Measure and Integration
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Lecture 21
4.2 - The Cantor function

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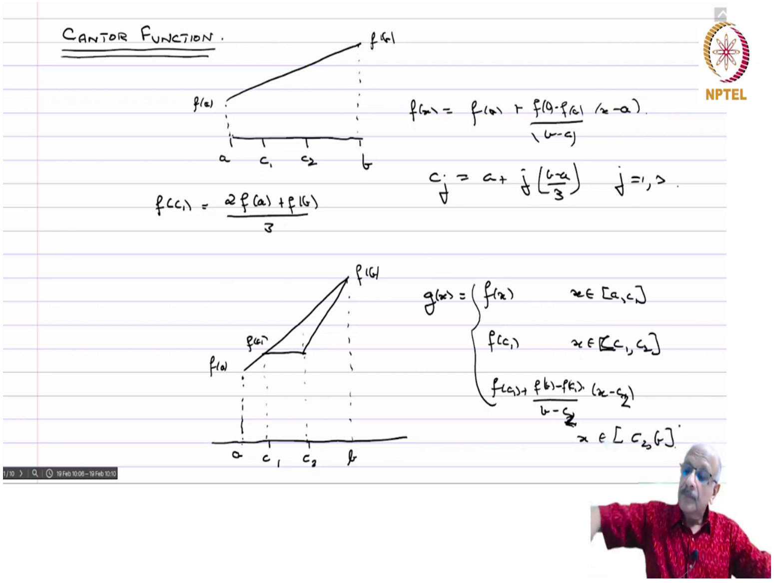

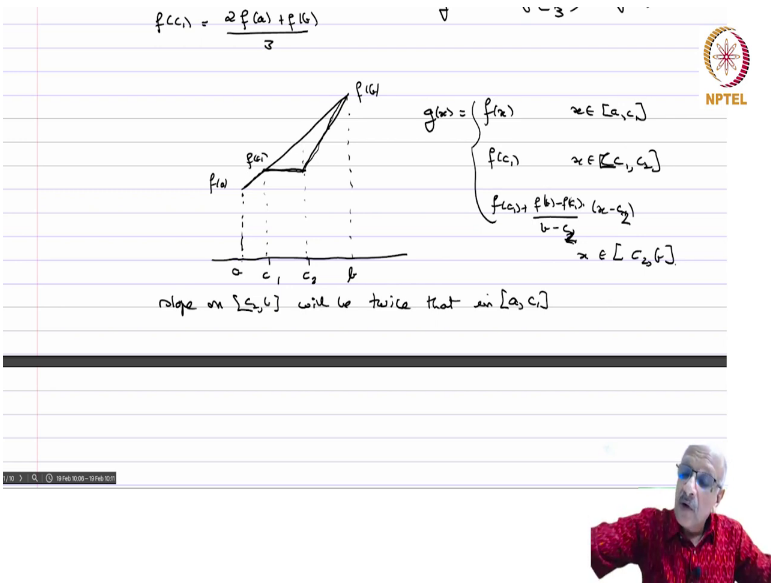

The Cantor function:

We will now look at the cantor function so please recall the cantor set, so we took 0, 1 and then we removed the middle third and then with the remaining intervals we removed the middle third and so on and what was left was the cantor set and this set was an uncountable set of measure 0 and which was nowhere dense, so now the counter functions like the cantor sets provides a lot of interesting examples and counter examples to illustrate various fine points in the theory of lebesgue integration.

Several constructions are possible and always the final properties of the function which you get are the same and they also serve the same purpose, so we will describe one such construction, so before we do that to make it easier let me just illustrate a certain basic construction, let us take an interval a & b and then you have a linear function so this is $f(a)$ and this is $f(b)$.

So, then so $f(x) = f(a) + \frac{f(b)-f(a)}{(b-a)}(x - a)$

for all x in a b, so now we divide this interval into three equal parts so these are c_1 andc_1 so

$$c_j = a + j\left(\frac{b-a}{3}\right);\ j = 1, 2,...$$

the value at c_1, $f(c_1) = \frac{2f(a)+f(b)}{3}$.

