Measure and Integration
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Lecture no-19
3.6 - Measurable functions

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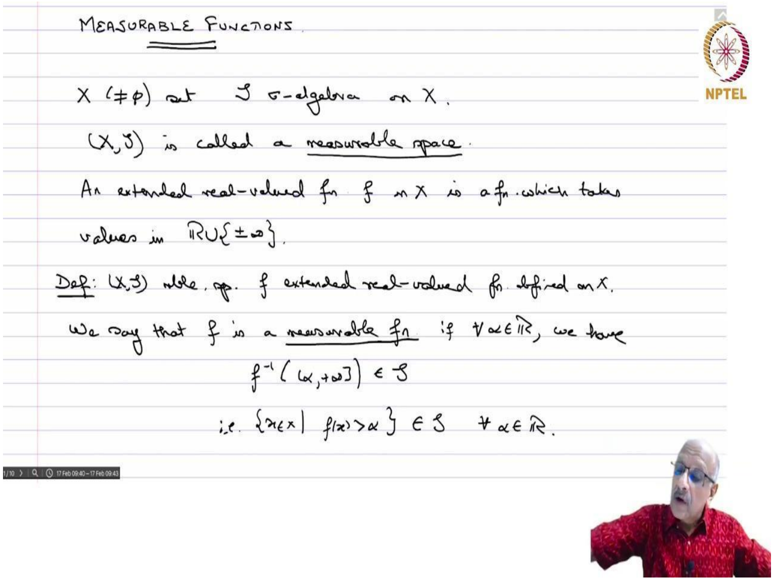

Measurable Functions:

Today we will start a new chapter. So, we will now talk about **Measurable Functions**. So, first we will study some of the basic properties and definition of measurable functions. So, X is a non empty set and S is a σ algebra will work only with sigma algebra in future on X. So, then we will say (X, S), is called a measurable space; it means it has all the structures necessary to define a measure. So, it has a non empty set and you have a sigma algebra and so, if possible one can define a measure on it.

So, an extended real valued function f on X is a function which takes values in $\mathbb{R} \cup \{+\infty, -\infty\}$ so, you allow infinite values also. So, now

Definition (X, S) measurable space, f extended real valued function defined on X. We say that f is a measurable function, if $\alpha \in \mathbb{R}$, *we have*

$$f^{-1}((\alpha, +\infty]) \in S$$

i.e., $\{x \in X : f(x) > \alpha\} \in S$, [redacted].

So, then such a function is called a measurable function.

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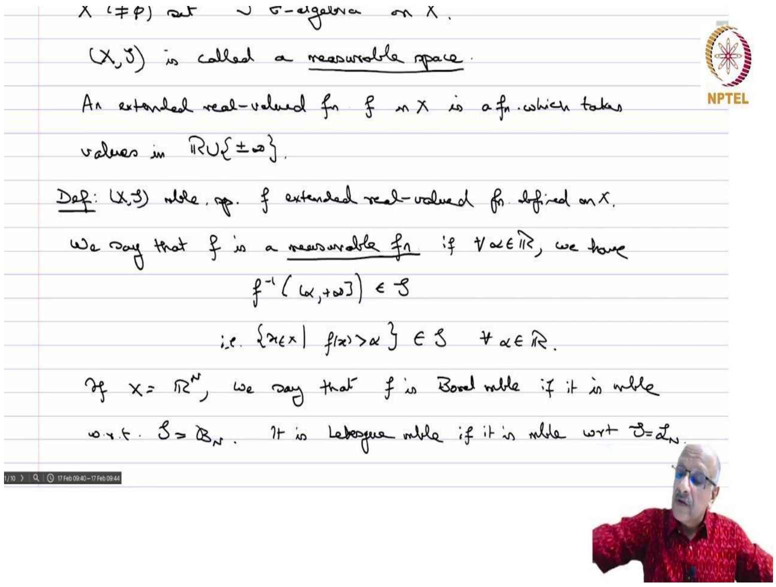

So, if $X = \mathbb{R}^N$. We say that f is Borel measurable if it is measurable with respect to $X = B_N$ It is Lebesgue measurable if it is measurable with respect to $X = L_N$. So, if you have the Lebesgue sigma algebra and the function is measurable. Then you say it is a Lebesgue measurable function and otherwise, if it is if the sigma algebra is the Borel sigma algebra then you see this Borel measurable.

