Measure and Integration
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Lecture 20
4.1 – Measurable functions

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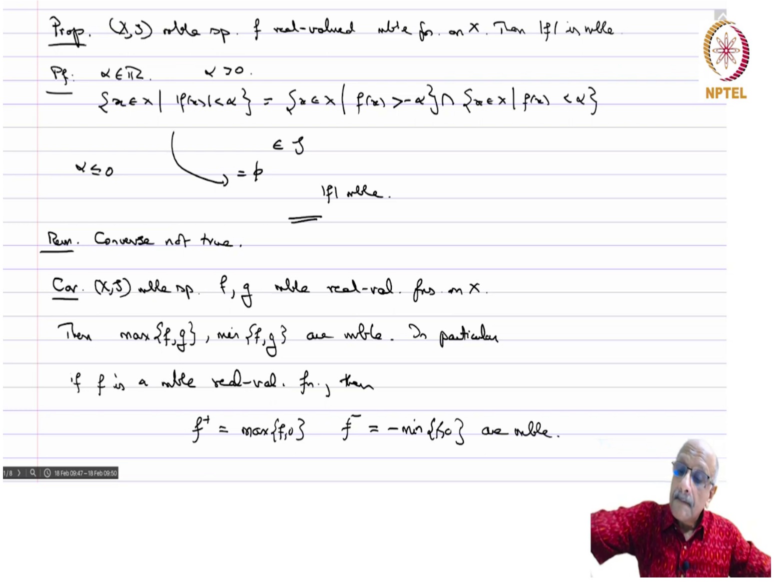

So, we define what is a measurable function so we continue with the properties of the measurable functions, so here is a

Proposition: (X, S) measurable space, f real valued measurable function on X then $|f|$ is measurable, so

proof: Let $\alpha \in \mathbb{R}$, $\alpha > 0$ so we have to look at

$$\{x \in X : |f(x)| < \alpha\} = \{x \in X : |f(x)| >- \alpha\} \cap \{x \in X : f(x) < \alpha\}$$

and therefore this belongs to S because f is measurable.

And if $\alpha \leq 0$ then the set is empty and therefore you have that so therefore $|f|$ is measured, remark I leave it as an exercise for you to check, converse not true you can easily construct a counter example so we will leave it that, now

Corollary: (X, S) measurable space f and g measurable real valued functions then $max\{f, g\}$, $min\{f, g\}$ are measurable, in particular if f is measurable real valued function then $f^{+} = \max\{f, 0\}$ and $f^{-} =- min\{f, 0\}$ are measurable.

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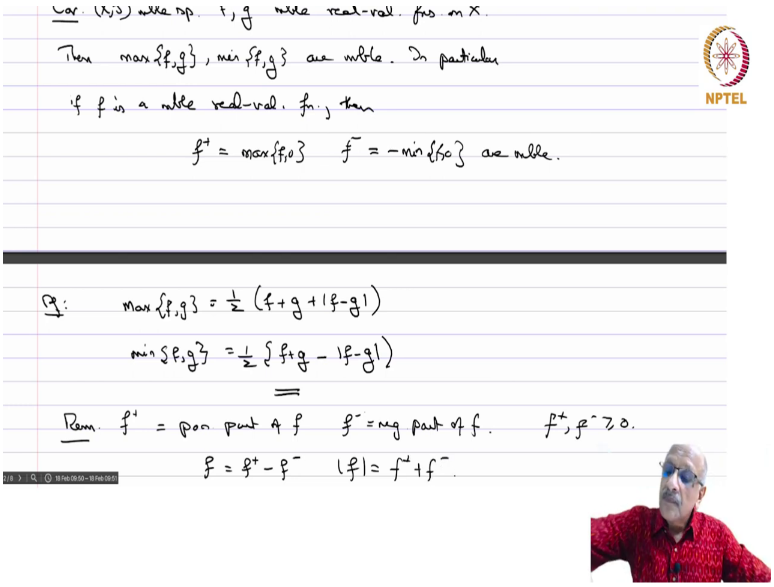

Pproof: $max\{f, g\} = \frac{1}{2}(f + g + |f - g|)$

$$min\{f, g\} = \frac{1}{2}(f + g - |f - g|)$$

so f and g measurable means $f - g$ measurable $|f - g|$ measurable $f + g$, $|f + g|$ measurable multiplied by half its measurable so this is measurable similarly mean is also measurable.