Now, we then define the next iterate of this function so what we do is we so again let me draw it again so this is f of a, f of b, this is a, this is b and you have c_1 and c_2 so up to c_1 we follow the function and then from c_1 to c_2 we go horizontally so this is $f(c_1)$ and then from here you climb up to get the new function so this is the new function so g of x is equal to f of x if x belongs to a c_1 and c equal to $f(c_1)$ it is a constant if x belongs to c_1, c_2 and then you have $f(c_1)$ plus $f(b)$ minus $f(c_1)$ by b minusc_1 into x minus c_1 so this is for x belonging to, sorry c_2, this is $f(c_1)$ because that is the same value. But b minus c_2 into x minus c_2 for x belong to c_2 b.

So, this will be the new function which you have so you follow the function up to c_1 then you do a constant and then you do this, so now a simple calculation will show that the slope on c_2 b will be twice that in ac 1 so this slope and this slope will be exactly double the slope of this that is a very standard straightforward calculation which you can do from this, from whatever is given to you.

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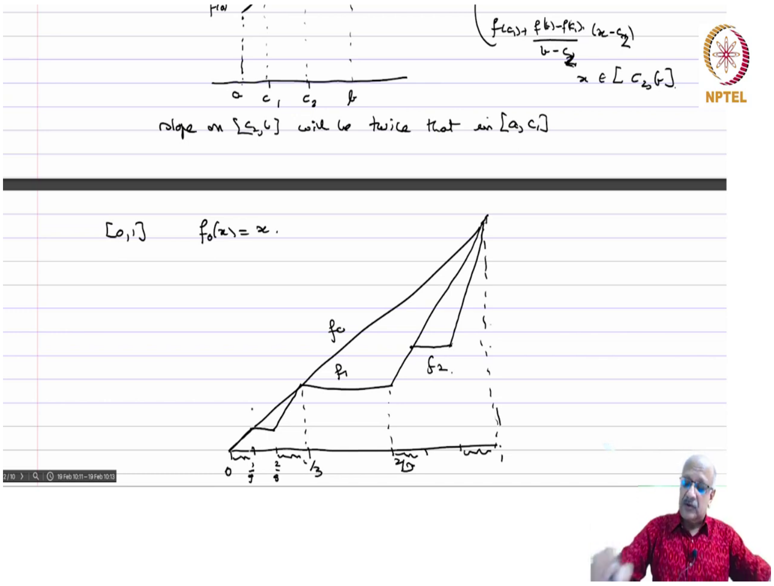

