Measure and Integration
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Lecture 22
4.3 – Exercises

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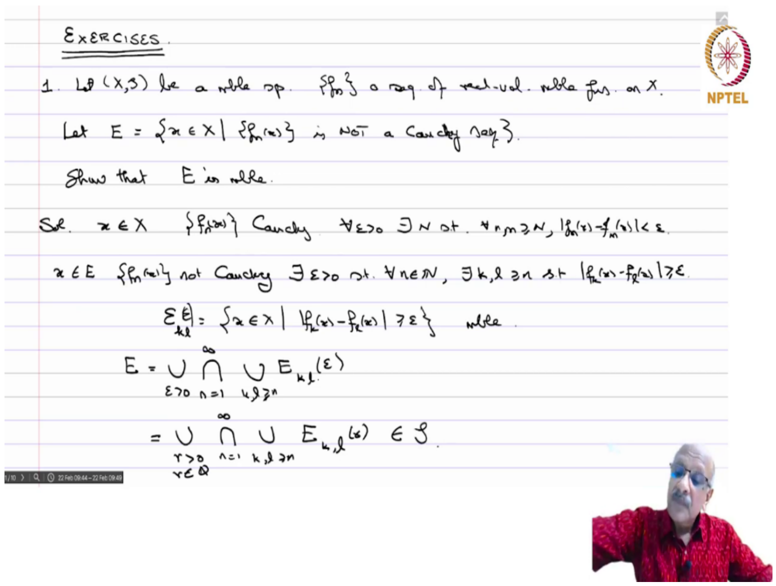

(1): Let $(X,\, S)$ be a measurable space. $\{f_n\}$ be a sequence of real-valued measurable functions on X. Let $E = \{x \in X\colon f_n(x)\ \textit{is not a cauchy sequence}\}$. Show that E is measurable. (So, you to show that E belongs to S).

Solution: Let $x \in X$. So, what do you mean by $\{f_n(x)\}$ Cauchy? So, that means for every $\varepsilon > 0\ \exists\ N$ such that for all $n,\, m \geq N$

$$|f_n(x) - f_m(x)| < \varepsilon.$$

So, what do we mean by $x \in E$? It means $\{f_n(x)\}$ not Cauchy. So, we have to take the contrapositive statement of whatever we have written here. So, that means, there exists an $\varepsilon > 0$ such that for every $n \in N\ \exists\ k,\, l \geq n$ such that

$$|f_k(x) - f_l(x)| \geq \varepsilon.$$

So, let us define

$$E_{kl}(\varepsilon) = \{x \in X : |f_k(x) - f_l(x)| \geq \varepsilon\} \text{ measurable.}$$

So, f_k, f_l measurable, so, the difference is measurable, absolute value is measurable and therefore, this set is measurable.

So, now what do you mean by E? E is nothing but

$$E = \bigcup_{\varepsilon>0} \bigcap_{n=1}^{\infty} \bigcup_{k,l\geq n} E_{kl}(\varepsilon)$$

$$= \bigcup_{r>0,\, r\in\mathbb{Q}} \bigcap_{n=1}^{\infty} \bigcup_{k,l\geq n} E_{kl}(r) \in S.$$

{ Why? Because every element in the first set if you take an r is less than epsilon will already be. So, called this $E_{kl}(\varepsilon)$ is $E_{kl}(r)$, so, $E_{kl}(\varepsilon)$. So, if it is bigger than ε will be bigger than r, therefore irrational which is greater than that and therefore, since it is rational, which is less than ε, so it will definitely be in this set. Conversely, this is a subset of this that is obvious and therefore these two sets are equal. Now, here you have everything countable. Rationals are countable, $n = 1$ to ∞ , k , $l \geq n$. All these are countable sets and they are all in S and therefore this belongs to S and therefore, it is measurable.}