remark, $f^{+} =$ is called the positive part of f and f^{-} = negative part of f, note that positive f^{+} , f^{-} are both non-negative functions,

$$f = f^{+} - f^{-} \qquad |f| = f^{+} + f^{-}.$$

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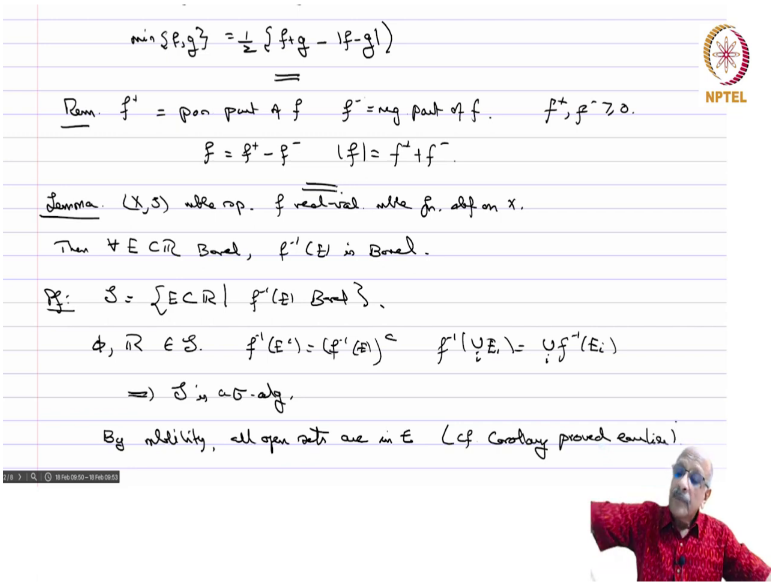

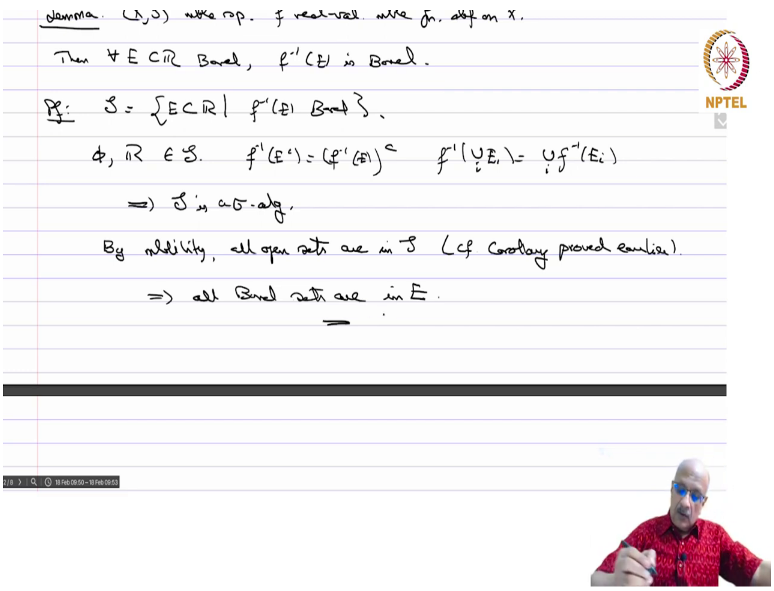

So lemma this we have, this kind of thing we have done many times,

Lemma: (X, S) measurable space. f real value measurable function defined on X. Then for every $E \subset \mathbb{R}$ Borel $f^{-1}(E)$ is Borel.

Proof: $S^{\sim} = \{E \subset \mathbb{R} \mid f^{-1}(E) \text{ is Borel}\}$. So empty set and $\mathbb{R} \in S^{\sim}$ and then

$$f^{-1}(E^c) = (f^{-1}(E))^c, \qquad f^{-1}(\bigcup_i E_i) = \bigcup_i f^{-1}(E_i)$$

From this you get that $S^{\sim}$ is a sigma algebra. By measurability all open sets are in E we proved this cf corollary proved earlier therefore open sets are all in $S^{\sim}$ so implies all open sets are in $S^{\sim}$, in place all Borel sets and that completes the proof of this theorem.

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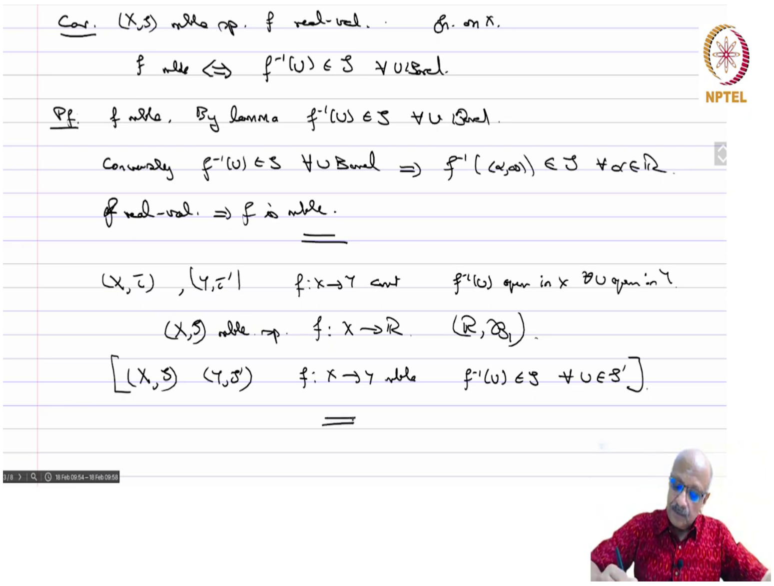

