


**Measure and Integration**  
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**Lecture no-23**  
**4.4 - Egorov's theorem**

(Refer Slide Time: 00:16)

CONVERGENCE



S EGOV'S THEOREM


Thm. (Egorov). Let  $(X, \mathcal{S}, \mu)$  be a FINITE meas. sp.  $\mu(X) < +\infty$ .

Let  $\{f_n\}$  be a seq. of real-val. mds. fns. converging a.e. to a real-val. mds. fn.  $f$ . Then given any  $\epsilon > 0$ ,  $\exists$  a mds. set  $F \subset X$  s.t.  $\mu(F^c) < \epsilon$ , and  $f_n \rightarrow f$  UNIFORMLY on  $F^c$ .

Pf. Let  $E \in \mathcal{S}$  s.t.  $\mu(E) = 0$  and  $f_n \rightarrow f$  on  $E^c$ .  $Y = E^c$ .

Given  $n, m \in \mathbb{N}$

$$E_{n,m} = \bigcap_{i=n}^{\infty} \left\{ x \in Y \mid |f_i(x) - f(x)| < \frac{1}{m} \right\}$$





Pf. Let  $E \in \mathcal{S}$  s.t.  $\mu(E) = 0$  and  $f_n \rightarrow f$  on  $E^c$ .  $Y = E^c$ .

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$$E_{n,m} = \bigcap_{i=n}^{\infty} \left\{ x \in Y \mid |f_i(x) - f(x)| < \frac{1}{m} \right\}$$


Clearly,  $E_{1,m} \subset E_{2,m} \subset \dots \subset E_{n,m} \subset \dots$  ✓

$x \in Y \implies f_n(x) \rightarrow f(x) \quad \forall n, \exists N \text{ s.t. } \forall i \geq N, |f_i(x) - f(x)| < \frac{1}{m}$

$x \in E_{N,m} \implies Y = \bigcup_{n=1}^{\infty} E_{n,m} \quad \forall m$


$\mu(Y) \subset \mu(X) < +\infty$ .

$$\mu(Y) = \lim_{n \rightarrow \infty} \mu(E_{n,m})$$







$x \in Y \quad f_n(x) \rightarrow f(x) \quad \forall n, \exists N \geq 1, \forall i \geq N, |f_n(x) - f(x)| < \epsilon_n$   
 $x \in E_{n,m} \Rightarrow Y = \bigcup_{n=1}^{\infty} E_{n,m} \quad \forall m$   
 $\mu(Y) = \mu(X) < +\infty$   
 $\mu(Y) = \lim_{n \rightarrow \infty} \mu(E_{n,m}) \quad \checkmark$   
 $\Rightarrow \exists n_0(m) \in \mathbb{N} \quad \forall n > n_0(m)$   
 $\mu(Y \setminus E_{n_0(m),m}) = \mu(Y) - \mu(E_{n_0(m),m}) < \epsilon/2^m$   
 $G = \bigcup_{m=1}^{\infty} (Y \setminus E_{n_0(m),m}) \quad G \text{ null set.}$   
 $\mu(G) < \sum_{m=1}^{\infty} \epsilon/2^m = \epsilon$




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 $\mu(G) < \sum_{m=1}^{\infty} \epsilon/2^m = \epsilon$   
 $F = E \setminus G \quad \mu(F) = \mu(X) < \epsilon$   
 $F^c = \bigcap_{m=1}^{\infty} E_{n_0(m),m}$



Let  $\eta > 0$  choose  $m, \frac{1}{m} < \eta$   
 $x \in F^c, \quad x \in E_{n_0(m),m} \subset E_{n,m} \quad \forall n \geq n_0(m)$   
 $x \in F^c \quad \forall n \geq n_0(m) \quad |f_n(x) - f(x)| < \frac{1}{m} < \eta$   
 $\Rightarrow f_n \rightarrow f$  unif on  $F^c$




We now do a new chapter where we will study various types of convergence. So, the first section in this is Egorov's theorem:

**Egorov's Theorem:** Let  $(X, S, \mu)$  be a finite measure space that means  $\mu(X) < \infty$ . Let  $f_n$  be a sequence of real-valued measurable functions converges almost everywhere to a real valued measurable function  $f$ . (So, we saw in the exercises that if  $\mu$  is not complete, then it is not necessarily that  $f$  be measurable. So, we are now specifying that the limit is also a measurement function.)

