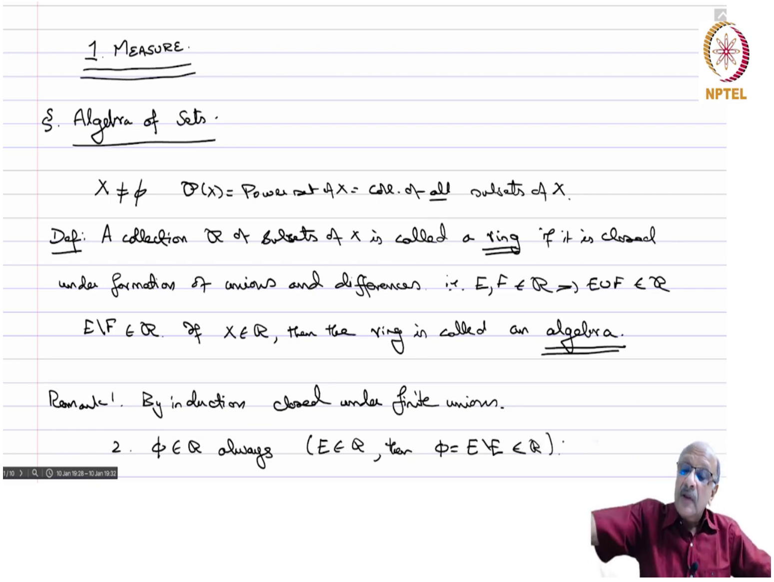
Measure and Integration
Professor S. Kesavan
Department of Mathematics
The Institute of Mathematical Sciences
Lecture No- 2
1.2 – Algebras of sets

(Refer Slide Time: 0:16)



So, we will now begin our first chapter, the first topic namely the notion of a measure in an abstract city. So, recall measure is supposed to generalize the notion of length, area, volume etc depending on the dimension, and when calculating complicated areas, what did we say people did namely, you break it up into smaller domains, add some area, subtract some areas and so on.

So, when we want to define the measure mimicking the properties of area, volume, or length, we need to do it on a class of subsets. If we can do it on all subsets, it is fine, but very often we may not be able to do that. So, we will restrict ourselves to a class of subsets. So, when you want this class of subsets on which you want to define, so, you want to add areas and subtract and so on, that means you have to be able to put together sets and you should be able to take away some sets and that leads us to the following sets.

So, the class of sets on which we will divide the measure, so, we want to look at that. So, the first sections will be algebra of sets.

Algebra of sets: So, throughout we will work with a non empty set 'X' is a non empty set and we will denote the power set of X as

$$P(X) = power\ set\ of\ X = collection\ of\ all\ subsets\ of\ X.$$

So, this notation we will always use. So, we will start with a definition.

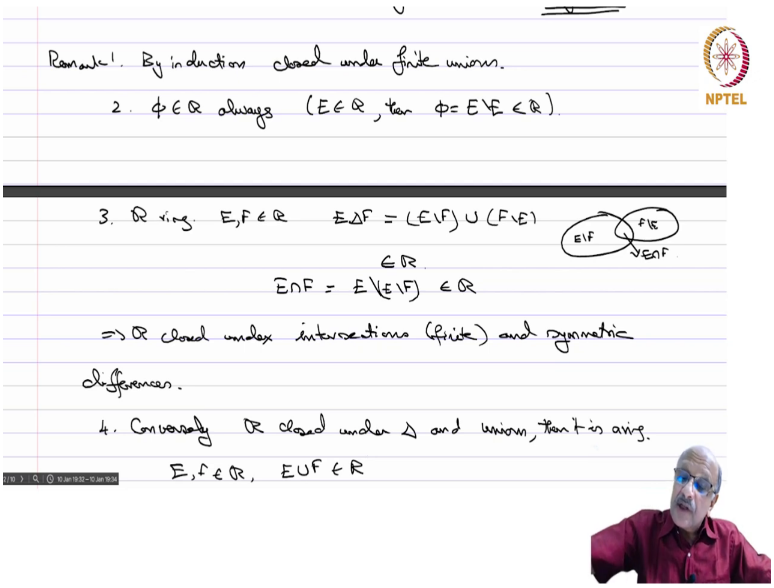
Definition: A collection R, of subsets of X, is called a ring if it is closed under formation of unions and differences. i.e., if E, F belongs to R, then this implies that $E \cup F \in R$, and also $E \setminus F \in R$. If $X \in R$ then the ring is called an algebra.