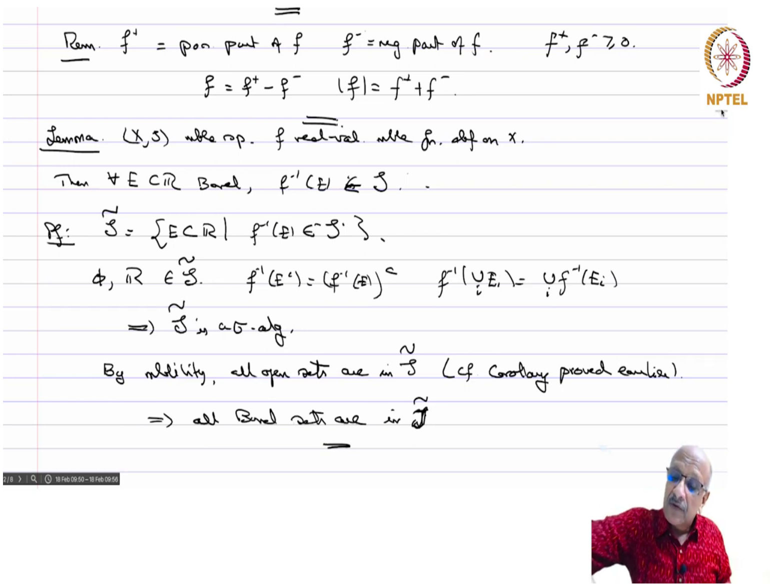

Corollary: (X, S) measurable space f real valued measurable function on X, then f is measurable if and only if $f^{-1}(U) \in S$ for every U Borel.

Proof: so f measurable then by lemma $f^{-1}(U) \in S$ for every U Borel

Conversely $f^{-1}(U) \in S$ for every U Borel implies

$$f^{-1}((\alpha,\infty)) \in S, \quad \forall\, \alpha \in \mathbb{R}$$

and then f is real valued implies f is measurable, so this proves,

So now measurability can be thought it can be seen is nothing but the imitation of continuity suppose I have two topological spaces (X, S) and $(Y, S^{'})$ then what do you say that

$f\colon X \to Y$ is continuous that means $f^{-1}(U)$ open for, in X, for all U open in Y.

So, similarly for measurability we are saying we have (X, S) measurable space and then f is a real valued function and so on $\mathbb{R}$ we take the Borel sigma algebra B 1 and then inverse image of every Borel set is a measurable set that is what we are saying so just like the definition of continuity in fact if you have two measurable spaces (X, S) and $(Y, S^{'})$ one could talk of $f\colon X \to Y$ measurable if $f^{-1}(U) \in S$ $for\ every\ U \in S^{'}$ so this could be an abstract definition of measurability between abstract measurable spaces but we are restricting our attention to real valued functions, so for real valued functions measurability essentially inverse image of every Borel set should be measurable that is all that we are saying.

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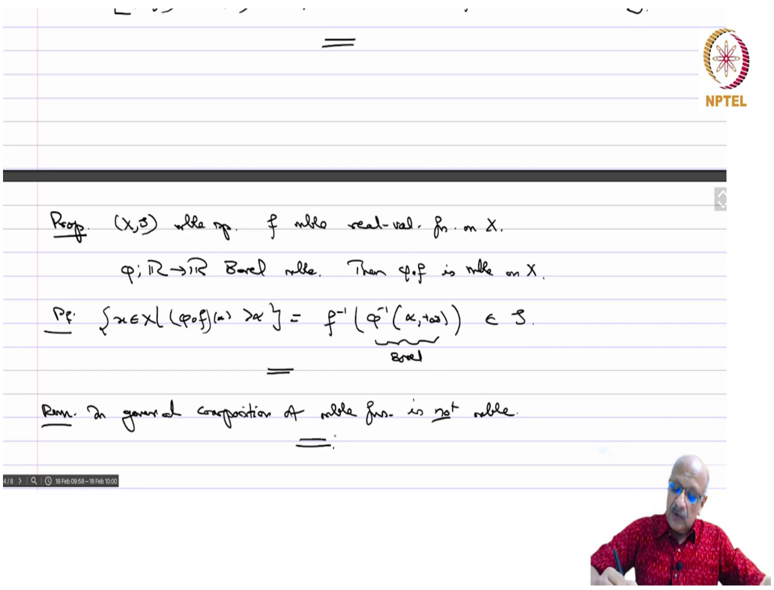