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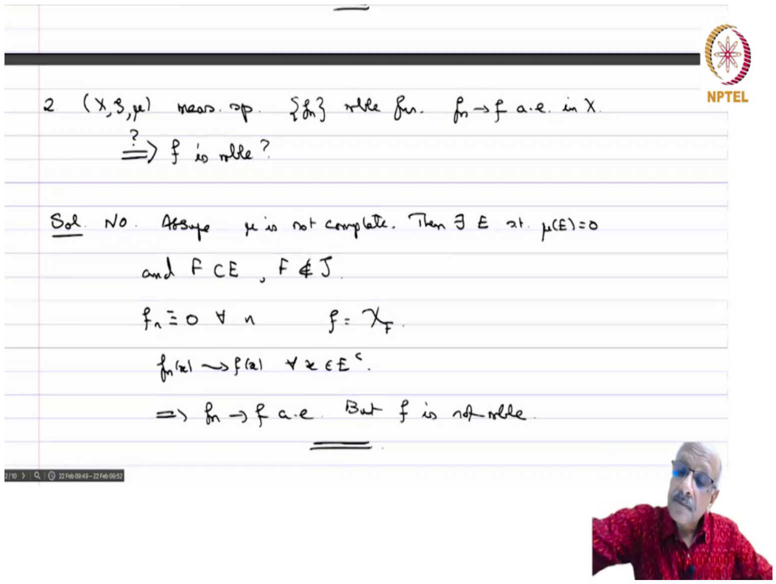

(2): Let (X, S) be a measurable space. $\{f_n\}$ be a sequence of measurable functions on X. $f_n \to f$ almost everywhere in X. Thus, this implies f is measurable.

Solution: No, it does not imply that f is measurable. So, we have shown that if f_n converges to f everywhere then f is measurable, but if f_n converges S almost everywhere then it may not be measurable.

So, let assume, μ is not complete. Then there exists E such that $\mu(E) = 0$ and $F \subset E$, F not in S.

So, now, you take $f_n = 0$ for all n, and you take $f = \chi_F$ which is 1 on F and 0 outside. So, then what happens, $f_n(x)$ converges to f of x for all x in E complement. Because outside E χ_F is 0 and f_n is 0 and therefore, you have this. But, and therefore, this implies that

$f_n \to f$ almost everywhere in X. But f is not measurable.

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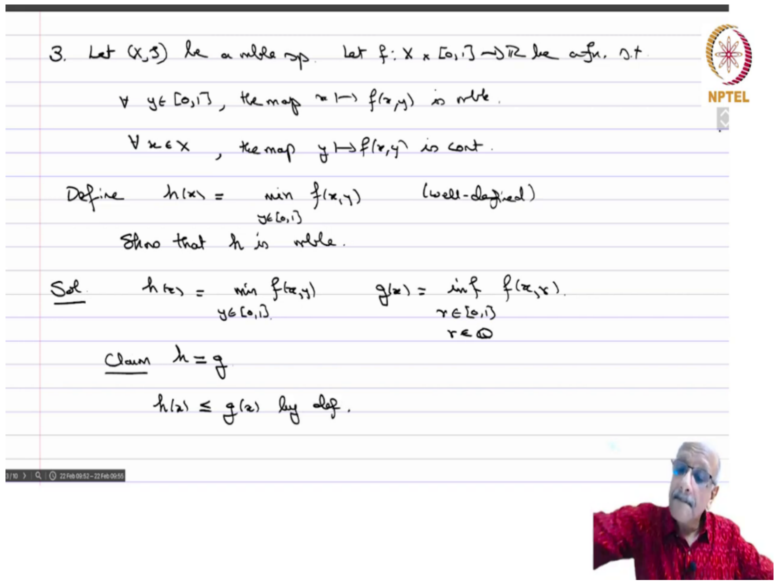