So, now, we have a series of remarks.

Remark: (1) So, by induction closed under finite unions.

(2) empty set always belongs to R always, because if $E \in R$, then $\phi = E \setminus E \in R$, and therefore, it should belong to R.

(Refer Slide Time: 4:23)



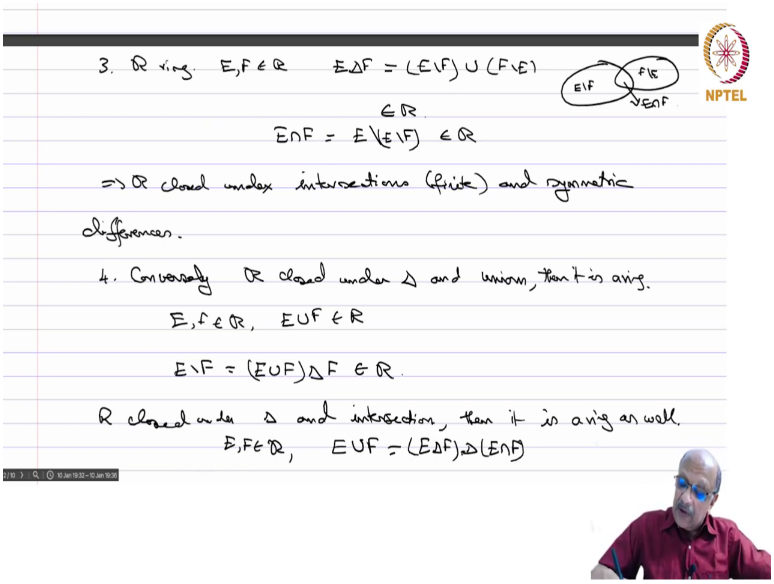
(3) If $E, F \in R$, then the symmetric difference $E \Delta F = (E \setminus F) \cup (F \setminus E) \in R$. Also,

$E \cap F = E \setminus (E \setminus F) \in R$.

Therefore, R is closed under intersections and by induction this finite number of intersections are also true and symmetric differences.

(4) Conversely, if R is closed under the symmetric difference and unions then it is a ring, because if E and F belong to R we already know it is closed under the union So, $E \cup F$ belongs to R.

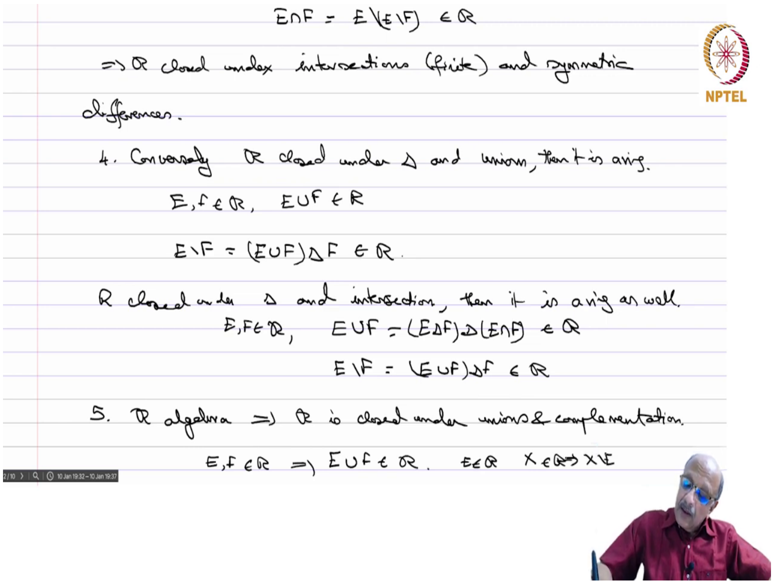
(Refer Slide Time: 6:44)



And then further $E \backslash F = (E \cup F)\Delta F \in R$. So, if it is closed under symmetric differences in unions then also it is a ring, the equivalent statement.