Proposition: (X, S) measurable space f measurable real valued function

$\Phi\colon \mathbb{R} \to \mathbb{R}$ Borel measurable then $\Phi \circ f$ is measurable on X,

Proof: $\{x \in X : (\Phi \circ f)(x) > \alpha\} = f^{-1}(\Phi^{-1}(\alpha, \infty)) \in S$

Now, this being a Borel measurable function is a Borel set and by the previous lemma this belongs to s because previous corollary therefore this belongs to s and that proves the result.

Remark: In general composition of measurable functions is not measurable, we will see an example of this but if you have that the second function is Borel measurable $\mathbb{R}$ to $\mathbb{R}$ then the composition is measurable so that is the moral of the story.

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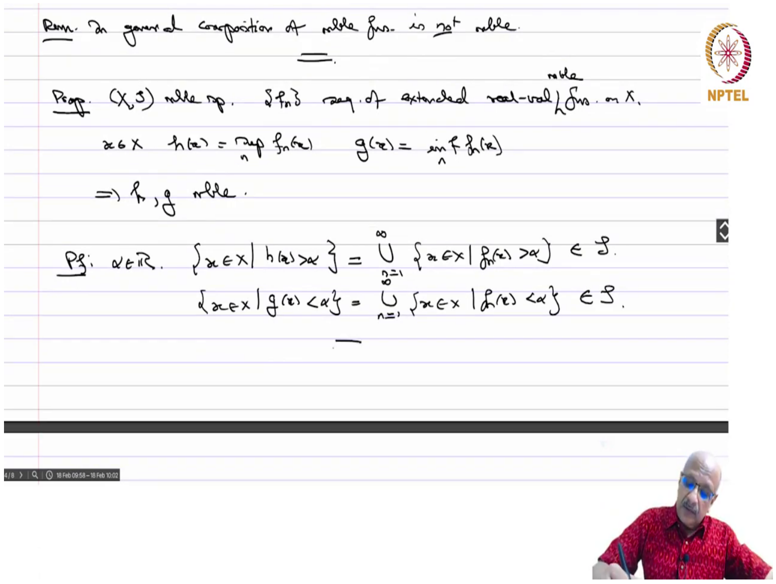

Proposition: (X, S) measurable space $\{f_n\}$ sequence of extended real valued functions on X, measurable functions, so you take for $x \in X$ you define $h(x) = sup\, f_n(x)$, $h(x) = inf\, f_n(x)$ because these are extended real valued implies f *and* g are measurable,

Proof: let $\alpha \in \mathbb{R}$ so you take

$$\{x \in X |\, h(x) > \alpha\} = \bigcup_{n=1}^{\infty} \left\{x \in X |\, f_n(x) > \alpha\right\} \in S,$$

$$\{x \in X |\, g(x) > \alpha\} = \bigcap_{n=1}^{\infty} \left\{x \in X |\, f_n(x) > \alpha\right\} \in S$$

and therefore h and g are measurable.

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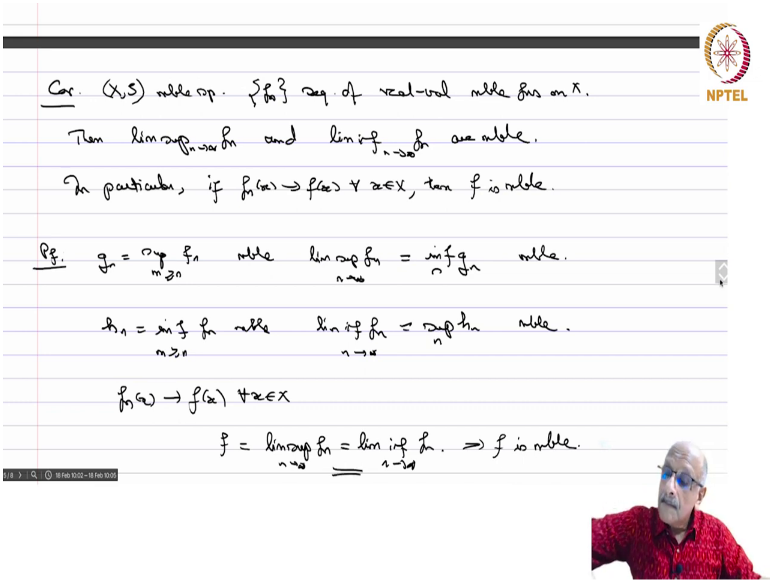

Corollary (X, S) measurable space $\{f_n\}$ sequence of real valued measurable functions then

$$\lim_{n\to\infty} sup\ f_n\ and\ \lim_{n\to\infty} inf\ f_n \text{ are measurable.}$$

In particular, if $f_n(x) \to f(x),\ \ \forall x \in X$ then f is measured, so the limit of measurable functions convergence point wise everywhere is measurable,

Proof: so you take $g_n = sup_{m\geq n} f_n$ this is measurable and then $\lim_{n\to\infty} sup\ g_n = inf_n g_n$, so measurable.