Then given any  $\epsilon > 0$ ,  $\exists$  a measurable set  $F \subset X$  such that  $\mu(F) < \epsilon$  and

$f_n \rightarrow f$  uniformly on  $F^c$ . (So, we have convergence almost everywhere, then this convergence is practically like uniform convergence except, it may not be everywhere, so, then it is not obvious you cannot have an arbitrary convergence sequence to be uniformly convergent. But what is remarkable is if you are in the finite measure space, then you can choose the smallest set, measure with set with as small measures as you like. So, set on the complement the sequence is in fact uniformly convergent.)

**Proof:** Let  $E \in \mathcal{S}$  such that  $\mu(E) = 0$   $f_n \rightarrow f$  uniformly on  $E^c$ . So, you set  $Y = E^c$ .

So, then given  $n, m \in \mathbb{N}$ . You define

$$E_{n,m} = \bigcap_{i=n}^{\infty} \{x \in Y \mid |f_i(x) - f(x)| < 1/n\}.$$

$$\text{Clearly, } E_{1,m} \subset E_{2,m} \subset \dots \subset E_{n,m} \subset \dots \quad \forall m.$$

If  $x \in Y$ , then  $f_n(x) \rightarrow f(x)$  and  $\forall m, \exists N$  such that for all  $i \geq N$ , we have

$$|f_i(x) - f(x)| < 1/m.$$

$$\text{That means, } x \in E_{n,m} \Rightarrow Y = \bigcup_{n=1}^{\infty} E_{n,m} \quad \forall m.$$

$$\text{So, } \mu(Y) = \mu(X) < +\infty.$$

(Because  $Y$  equals  $Y$  union  $E$  is the whole  $f(X)$  and  $E$  has measure 0.)

Now, therefore, you have

$$\mu(Y) = \lim_{n \rightarrow \infty} \mu(E_{n,m}).$$

(And because of the finiteness you have)

$$\Rightarrow \exists n_0(m) \in \mathbb{N} \text{ such that}$$

$$\mu(Y \setminus E_{n_0(m),m}) = \mu(Y) - \mu(E_{n_0(m),m}) < \varepsilon/2^m.$$

Now, you say  $G = \bigcup_{n=1}^{\infty} (Y \setminus E_{n_0(m),m})$ , then  $G$  measurable.

And you have

$$\mu(G) < \sum_{m=1}^{\infty} \varepsilon/2^m = \varepsilon.$$

Now, you say  $F = E \cup G$ ,  $\mu(F) = \mu(G) < \varepsilon$ .

And what is  $F^c$ ?  $F^c$  is precisely

$$F^c = \bigcap_{m=1}^{\infty} E_{n_0(m),m}.$$

Let  $\eta > 0$ , choose  $m$ , says that  $1/m < \eta$ .

So, now, if  $x \in F^c$ , therefore  $x \in E_{n_0(m),m} \subset E_{n,m} \forall n \geq n_0(m)$ .

So, if  $x \in F^c$ ,  $\forall n \geq n_0(m)$ , you have

$$|f_n(x) - f(x)| < 1/m < \eta.$$

So, this is true  $\forall x \in F^c$ , and therefore, you have



$$f_n \rightarrow f \text{ uniformly on } F^c \text{ and } \mu(F) < \varepsilon,$$

and therefore, that completes the proof of Egorov's theorem.

(Refer Slide Time: 09:30)

Let  $\eta > 0$  Choose  $m, \frac{1}{m} < \eta$   
 $x \in F^c, \dots x \in E_{\eta, m}, \dots \subset E_{\eta, m} \text{ for } n \geq \eta, m$ .  
 $x \in F^c \text{ for } n \geq \eta, m \implies |f_n(x) - f(x)| < \frac{1}{m} < \eta$ .  
 $\implies f_n \rightarrow f$  unif on  $F^c$ .

Ex: Not true for inf. meas. spaces.  
 $X = \mathbb{N}, \mathcal{S} = \mathcal{P}(\mathbb{N}), \mu = \text{counting measure}$   
 $\mu(F) < \varepsilon < 1 \implies \mu(F) = 0 \implies F = \emptyset$ .  
 $\implies$  unif. conv. on  $F^c = \mathbb{N}$ .  
 $\{f_n\} f_n = \chi_{\{1, 2, \dots, n\}} \implies f_n \rightarrow f \equiv 1$ .  
 But this conv. is not uniform.