So, it is closed and the intersection so, this belongs to R is close and symmetric differences so, this belongs to R and then you have two elements of R and the symmetric difference is also in R.

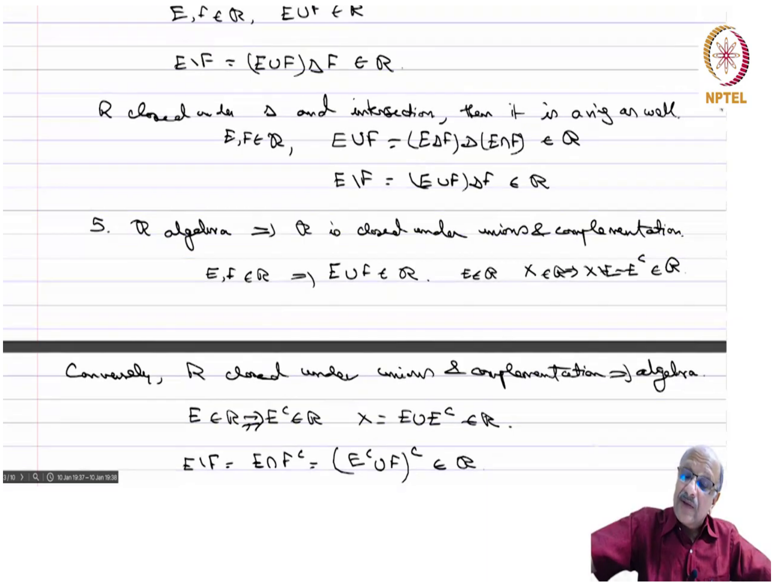
(Refer Slide Time: 8:18)



And then E minus F equal to E union F delta F we already saw E union F we just proved this in R and then delta is already a closed operation and therefore, this belongs to R. So, these are then one more remark.

(5) So, if R is an algebra ⇒ R is closed under unions and complementation, so, E F belongs to R so, then you have $E \cup F \in R$, that is true and you have E is in R, then X is also in R. So, $X \backslash E$ which is nothing but E^c belongs to R.

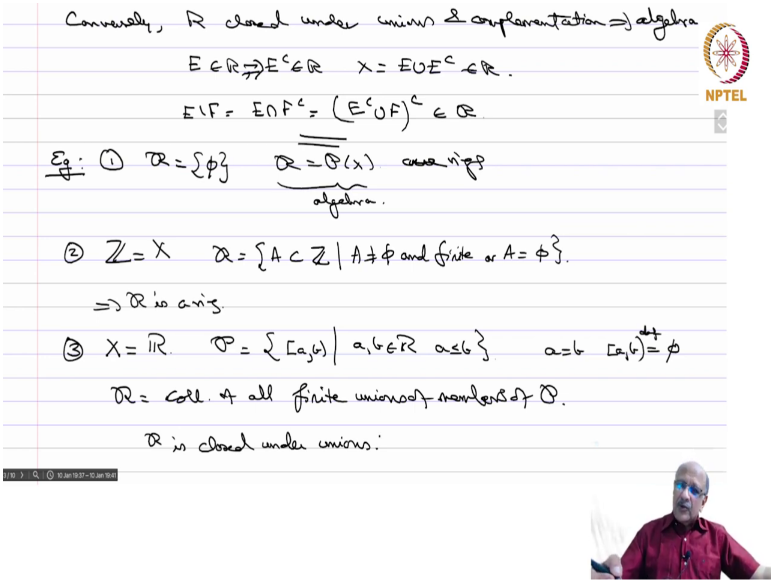
(Refer Slide Time: 9:30)



Conversely R closed under unions and complementation implies it is a ring, because if you have E is in R, then E^c is in R and $X = E \cup E^c \in R$. So, R is already there, it is closed under unions and then what about the differences? $E\backslash F = E \cap F^c = (E^c \cap F^c) \in R$.