Similarly, you take $h_n = inf\ f_n$ is measurable therefore $\lim_{n\to\infty} inf\ h_n = sup_n h_n$ this is measurable, so $f_n(x) \to f(x),\ \ \forall x \in X$ then you have f is nothing but

$$f_n(x) \to f(x),\ \ \forall x \in X$$

$f = \lim_{n\to\infty} sup\ f_n = \lim_{n\to\infty} inf\ f_n$ and therefore this implies that f is measurable.

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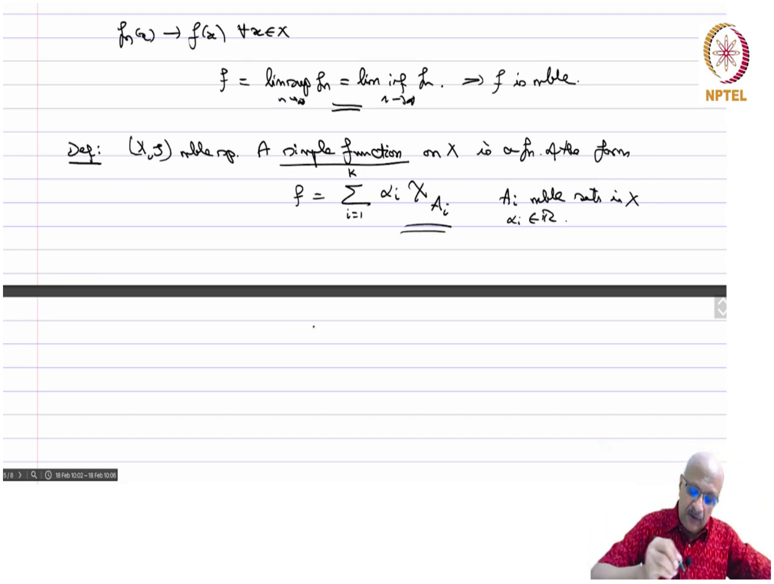

Definition: (X, S) measurable space,. A simple function on X is a function of the form we have seen this before

$$f = \sum_{i=1}^{k} \alpha_i \chi_{A_i}, \quad A_i \ are\ measurable\ functions\ on\ X.$$

So when we say a simple function it is automatically measurable because of the example where we saw chi of A_i is measurable if and only if A_i is measurable.

So, these simple functions are the building blocks of integration theory so we will lebesgue integration theory starts with defining the integral on simple functions and then going on to non-negative functions, general functions and so on, so we have the following very very important theorem which you must remember very well.

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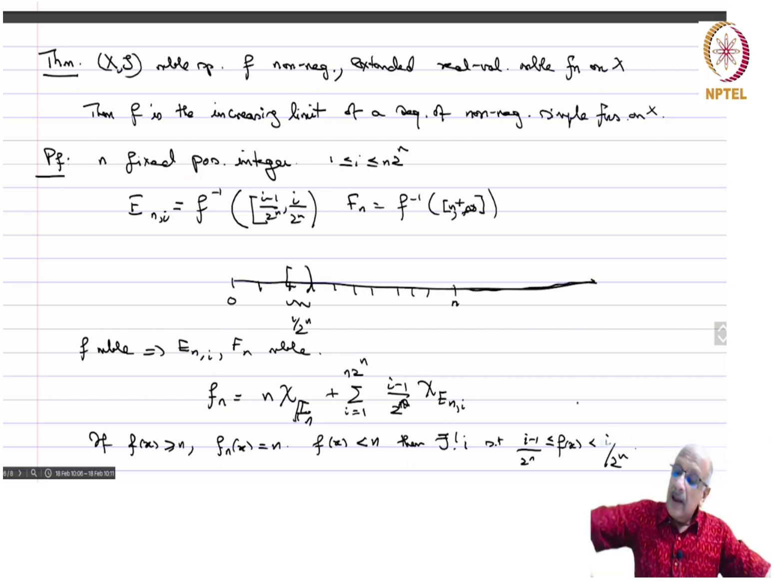

