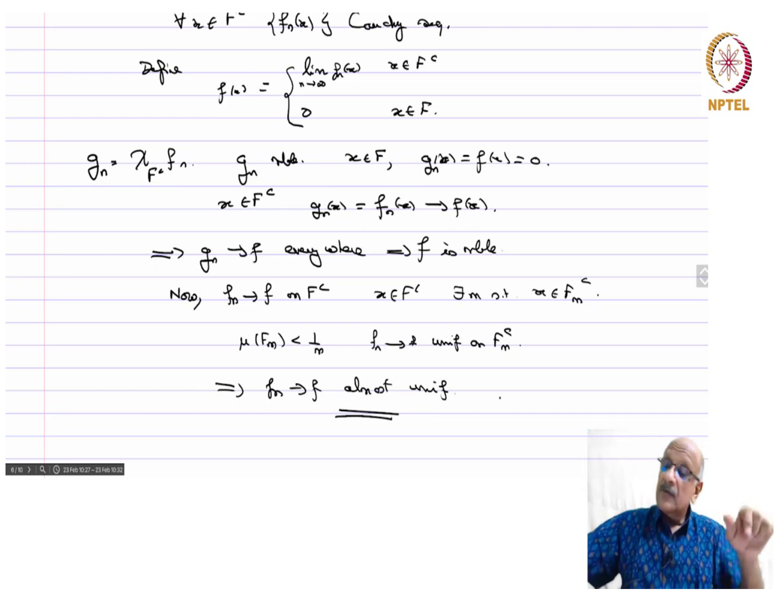

Proposition. Let (X, S, μ) measure space $\{f_n\}$ almost uniformly cauchy sequence of real valued measurable functions. Then there exists f real-value measurable function on X such that $f_n \to f$ almost uniformly.

Proof: Let $m \in \mathbb{N}$There exists $F_m \in S$, $\mu(F_m) < 1/m$, $\{f_n\}$ almost uniformly cauchy on F_m^c.

So, you said $F = \bigcap_{n=1}^{\infty} F_m,\ \mu(F\ ) = 0,\ F^c = \bigcup_{n=1}^{\infty} F_m$ ……..

$\forall\ x\epsilon F^c,\ \{f_n(x)\}$uniformly cauchy.

Define

$$f(x) = \{ {}_{0\ \ if\ \ x\epsilon F}^{lim_{n\to\infty}f_n(x)\ \ if\ \ x\epsilon F^c} \ .$$

Now, you define

$$g_n = \chi_{F^c} f_n$$

so, g_n is measurable.

If $x\epsilon F$, you have $g_n(x) = f(x) = 0$.

If $x\epsilon F^c$, $g_n(x) = f_n(x) = f(x)$.

$\Rightarrow g_n \to f$ everywhere $\Rightarrow f$ is measurable.

Now, $f_n \to f$ on F^c, $x\epsilon F^c$ $\exists\ m$ such that $x\epsilon F^c_m$.

$\mu(F_m\ )$<1/m, $f_n \to f$ on F^c_m.

$\Rightarrow f_n \to f$ almost uniformly.

So, given any η you can choose m large enough such that it is less than $\frac{1}{m}$ is less than η, and then therefore, you will have that this belongs to it converges uniformly on the set. Therefore, this implies that $\Rightarrow f_n \to f$ almost uniformly. So, this completes about almost uniform convergence and almost cauchyness. Next time we will take up another notion of convergence and examine its properties.