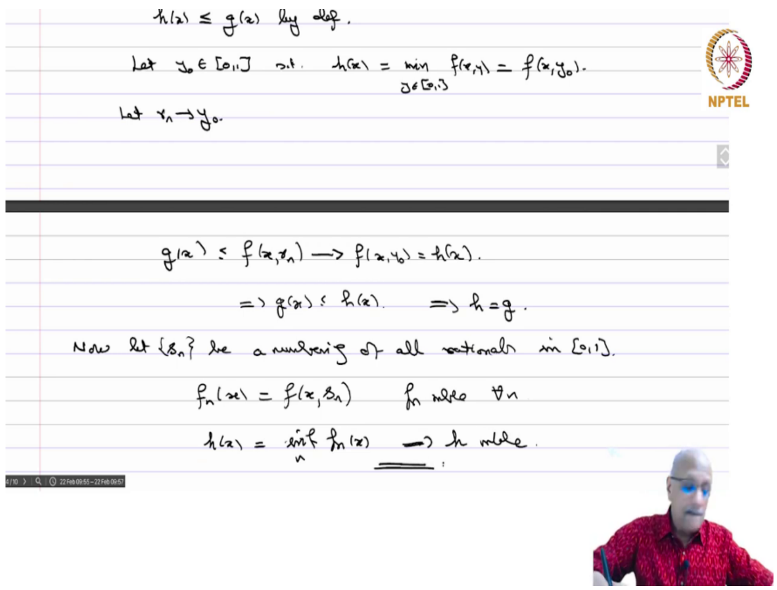

(2): Let (X, S) be a measurable space. Let $f: X \times [0,1] \to \mathbb{R}$ be a function, such that,

$\forall\ y \in [0,1]$, the map, $x \to f(x,y)$ is measurable.

And $\forall\ x \in X$, the map, $y \to f(x,y)$ is continuous.

(Because we can talk of the continuity it is $[0,1]$, here and $\mathbb{R}$ on the other side. So, we can talk about continuity).

Define $h(x) = min_{y \in [0,1]} f(x,y)$, this is well-defined.

(Because $\forall \ x \in X$, y going to $f(x, y)$ a continuous function defined on the closed interval $[0, 1]$ and therefore, $[0, 1]$ is a compact interval and therefore, f attains its minimum. So, the f minimum is actually a minimum.)

Show that h is measurable.

Solution: $h(x) = min_{y\epsilon[0,1]} f(x, y)$, $g(x) = min_{r\epsilon[0,1],\, r\epsilon\mathbb{Q}} f(x, r)$.

Then we claim, $h = g$.

(So, obviously, you are taking the minimum over all y's and here you are taking only the infimum over the rationales.)

Therefore, $h(x) \leq g(x)$ by definition, no minimum is attained.

Let $y_0 \epsilon [0, 1]$ such that $h(x) = min_{y\epsilon[0,1]} f(x, y) = f(x, y_0)$.

Let $r_n \to y_0$. Then

$$g(x) \leq f(x, r_n) \to f(x, y_0 = h(x).$$

$$\Rightarrow g(x) \leq h(x) \Rightarrow h = g.$$

Now, let $\{s_n\}$ be a numbering of all rationals in $[0, 1]$. And you define

$f_n(x) = f(x, s_n)$, this is f_n measurable for all n, then

$h(x) = min_{n\in \mathbb{N}} f_n(x) \Rightarrow h$ is measurable.

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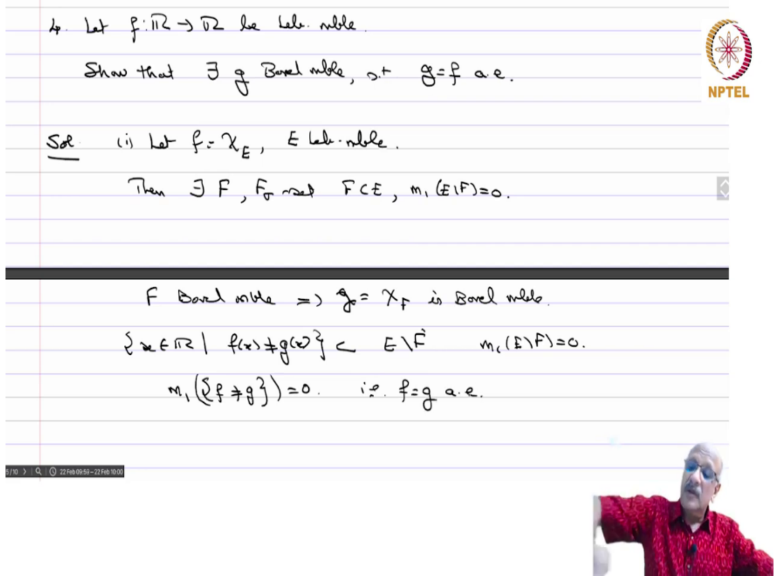