**Example:** Not true for infinite measure spaces. So, let us take

$X = \mathbb{N}, X = P(\mathbb{N})$  and  $\mu = \text{counting measures}$ .

$$\mu(F) < \varepsilon < 1 \implies \mu(F) = 0 \implies F = \emptyset$$

Therefore, uniform convergence on  $F^c$  is the same as uniform convergence on  $\mathbb{N}$ . So, now, we will look at  $\{f_n\}, f_n = \chi_{\{1, 2, \dots, n\}}$ .



Then  $f_n \rightarrow f \equiv 1$ . But this convergence is not uniform. So, Egorov's theorem fails in the case of infinite measures.

(Refer Slide Time: 11:16)

$\{f_n\} f_n = \chi_{(1/2, \dots, n)} \quad f_n \rightarrow f \equiv 1.$   
But this case is not uniform.

**Def:**  $(X, S, \mu)$  meas. sp.  $\{f_n\}$  real-val. mble fun. def. on  $X$ .  
We say that the seq. converges almost uniformly to a real-val. mble fun.  $f$  if  $\forall \epsilon > 0 \exists F \subset X, \mu(F) < \epsilon, f_n \rightarrow f$  unif on  $F^c$ .

**Prop**  $(X, S, \mu)$  meas. sp.  $\{f_n\}$  seq. of real-val. mble fun.  $f$  is the  $f$  (real-val. mble) almost unif  $\Rightarrow f_n \rightarrow f$  a.e.



So, inspired by Egorov's theorem we make the following:

**Definition:** Let  $(X, S, \mu)$  measure space  $\{f_n\}$  real-valued measurable functions defined on  $X$ .

We say that the sequence **converges almost uniformly** (here you must, this almost is not in the sense of what we have been saying before about almost everywhere) to real-valued measurable function  $f$  if  $\epsilon > 0 \exists F \subset X, \mu(F) < \epsilon$  and  $f_n \rightarrow f$  uniformly on  $F^c$ .

So, what does Egorov's theorem say that if you have a finite measure space convergence almost everywhere is the same as convergence almost uniformly.

Converse of Egorov's theorem is true in any measure space. So, if you have almost uniform convergence then you have convergence almost everywhere.

**Proposition:** Let  $(X, S, \mu)$  measure space  $\{f_n\}$  real-valued measurable functions defined on  $X$ .

$f_n \rightarrow f$  (real valued measurable function) almost uniformly  $\Rightarrow f_n \rightarrow f$  almost everywhere.

(Refer Slide Time: 14:00)

We say that the seq. converges almost uniformly to a real-val. mble  $f_0, f$  if  $\forall \epsilon > 0 \exists F \subset X, \mu(F) < \epsilon, f_n \rightarrow f$  unif on  $F^c$ .



Prop  $(X, \mathcal{B}, \mu)$  meas. sp. if  $\{f_n\}$  seq. of real-val. mble fun.  $\{f\}$  sp. to  $f$  (real-val. mble) almost unif  $\Rightarrow f_n \rightarrow f$  a.e.

Pf:  $m \in \mathbb{N} \exists F_m \in \mathcal{S}$  st  $\mu(F_m) < \frac{1}{m}$   $f_n \rightarrow f$  unif on  $F_m^c$ .

$$F = \bigcap_{n=1}^{\infty} F_n \quad \mu(F) = 0 \quad F^c = \bigcup_{n=1}^{\infty} F_n^c$$

$$x \in F^c \rightarrow f_n(x) \rightarrow f(x).$$

$\therefore f_n \rightarrow f$  a.e.

**Proof:** Let  $m \in \mathbb{N}$ . You chose  $F_m \in \mathcal{S}$  such that  $\mu(F_m) < 1/m$  and  $f_n \rightarrow f$  uniformly on  $F_m^c$ .

$$\text{So, } F = \bigcap_{n=1}^{\infty} F_n, \mu(F) = 0,$$

(because it is less than  $\mu(F_m) < 1/m$ . So,  $\mu(F_m) < 1/m$  for each  $m$ . So,  $\mu(F) = 0$ .)

$$\text{and } F^c = \bigcup_{m=1}^{\infty} F_m^c.$$

And in any of these sets, so if  $x \in F^c$  It belongs to some  $F_m^c$  and this implies

$$f_n(x) \rightarrow f(x)$$

that is,  $f_n \rightarrow f$  a. e.