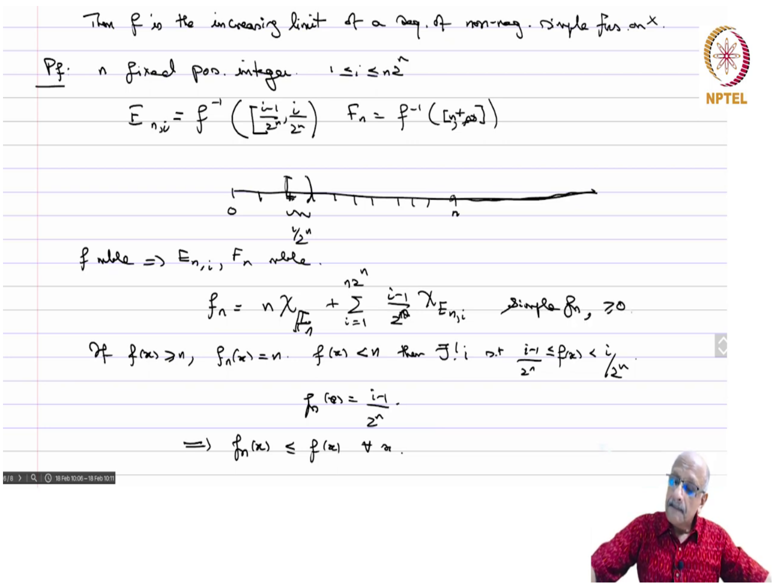

So theorem, it is a very important theorem

Theorem: *Let* (X, S) measurable space and f non-negative extended real valued measurable function. Then f is the increasing limit of a sequence of non-negative simple functions on X, so we are going to so every non-negative measurable function which is extended real value can be written as the increasing limit that means you can find the sequence f_n of X which is monotonic increasing and the limit is $f(x)$.

Proof: n fixed positive integer so $1 \leq i \leq n2^n$;

$$E_{n,i} = f^{-1}\left([\frac{i-1}{2^n}, \frac{i}{2^n})\right), \qquad F_n = f^{-1}([n, \infty)).$$

so what are you doing here so I am taking 0 and then I am taking n so whenever x inverse image of n plus infinity is given by fn and this portion here I am going to divide each sub interval of length 1 by 2 power n so there are n times 2 power n such intervals and if f falls in any particular interval here then I put $E_{n,i}$ is the inverse image of f here so this is how I am defining this thing.

So, f measurable, so f measurable implies $E_{n,i}$ F_n are all measurable and then of course if I define

$$f_n = n\chi_{F_n} + \sum_{i=1}^{n2^n} \frac{i-1}{2^n}\chi_{E_{n,i}}$$

so what am I saying so in other words if $f(x)$ is greater than equal to n then we say $f_n(x)$ equals n, if f x is less than n then there exists a unique i such that i minus 1 by 2 power n is less than equal to $f(x)$ is less than i by 2 power n and then we are going to put

$$f_n(x) = \frac{i-1}{2^n}.$$

So, if $f(x)$ falls after then then you put this lower value if it is before n then it will fall in one of these intervals and then you take the lower end of that interval to do that so this implies of course that $f_n(x) \le f(x)$ for all $f(xX$ and this is of course a simple function.

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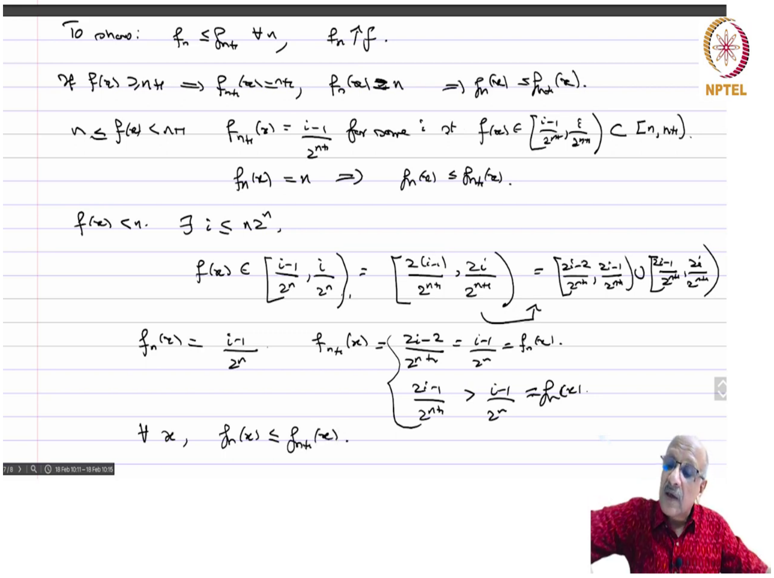

So, we just have to show two things that to show

$$f_n \leq f_{n+1}, \quad \forall n, \quad f_n \uparrow f, \quad ,$$

so if $f(x)$ is greater equal to n plus 1 so this implies that $f_{n+1}(x) = \frac{i-1}{2^{n+1}}$ and then $f_n(x) = n$ since it is bigger than equal to n plus 1 is bigger than n and therefore this is equal to n so this implies that $f_n(x)$ is less than equal to $f_{n+1}(x)$, so if $f_n(x)$ less than f x less than n plus 1 then you have $f_{n+1}(x)$ equal to $\frac{i-1}{2}$ to the n plus 1 for some i such that $f\ (x)$ belongs to $\frac{i-1}{2}$ to the n plus 1, $\frac{i}{2}$ to the n plus 1 which will be contained in n n plus 1.