So, it is closed under the unions X is there, and it is also so, so, this is not just a ring it is algebra, closed under the unions, closed under differences and X is there. So, this becomes an algebra.

(Refer Slide Time: 10:45)



So, now, let us see some examples.

Example: (1) you take $R = \{\phi\}$ or $P(X)$. So, these are trivial examples.

(2) So, $Z = X$, Z is a set of all integers and then you take

$$R = \{A \subset Z: A \neq \phi \; and \; finite \; or \; A = \phi\}$$

Then R is a ring because you either have empty sets or you have a finite subset non empty subset.

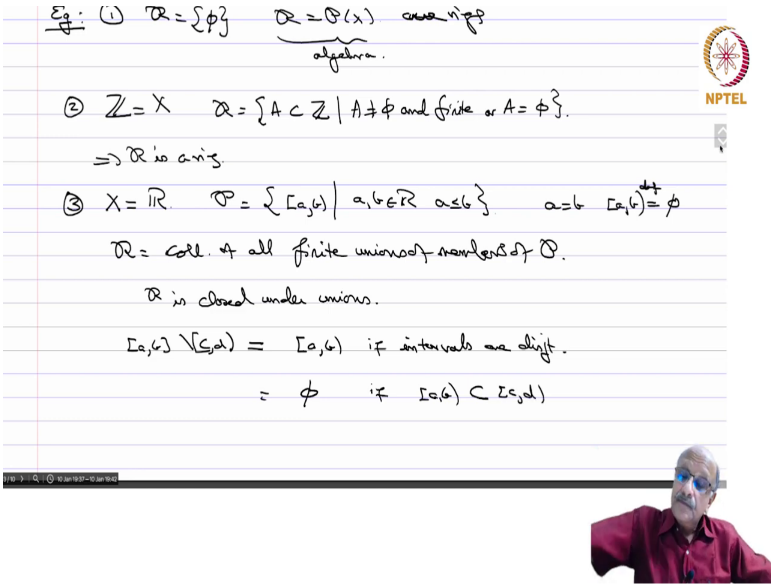
(3) So, let us take $X = \mathbb{R}$, and then we define

$$P = \{[a, b): a, b \in \mathbb{R}, a \leq b\}.$$

So, if $a = b$, then the convention is that $[a, b) = \phi$.

So, then you take R= collection of all finite unions of members of P, so, R to be the collection of all finite unions of members of P. So, obviously R is closed under unions because you take two finite unions and put them together you are going to get another finite union so, there is nothing to do. So, only we have to check for differences.

(Refer Slide Time: 13:52)



Now, if you take $[a, b)\backslash[c, d) \ = \ [a, b) \ if \ intervals \ are \ disjoint,$