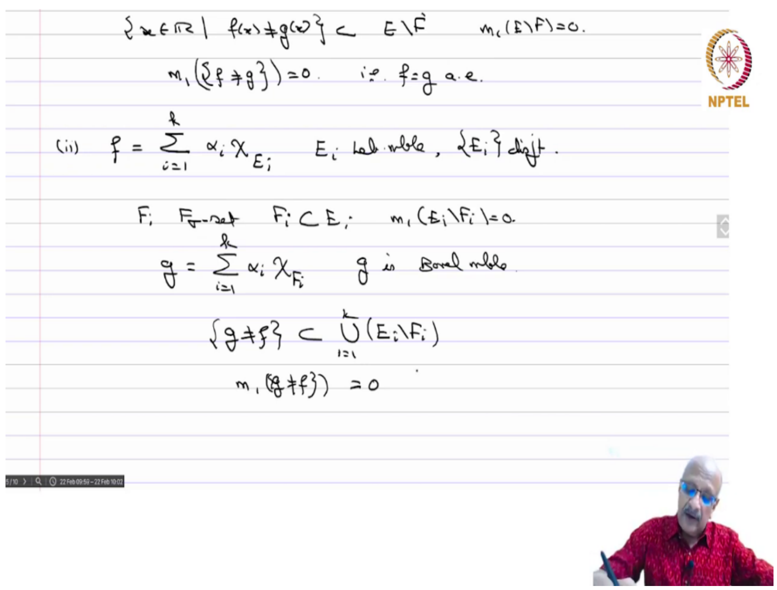

(2): Let $f: \mathbb{R} \to \mathbb{R}$ be Lebesgue measurable, show that there exists a g Borel measurable, such that

$$g = f \text{ a.e.}.$$

(So, given any Lebesgue measurable function, we can find a Boreal measurable function which is equal to it, almost everywhere. So, this method we will use very often in integration theory. So, we will first prove it for characteristic functions then for simple functions, then use a limiting argument to prove it for an arbitrary measurable function).

Solution: (i) Let $f = \chi_E$ Lebsegue measurable then, there are several equivalent characterizations of Lebesgue measurability which we have seen one of them is there exists F, F_σ set $F \subset E$ and $m_1(E - F) = 0$.

So, F is Borel measurable because it is an F_σ set it is a countable union of closed sets. And therefore, this implies it $g = \chi_F$ is Borel measurable.

Now, $\{ x \in R : f(x) \neq (x) \} \subset E - F$ (since outside E, both of them f and g is 0, inside F, both of them are 1, and therefore), and $m_1(E - F) = 0$.

So, $m_1(\{ x \in R : f(x) \neq (x) \}) = 0$.

That is, $g = f$ a.e..

So, this is true for the characteristic function.

(ii) So, the second step is to take f to be a simple function.

So, $f = \sum_{i=0}^{k} \alpha_i \chi_{E_i}$, E_i are Lebesgue measurable and E_i are all disjoint, because we can always write, rewrite simple functions in terms of disjoint sets. Now, we will take

F_i F_σ set , $F_i \subset E_i$, and $m_1(F_i - E_i) = 0$.

And you define

$g = \sum_{i=0}^{k} \alpha_i \chi_{F_i}$, then g is Borel measurable.

And now, what about the set

$$\{g \neq f\} \subset \bigcup_{i=1}^{k} F_i - E_i .$$

And therefore, $m_1(\{g \neq f\})=0$

because each one of these is 0 and you have a finite union by some positivity the measure has to be 0.