Because we are going to divide like that and now further and f n of x will still be n and therefore this implies that $f_n(x) \leq f_{n+1}(x)$, now finally if $f\ (x)$ is less than n then there exists an i this is less than or equal to n 2 power n such that $f(x)$ belongs to $\frac{i-1}{2}$ power n i by 2 power n, but this is also equal to $\frac{2i-1}{2}$ power n plus 1 $\frac{2i}{2}$ to the n plus 1.

So, what is $f_n(x)$, $f_n(x) = \frac{i-1}{2^n}$,what is $f_{n+1}(x)$, now this interval here is can be written as $\frac{2i-2}{2^{n+1}}$ to, $\frac{2i-1}{2}$ to the n plus 1, union $\frac{2i-1}{2}$ to the n plus 1, $\frac{2i}{2}$ to the n plus 1, so $f_{n+1}(x)$ equal to either if it falls in this interval it is $\left[\frac{2i-2}{2^{n+1}}, n+1\right]$ which is $\left[\frac{i-1}{2^{n}}, n+1\right]$ which is $f_n(x)$ or it can be $\frac{2i-1}{2}$ to the n plus 1 which is bigger than $\frac{i-1}{2}$ power equals $f_n(x)$. Therefore, for all

$\forall x, \; f_n(x) \leq f_{n+1}(x)$ so this is a monotonically increasing sequence.

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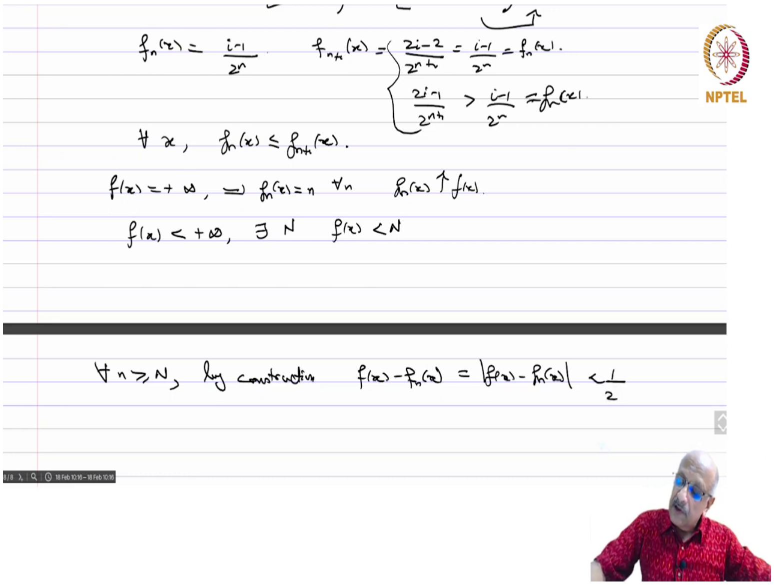