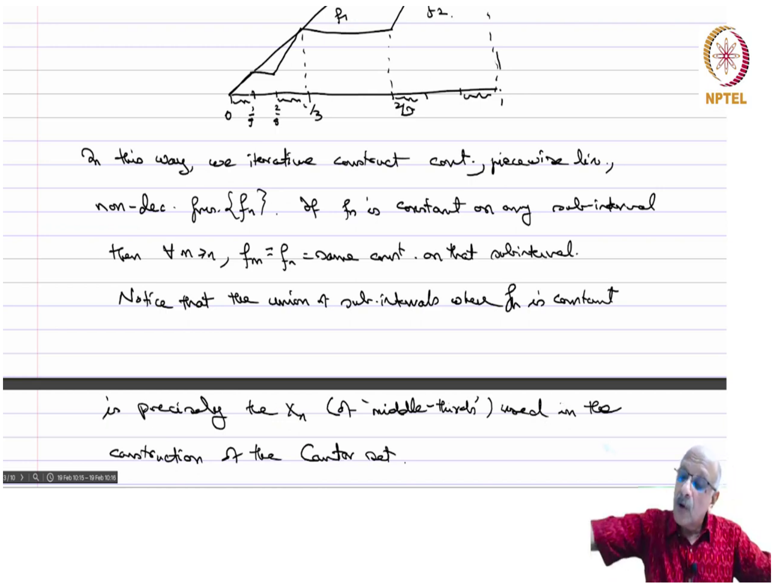

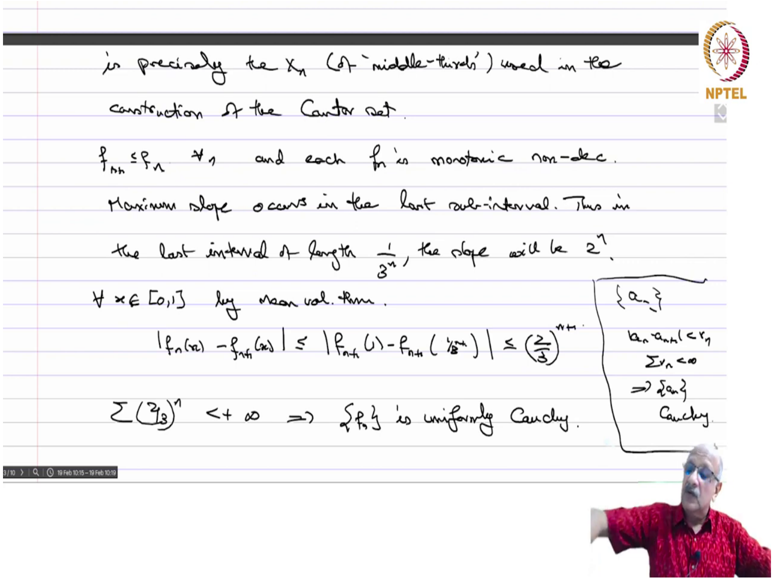

So, now let us consider the interval [0 1] and we take the function $f_0(x) = x$ and then we apply this procedure so we have here [0 1], so now we apply this procedure so we have 1 by 3 here and you have 2 by 3 so the in this part up to this part we will keep the same and then we will go horizontally up to 2 by 3 and from 2 by 3 we will go climb up now what we will do is we will leave this constant portion simple.

So, here we have you have 1 by 9 and then 2 by 9 and then similarly you have two portions here, so here we will go along this function in this interval and then we will go as a constant here and then climb up to this function, similarly here we will go along this function up to this point up to the one third point then go horizontally up to this and then you construct so this is f_1 this is f_0 and this is f_2 so once it is a constant it is always constant.

So, now f_3 will be you will have to divide these intervals into three equal parts, follow the function on the first part go horizontally in the second part and climb up to the third part so this is the construction which we are going to repeatedly do at different scales. So, in this manner we can construct, so in this way we iteratively construct continuous piecewise linear non decreasing functions fn, if fn is constant on any sub interval then for all m greater equal to n fm equals fn equals same constant on that sub interval, once it is a constant you do not meddle with it you always apply this construction only in the non-constant parts that you divide by 3 and go on.

So, the union, notice that the union of sub intervals where fn is constant is precisely the set x n of middle thirds used in the construction of the cantor set so go back we are I am using the same notation here, so the maximum slope, so also f_n is less than or equal to f_{n+1} for all n because when you do this construction you will see that you are following then you are falling below and then you are climbing up so each time the function is less than the previous sorry f_{n+1} and each f_n is monotonic non decreasing.

So, sequence is decreasing point wise but the function is increasing is an increasing function. Now, maximum slope occurs in the last subinterval, each time we apply the procedure the slope is doubling and therefore thus in the last sub interval of length 1 by 3 power n the slope will be 2 power n because we started with the slope 1 in the original function and every time in the last interval it is going to double we applied it n times and therefore we get a 2 power n as a slope there.

Therefore, for all x in [0 1] by the mean value theorem we have that mod $f_n(x) - f_{n+1}(x)$ this need not be a modulus because this is a non negative quantity this is less than or equal to the maximum slope which you have in f_{n+1}, 1 minus f_{n+1} 1 by 3 power n plus 1 and that is less than equal to 2 by 3 power n plus 1, now sigma 2 by 3 power n is convergent because of the geometric series and therefore this implies that fn is uniformly Cauchy.

So, if you have sigma a n such that a n minus a n plus 1 is less than r n sigma r n is convergent this means that a n is Cauchy, sorry not sigma a n you have a sequence a n subsequent terms are in a convergent series and this is Cauchy this is a easy fact to prove so you should be able to convince yourself about this.