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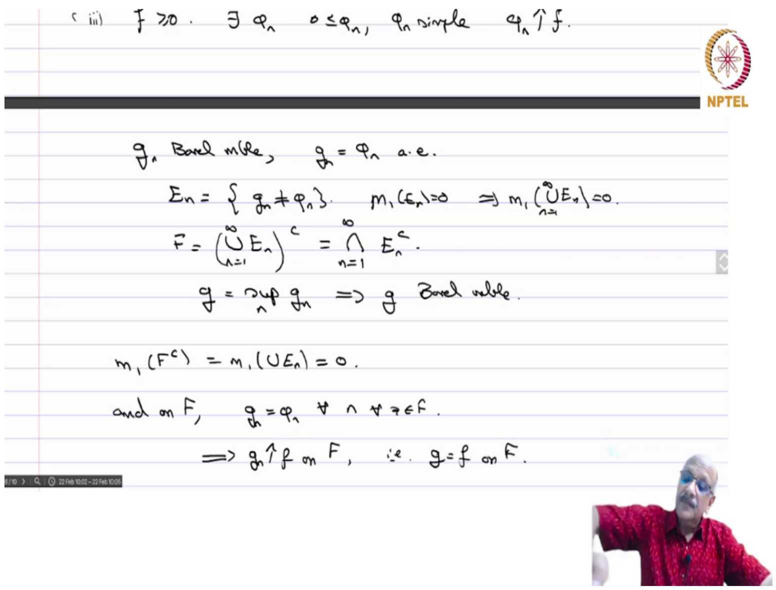

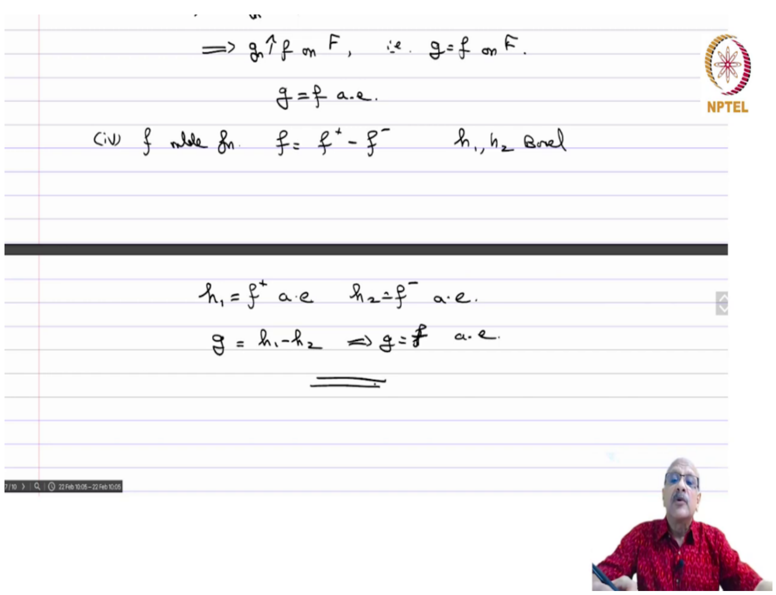

(iii): Let us take $f \geq 0$ be measurable function, then we know that exists $\varphi_{n,}$ $\varphi_{n,} \geq 0,$ φ_n simple and $\varphi_{n,} \uparrow f$.

Let g_n be Borel measurable, $g_n = \varphi_n$ almost everywhere.

So, let $E_n = \{g_n \neq \varphi_n\}$, $m_1(E_n) = 0$ implies $m_1(\bigcup_{n=1}^{\infty} E_n) = 0$.

And then what is defined

$$F = (\bigcup_{n=1}^{\infty} E_n)^c = \bigcap_{n=1}^{\infty} E_n^{\ c}.$$

Now, we are defined

$$g = sup_n g_n \Rightarrow \text{g is Borel measurable.}$$

Since g_n are all Borel measurable you are taking the supremum.

Now, $m_1(F^c) = m_1(\cup E_n) = 0$

and on F, we have $g_n = \varphi_n, \forall n, \forall x \in F.$

$$\Rightarrow g_n \uparrow f \text{ on } F, \text{ and } F^c \text{ is of measure 0.}$$

So, $g = f$ a. e.

(iv): Let f measurable function. Then you can write f as

$$f = f^+ - g^-$$

and then you have h_1, h_2 Borel measurable.

$$h_1 = f^+ \ a.e.\ ,\ h_2 = f^- \ a.e.$$

and then you define

$$g = h_1 - h_2$$

$\Rightarrow$ g equals f $a.e.$, that completes.

So, this is a technique which is very useful and therefore, that is why the fact that non-negative measurable functions see increasing limit of continuous simple functions is very important.