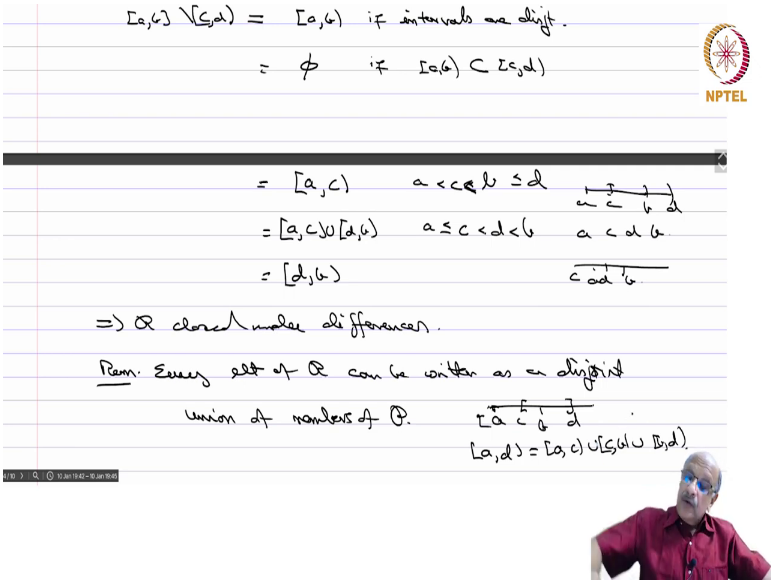
$$= \ \phi, \quad if \ [a, b) \ \subset \ [c, d),$$

$$= \ [a, c) \ if \ a \ < \ c \ < \ b \ \leq \ d,$$

$$= \ [a, c) \ \cup \ [d, b), \ \ if \ a \ \leq \ c \ < \ d \ < \ b,$$

$$= [d, b) \, .$$

(Refer Slide Time: 14:51)

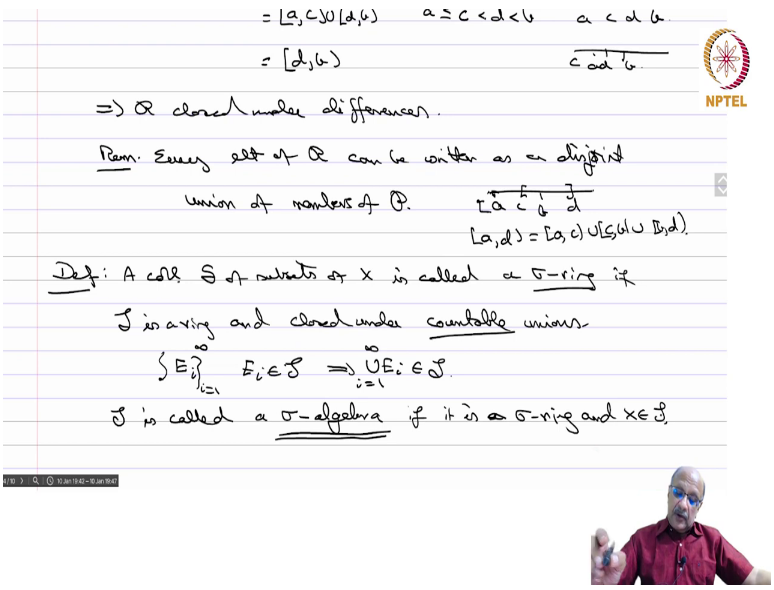


Whatever kind of difference you make, you are only going to get another either the same thing or the union of such intervals and therefore.

Now, if you take any finite unions they will be again combinations of the sets and therefore, this implies that R is closed under differences and therefore, this is a ring. So, and also remark.

Remark: Every element of R can be written as a disjoint union of members of P.

(Refer Slide Time: 17:59)