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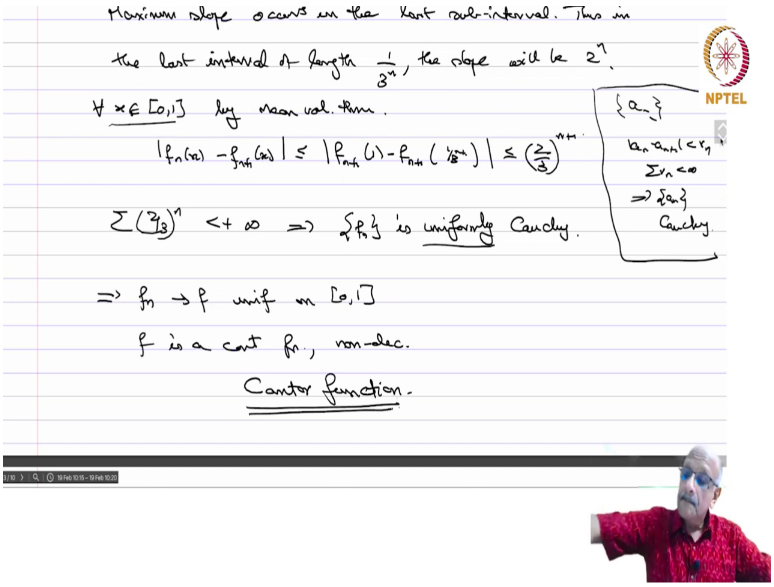

And this is uniform for all x in x we have and therefore you have uniformly Cauchy. So, you have a uniformly Cauchy sequence on an interval that means that f n converges to f uniformly on [0 1] and f is continuing because it is a uniform limit of continuous function f is a continuous function and it is also non-decreasing and this function is called the cantor function.

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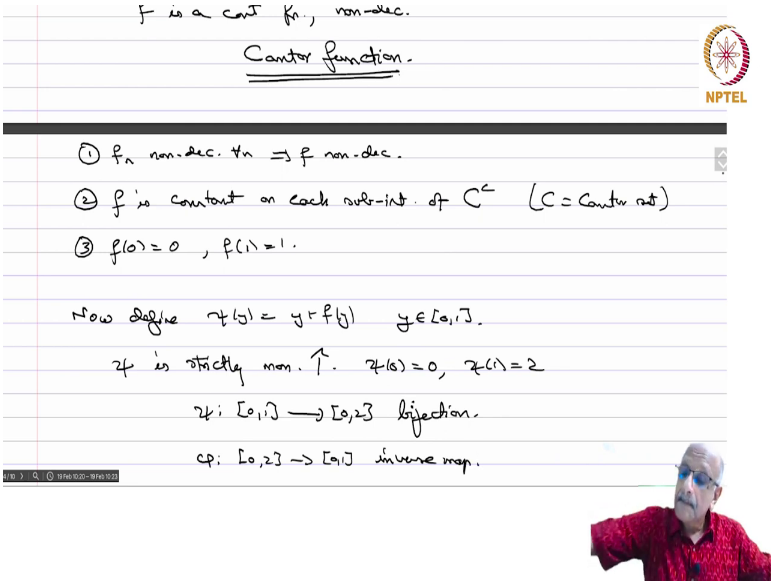