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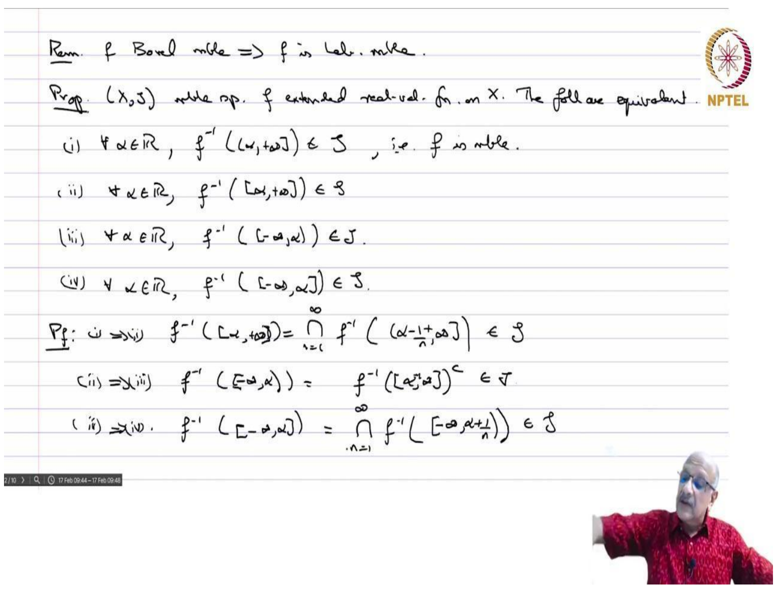

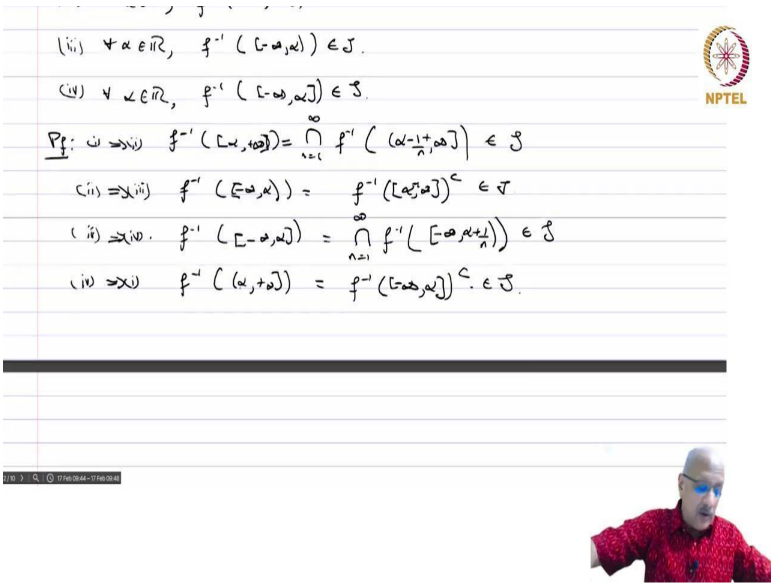

Remark, f Borel measurable in place of course, f is Lebesgue measure but not the converse. So,

first proposition, there is nothing sacred about the way we have defined measurability. So, you have the following various equivalent forms access measurable space f extended real valued function on X, the following are the equivalent.

Proposition:, (X, S) measurable space. f extended real valued function on X. The following are equivalent:

(i) $\forall\, \alpha \in \mathbb{R}$, $f^{-1}((\alpha, +\infty]) \in S$, $i.e., f\ is\ measurable.$

(ii) $\forall\, \alpha \in \mathbb{R}$, $f^{-1}([\alpha, +\infty]) \in S$

(iii) $\forall\, \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha)) \in S$

(iv) $\forall\, \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha]) \in S$

So, one of each of these in place others and therefore, you could have defined measurability using any of them.

Proof, (i) implies (ii), so

$$f^{-1}([\alpha, +\infty]) = \bigcap_{n=1}^{\infty} f^{-1}((\alpha - \frac{1}{n}, \infty]) \in S.$$

so (ii) implies (iii), so $f^{-1}([-\infty, \alpha)]) = f^{-1}([\alpha, +\infty])^{c} \in S.$

. Let us say plus infinity always because this belongs to S, so, this complement is also S.

So, (iii) implies (iv)

$$f^{-1}([-\infty, \alpha]) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, \alpha - \frac{1}{n})) \in S$$

and therefore, again this belongs S because by (iii) each of this is in S and intersection is in S . And then

(iv) to (i) $f^{-1}((\alpha, +\infty]]) = f^{-1}([-\infty, \alpha])^{c} \in S.$

and consequently you have all these things are equal.

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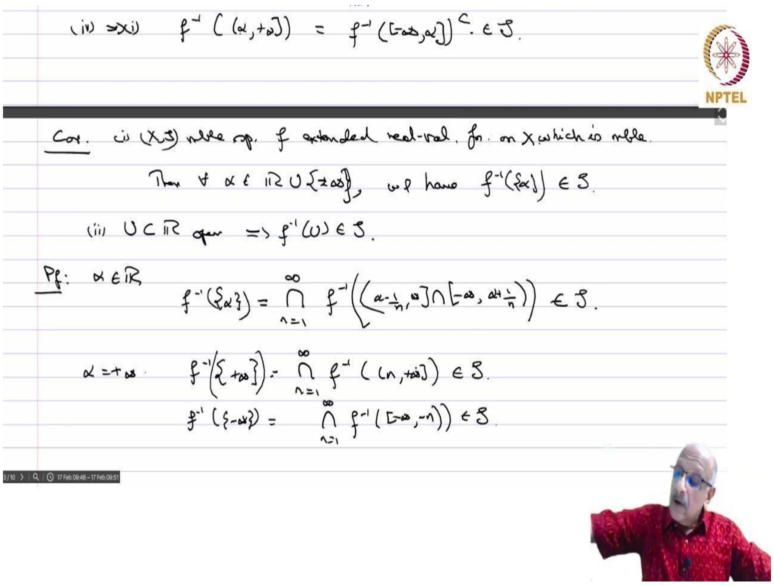