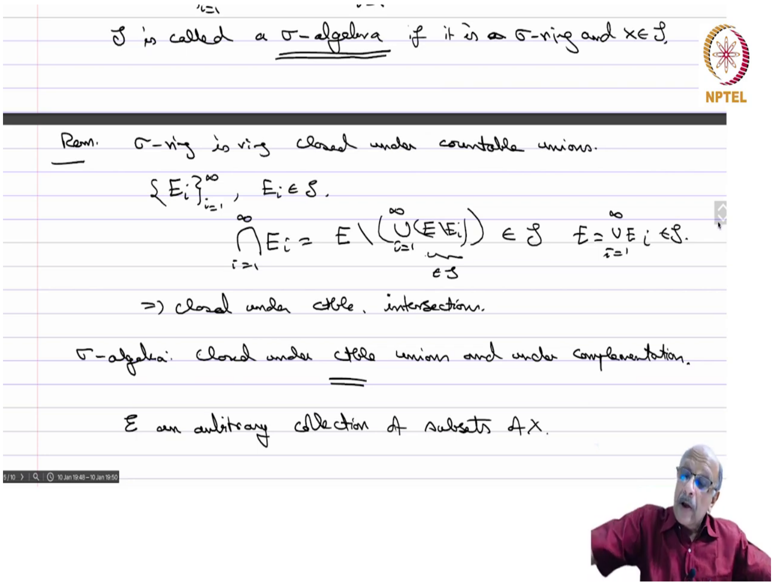


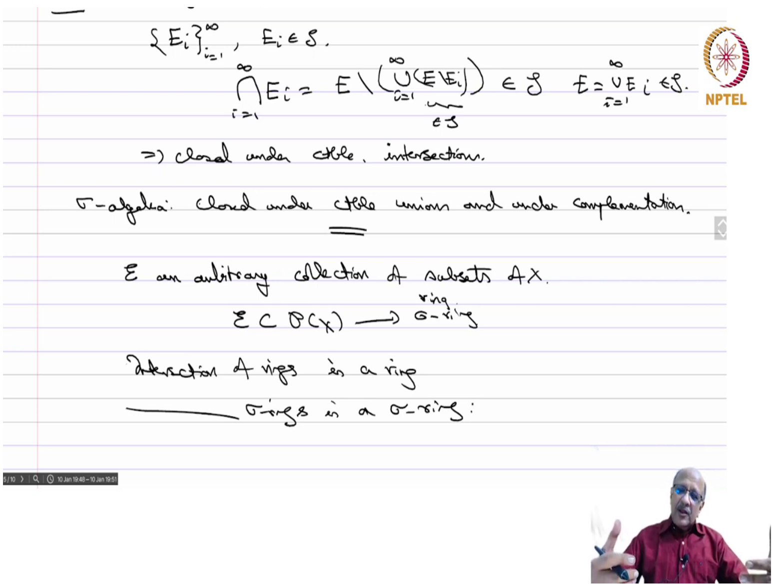
Definition: A collection S of subsets of X is called a σ −ring if S is a ring and closed under countable unions. So, if $\{E_i\}_{i=1}^{\infty}$, $E_i \in S$, $\Rightarrow \cup_{i=1}^{\infty} E_i \in S$.

S is called a σ −algebra if it is a σ −ring and $X \in S$.

Whenever you want an algebra the whole set must belong to it. So, throughout this course, you will find that if we will put this sigma in front of some definition then we will mean that something countable is allowed whereas previously only finite operations were allowed. Now, we are going to use countable operations so, countable unions will be allowed and that is what we have this.

(Refer Slide Time: 20:05)





Remark: σ −ring is a ring closed under countable unions. Now, if you take $\{E_i\}_{i=1}^{\infty}$, $E_i \in S$,

$$\cap_{i=1}^{\infty} E_i = E\backslash(\cup_{i=1}^{\infty}(E\backslash E_i)) \in S, \ E = \cup_{i=1}^{\infty} E_i \in S.$$

So, it is closed under countable intersections.

So, in the same way sigma algebra will be closed, you can define it this way under countable unions and under complementation, so this will give you a sigma algebra.