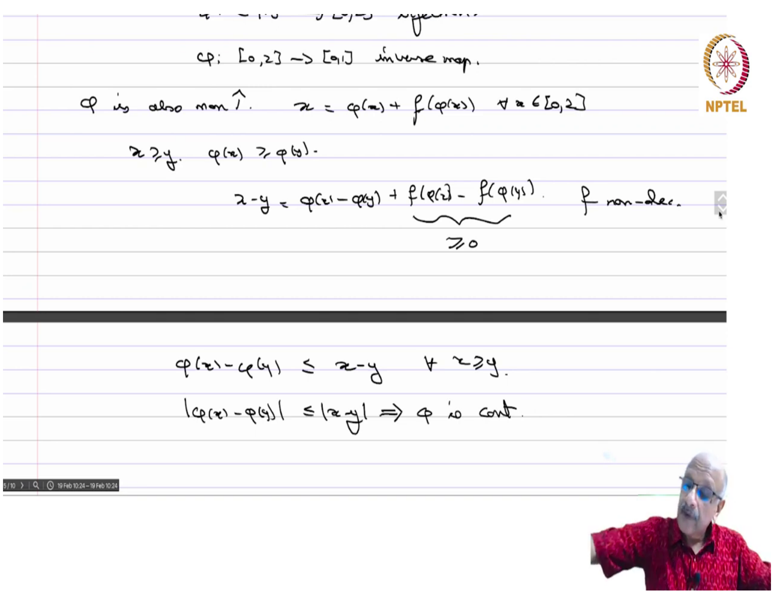

So, first f_n non-decreasing for every n implies is not decreasing to f is constant on each subinterval of c complement where c equals cantor set, we saw that once you it is a constant its constant in fn all along and these sets are precisely the middle third sets and therefore they are the complement of the cantor set and then 3, $f(0) = 0$ and $f(1) = 1$ so now define $\varphi(y) = y + f(y)$, y in 0 then psi is strictly monotonically increasing because I have added y here this is a non decreasing function I have added a strictly increasing function therefore this is you have.

And you have $\varphi(0) = 0$ and $\varphi(1) = 2$ therefore psi is from [0 1] onto [0 2] and this is a bijection so we have phi from [0 2] to [0 1] inverse then φ is also monotonically increasing and you have that $x = \varphi(x) + f(\varphi(x))$ for every x in 0 because you have φ is the inverse function now if x is becoming equal to y then you have $\varphi(x)$ is greater than equal to phi of y and $x - y = \varphi(x) - \varphi(y)$, f is non decreasing, f non-decreasing and therefore this is greater than equal to 0 and therefore this shows that $\varphi(x) - \varphi(y) \leq x - y$, for all x greater than or equal to 1 or in other words mod $|\varphi(x) - \varphi(y)| \leq |x - y|$ and therefore φ is continuous as well.

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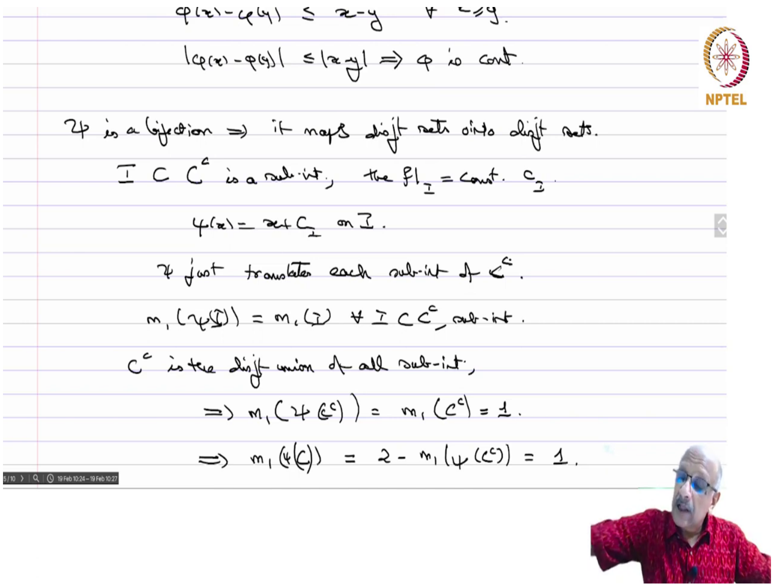