(Refer Slide Time: 15:13)

Def.  $(X, S, \mu)$  meas. sp.  $\{f_n\}$  a seq. of real-val. measurable on  $X$ .  
We say  $\{f_n\}$  is almost unif. Cauchy if  $\forall \epsilon > 0 \exists F \in S$ ,  
 $\mu(F^c) < \epsilon$ ,  $\{f_n\}$  is unif. Cauchy on  $F^c$ .

Clearly,  $\{f_n\}$  conv. almost unif.  
 $\Rightarrow \{f_n\}$  unif. Cauchy on  $F^c$ .  
 $\Rightarrow \{f_n\}$  unif. Cauchy on  $F^c$ .  
Conv. almost unif.  $\Rightarrow$  Cauchy almost unif.

NPTEL

So, whenever you have a notion of convergence you also have associated notion of cauchy and then usually convergence will always implies cauchy and one is always interested in knowing whether cauchy implies convergence that is like important thing which we had the completeness of the reals in the complex numbers namely if the cauchy sequence has a limit.

So, the same idea we will always see whenever we introduce a notion of convergence, we will have an associated notion of cauchyness and then convergence should imply cauchy and cauchy should imply convergence.

**Definition:** Let  $(X, S, \mu)$  measure space  $\{f_n\}$  real-valued measurable functions defined on  $X$ .

We say  $\{f_n\}$  is almost uniformly cauchy if  $\forall \epsilon > 0, \exists F \in S, \mu(F^c) < \epsilon, \{f_n\}$  is uniformly cauchy on  $F^c$ .

Clearly if  $\{f_n\}$  converges almost uniformly then there exists an  $F$  such that  $\mu(F^c) < \epsilon, f_n$  converges to  $f$  uniformly on  $F^c$ .

$\Rightarrow \{f_n\}$  uniformly cauchy on  $F^c$ .

So, convergence almost uniformly  $\Leftrightarrow$  Cauchy is almost uniform.

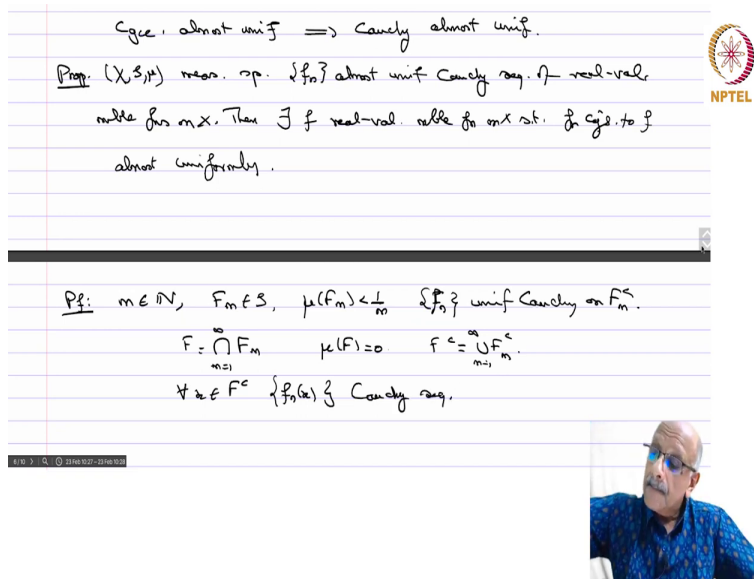


So, we want to know about the converse. Suppose you have a sequence which is cauchy almost uniformly, is it going to be convergent almost uniformly?

(Refer Slide Time: 18:17)

$\text{Cgt. : almost unif} \implies \text{Cauchy almost unif.}$   
Prop.  $(X, S, \mu)$  meas. sp.  $\{f_n\}$  almost unif Cauchy seq. of real-val. rble fns on  $X$ . Then  $\exists f$  real-val. rble fn on  $X$  st.  $f_n \text{ cgt. to } f$  almost uniformly.

Pf:  $m \in \mathbb{N}, F_m \in S, \mu(F_m) < \frac{1}{m}, \{f_n\}$  unif Cauchy on  $F_m^c$ .  
 $F = \bigcap_{m=1}^{\infty} F_m, \mu(F) = 0, F^c = \bigcup_{m=1}^{\infty} F_m^c$ .  
 $\forall x \in F^c, \{f_n(x)\}$  Cauchy seq.