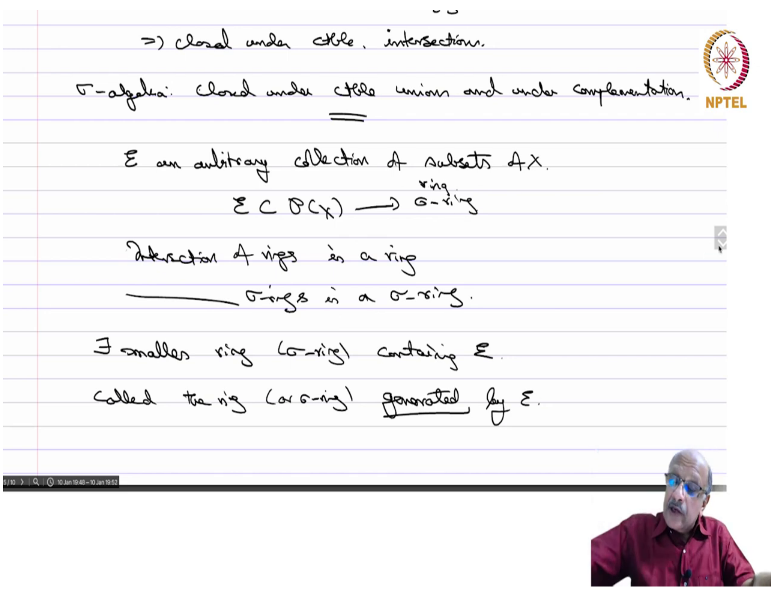
So, now, finally, let us take E an arbitrary collection of sets of subsets of X, so then of course,

$E \subset P(X)$, which is of course, at the same time a sigma ring and also it is a ring, it is a sigma ring whatever you want to say.

Now, intersection or rings is a ring obvious. So, you take various rings and take the common elements of these rings common subsets. Now, if two elements have this common subset, then the union is in all of them. And therefore, it is an intersection difference in all of them and therefore, this similar intersection of sigma rings is a sigma ring.

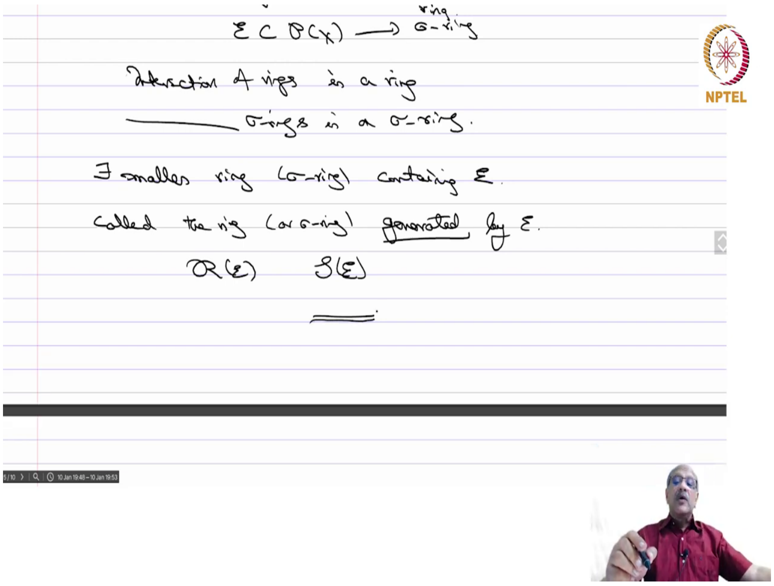
For the same reason, you take a countable set in the common collection, then countable union is going to be in all of them and therefore, it will be in the common one and then differences are also there. So, the intersection of rings sigma now, and always you have a power set is a ring or a sigma ring which contains S.

(Refer Slide Time: 24:02)



So, you can take the intersection of all rings containing E or the intersection of all sigma rings containing E and that will be a ring that will be so, there exists smallest ring or sigma ring containing E so, this is called the ring or sigma ring generated by E.

(Refer Slide Time: 24:42)



So, and then the notation for this is if it is a ring you have $R(E)$ and if it is a sigma ring, we will call it $S(E)$. So, given any arbitrary collection of sets, we will choose the smallest ring which contains it and all of the smallest sigma ring which contains it and that will be called $R(E)$ or $S(E)$. So this is the basic definition. So, we will now see some examples. So, next our aim is to construct a measure on a ring or a sigma so that is what we will do.