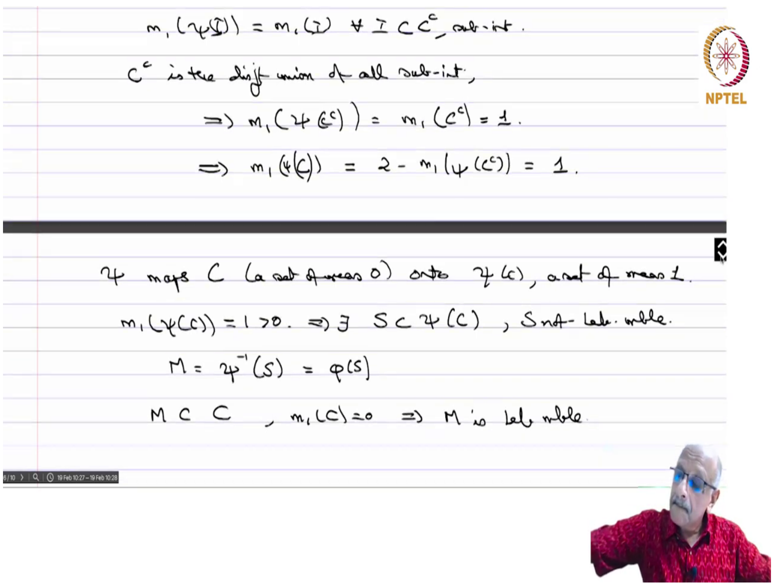

So, now φ is a bijection implies it maps disjoint sets into disjoint sets, onto, so if I contained in C complement is a sub interval then f restricted to I equals some constant which is some let us say c of I therefore $\varphi(x)$ equals x plus c I on I, so psi just translates each sub interval of c complement.

So, $m_1(\varphi(I)) = m_1(I)$ for each I contained in c complement sub interval and c complement is the disjoint union of all such sub intervals and φ maps disjoint sets into disjoint sets and therefore you have m 1 of c complement equals sorry, $\varphi(c)$ complement equals m 1 of c complement and that is equal to 1.

So, this implies that $m_1(c)$ itself has to be equal to 2 minus m 1 of φ(c) , φ(c) complement and that is equal to 1. So, in other words this function φ maps c a set of measure 0 on to φ(c) a set of measure 1 so m_1 of φ(c) equal to 1 which is strictly positive therefore there exists S contained in size c, S not lebesgue measurable, we saw that every positive set contains a non lebesgue measurable subset, so let M equal to φ inverse of c S and of course you can call this as φ of S because φ is nothing but psi inverse so M is this and so M is c but M is contained in C, measure of C is 0 the cantor set therefore by completeness of the lebesgue measure M is lebesgue measurable.

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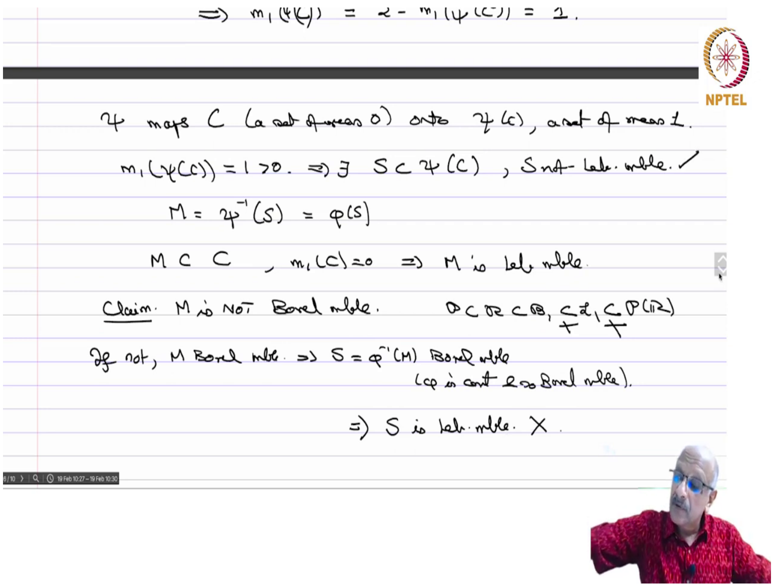