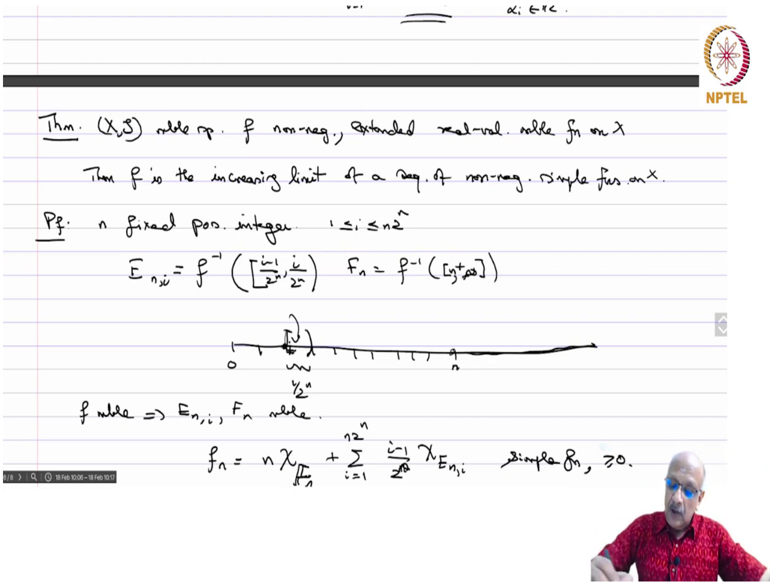

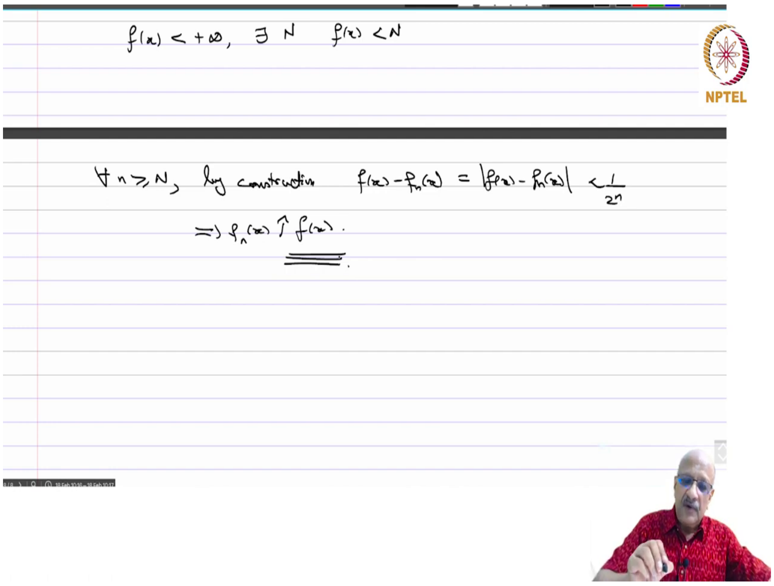

So, now if $f\ (x) = \infty$ this implies that $f_n(x) = n$ for all n and so $f_n(x) \uparrow f(x)$, if

$f\ (x) < \infty$ is strictly less than plus infinity there exists a positive integer N such that

$$f\ (x) < N$$

, then for all n greater equal to N by construction you have that $f(x) - f_n(x)$ which is the same as $|f(x) - f_n(x)|$ because this is non negative is less than 1 by 2 power n.

Because you see its in this interval if f falls in this interval you are putting the lower end point as a value as $f_n(x)$ so the distance between$f\ (x)$ and $f_n(x)$ is always less than 1 by 2 power n in this case and therefore this implies that $f_n(x)$ increases to $f\ (x)$, so this proves the theorem, very important theorem, so every non negative measurable function is the increasing limit of non negative simple functions.

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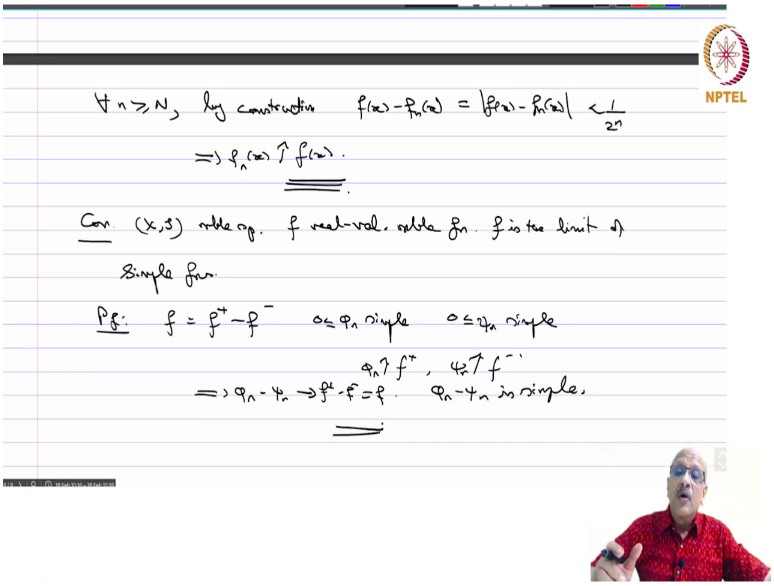

Corollary: Let (X, S) measurable space and f real valued measurable function then f is the limit of simple functions,

Proof: $f = f^{+} - f^{-}$

because its real value you do not have ambiguity in the definition both of these cannot be infinity and therefore this is well defined and now you take

$$0 \le \varphi_n \ \textit{are simple}\,,\ 0 \le \psi_n \ \textit{are simple}\,,\ \varphi_n \uparrow f^{+}\,,\ \psi_n \uparrow f^{-}$$

therefore you have that $\varphi_n - \psi_n \to f^{+} - f^{-} = f$ and then φ_n minus ψ_n is a simple. So, that proves this corollary.

So, this completes the fundamental properties of measurable functions, next time we will take up a very particular case of a function called the cantor function which is like the cantor set the cantor function they are closely related anyway, they are immeasurable source of counter examples.