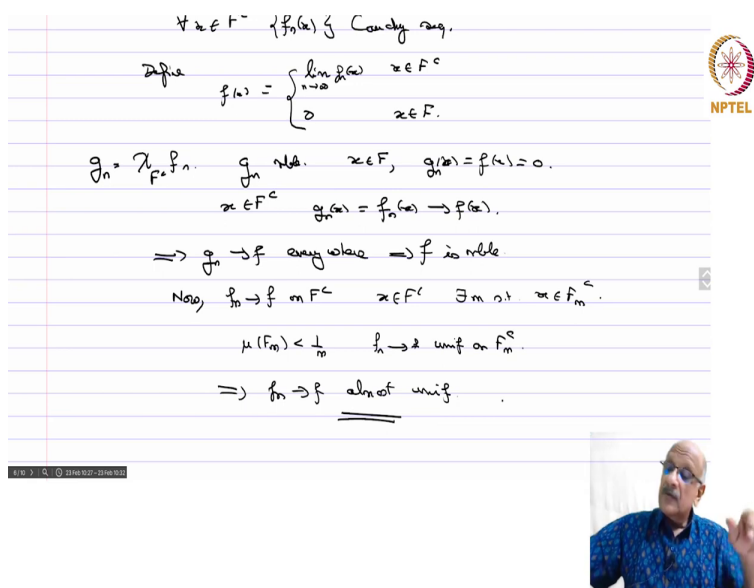
$\forall x \in F^c, \{f_n(x)\}$  Cauchy seq.

Define  $f(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & x \in F^c \\ 0 & x \in F. \end{cases}$

$g_n = \chi_{F_m^c} f_n, g_n$  rble.  $x \in F, g_n(x) = f(x) = 0$ .  
 $x \in F^c, g_n(x) = f_n(x) \rightarrow f(x)$ .

$\implies g_n \rightarrow f$  every where  $\implies f$  is rble.

Now,  $f_n \rightarrow f$  on  $F^c, x \in F^c, \exists m > 1/x \in F_m^c$ .  
 $\mu(F_m) < \frac{1}{m}, f_n \rightarrow f$  unif on  $F_m^c$ .  
 $\implies f_n \rightarrow f$  almost unif.



**Proposition.** Let  $(X, S, \mu)$  measure space  $\{f_n\}$  almost uniformly cauchy sequence of real valued measurable functions. Then there exists  $f$  real-value measurable function on  $X$  such that  $f_n \rightarrow f$  almost uniformly.

**Proof:** Let  $m \in \mathbb{N}$  There exists  $F_m \in S, \mu(F_m) < 1/m, \{f_n\}$  almost uniformly cauchy on  $F_m^c$ .

So, you said  $F = \bigcap_{n=1}^{\infty} F_n$ ,  $\mu(F) = 0$ ,  $F^c = \bigcup_{n=1}^{\infty} F_n^c$  .....

$\forall x \in F^c$ ,  $\{f_n(x)\}$  uniformly cauchy.

Define

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} f_n(x) & \text{if } x \in F^c \\ 0 & \text{if } x \in F \end{cases}$$

Now, you define

$$g_n = \chi_{F^c} f_n$$

so,  $g_n$  is measurable.

If  $x \in F$ , you have  $g_n(x) = f(x) = 0$ .

If  $x \in F^c$ ,  $g_n(x) = f_n(x) = f(x)$ .

$\Rightarrow g_n \rightarrow f$  everywhere  $\Rightarrow f$  is measurable.

Now,  $f_n \rightarrow f$  on  $F^c$ ,  $x \in F^c \exists m$  such that  $x \in F_m^c$ .

$$\mu(F_m^c) < 1/m, f_n \rightarrow f \text{ on } F_m^c$$

$\Rightarrow f_n \rightarrow f$  almost uniformly.

So, given any  $\eta$  you can choose  $m$  large enough such that  $\frac{1}{m}$  is less than  $\eta$ , and then therefore, you will have that this belongs to it converges uniformly on the set. Therefore, this implies that  $\Rightarrow f_n \rightarrow f$  almost uniformly. So, this completes about almost uniform convergence and almost cauchyness. Next time we will take up another notion of convergence and examine its properties.