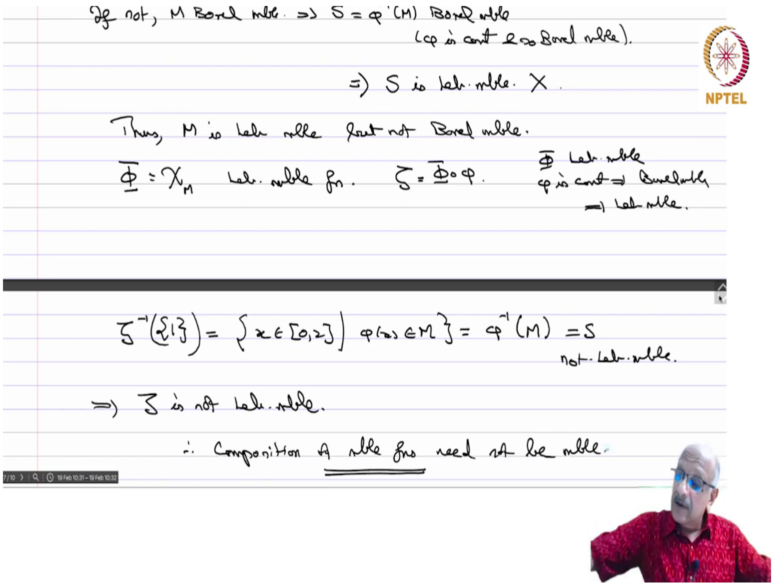

Claim, M is not Borel measurable, so you see we have if this claim is satisfied so we had P and then we had R and then we had B1 then we had L1 and then we had power set of R and we saw explicitly there are non measurable subsets so this is here we gave a unsubstantiated but cardinality argument to show that this is also strict but now we are giving an example of M which is lebesgue measurable but it is not Borel measurable.

So, if not, M Borel measurable implies s equals phi inverse of M borel measurable because phi is continuous and so borel measurable that implies if some set is borel measurable so automatically lebesgue measurable, S is lebesgue measurable but that is not true because we have S is not limit measurable so this is not true.

Therefore, the claim is substantiated so we have M is lebesgue measurable thus but not Borel measurable, finally let us set capital phi equals chi of M so this is a lebesgue measurable function because M is a lebesgue measurable set therefore it is a lebesgue measurable function.

Now, you take $\zeta = \Phi \circ \varphi$, so zeta is a composition of lebesgue measurable function and a continuous function, continuous functions are both so φ lebesgue measurable, small phi is continuous implies Borel measurable implies lebesgue measurable and you are taking the composition of two lebesgue measurable functions, so now if you took zeta inverse of the singleton 1 this equal to set of all x in 0 2 what is zeta is capital φ of small $\varphi(x)$, this is phi of x belongs to M which is equal to φ inverse of M and that is S but this is not lebesgue measurable.

But so zeta is not lebesgue measurable because we did this exercise we know that if the converse is not true but this proposition if a function is measurable then the inverse of every singleton has to be measurable and here you have a function which is 1 the inverse of the singleton 1 is not measurable and therefore the original function cannot be measurable therefore composition of measurable functions need not be measurable, so this cantor set will have many more applications.

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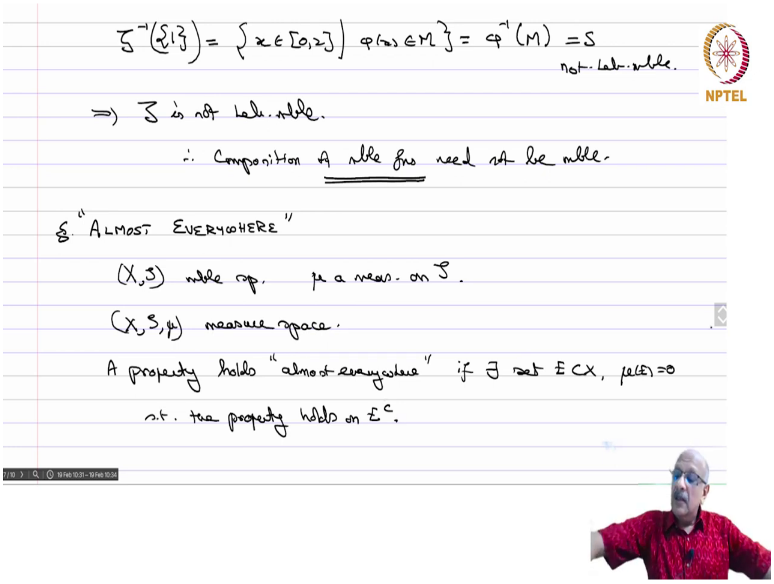

So, before we close I want to make some this thing about almost everywhere now that we have, what do we mean by this statement which we will have, so (X, S, μ) measurable space μ a measure on S then we say (X, S, μ) is a measure space, we have a measurable space (X, S, μ) which means you have x and sigma algebra and now if you specify the measure also then you call it a measure space. So, we say a property holds almost everywhere if there exists a set E contained in x $\mu(E)$ equal to 0 such that the property holds on E complement so when a property holds almost everywhere we mean it is true except possibly on a set of measure 0.

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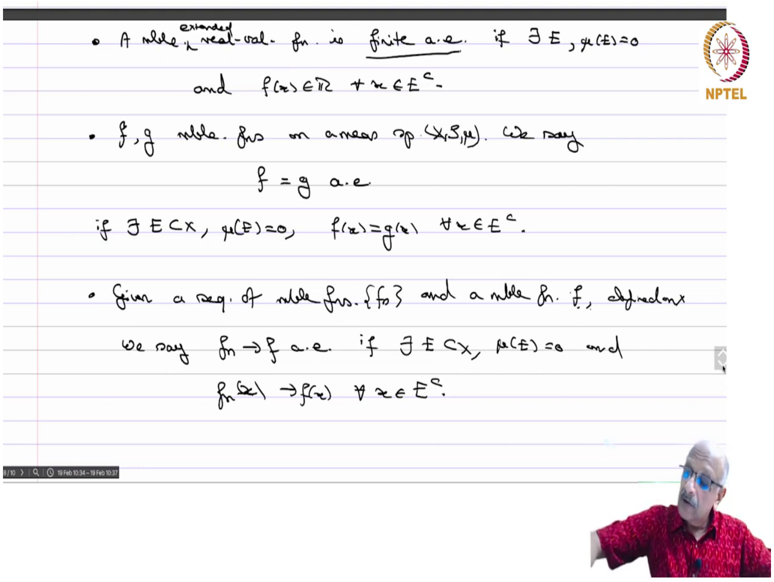

So, let us give me, give some examples, so a measurable real valued function, extended real valued function is finite almost everywhere so when you say almost everywhere the short form is a is finite almost everywhere if there exists a set E $\mu(E) = 0$ and f x belongs to R for all x in E complement, f g measurable functions on a measure space x is mu we say f equal to g almost everywhere if there exists a E contained in x $\mu(E) = 0$ and $f(x) = g(x)$ for all x in E complement.

Given a sequence of measurable functions fn and a measurable function f defined on x we say f_n converges to f almost everywhere if there exists a E contained in x $\mu(E) = 0$ and $f_n(x)$ converges to $f(x)$ for every x in e complement so this gives you an idea of how we use the word almost everywhere.

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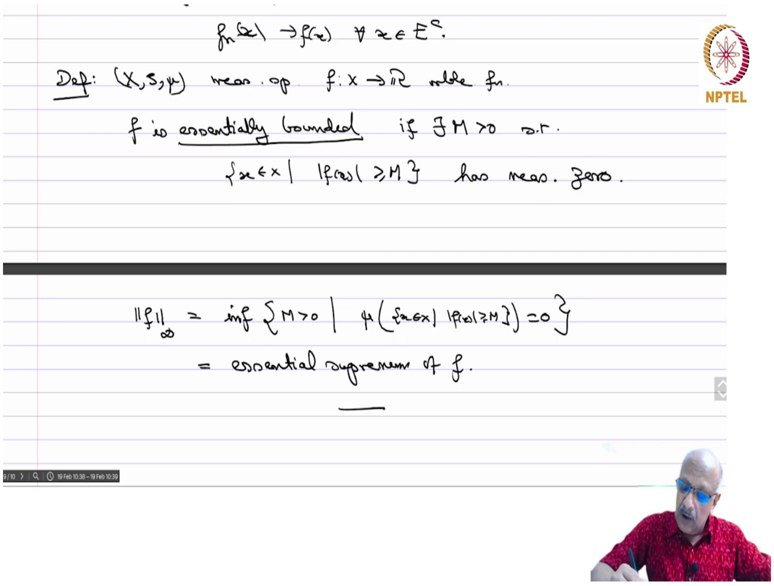

So, in this context we will also make the following

Definition:

(X, S, μ) is a measure space $f: X \to \mathbb{R}$ measurable function, f is essentially bounded if there exists a $M > 0$ such that set of

$\{x \in X \mid f(x) \geq M\}$ has measure zero.

and then we define norm

$$||f||_{\infty} = inf\{M > 0 \mid \mu(\{x \in X \mid |f(x)| \geq M\}) = 0\} = essential\ supremum\ of\ f.$$

is called the essential supremum that means except for the set of measure 0 this is the smallest number, smallest lower bond, upper bound sorry.