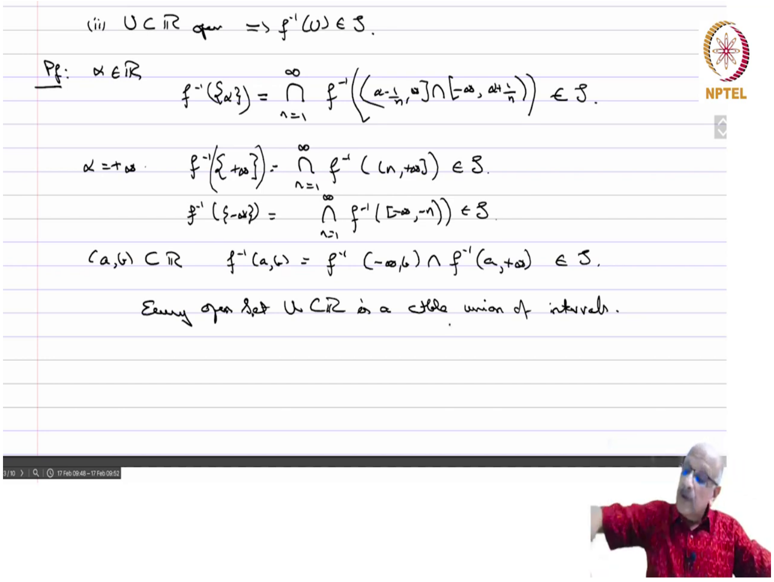

Corollary 1:, (X, S). measurable space, f extended real valued function which is measurable then $\alpha \in \mathbb{R}$, *we have*

(i) $f^{-1}(\{\alpha\}) \in S$

(ii) $U \subset \mathbb{R}\ open \Rightarrow f^{-1}(U) \in S$.

Proof, so let us take $\alpha \in \mathbb{R}$ then

$$f^{-1}(\{\alpha\}) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left(\alpha - \frac{1}{n}, +\infty\right) \cap \left[-\infty, \alpha + \frac{1}{n}\right]\right) \in S.$$

$$\text{If } \alpha = +\infty,\ f^{-1}(\{+\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, +\infty]) \in S.$$

$$\text{If } \alpha = -\infty,\ f^{-1}(\{-\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, -n)) \in S.$$

Now, if $(a, b) \subset \mathbb{R}$ then you have

$$f^{-1}(a, b) = f^{-1}(-\infty, b) \cap f^{-1}(a, +\infty) \in S.$$

Now, every open set of U is a countable union of intervals and therefore, this corollary is proved.

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Example 1. $X = \mathbb{R}^N$ $f: X \to \mathbb{R}$ cont. f is both Leb & Borel mble.

$$f^{-1}((\alpha, +\infty]) = f^{-1}((\alpha, +\infty)) = \text{open} \in \text{Borel \& Leb}.$$

Eg. 2 $E \subset [0,1)$ E not Leb mble. $E \notin \mathcal{L}_1$.

Define $f(x) = \begin{cases} x & \text{if } x \in E. \\ -x & \text{if } x \in [0,1) \setminus E. \\ -2 & x \notin [0,1). \end{cases}$

$$\begin{aligned} f^{-1}(\{\alpha\}) &= \mathbb{R} \setminus [0,1) \quad \text{if } \alpha = -2. && \in \mathcal{L}_1 \\ &= \{-\alpha\} \quad \text{if } -\alpha \in [0,1) \setminus E. && \in \mathcal{L}_1 \\ &= \{\alpha\} \quad \text{if } \alpha \in E. && \in \mathcal{L}_1 \\ &= \phi \quad \text{otherwise}. && \in \mathcal{L}_1. \end{aligned}$$

Cor. (i) $(X, \mathcal{S})$ mble sp. f extended real-val. fn. on X which is mble.

Then $\forall\, \alpha \in \mathbb{R} \cup \{\pm\infty\}$, we have $f^{-1}(\{\alpha\}) \in \mathcal{S}$.

(ii) $U \subset \mathbb{R}$ open $\Rightarrow f^{-1}(U) \in \mathcal{S}$.

Pf: $\alpha \in \mathbb{R}$

$$f^{-1}(\{\alpha\}) = \bigcap_{n=1}^{\infty} f^{-1}\left(\left(\alpha - \tfrac{1}{n}, \infty\right] \cap \left[-\infty, \alpha + \tfrac{1}{n}\right)\right) \in \mathcal{S}.$$

$\alpha = +\infty$. $f^{-1}(\{+\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}((n, +\infty]) \in \mathcal{S}$.

$f^{-1}(\{-\infty\}) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty, -n)) \in \mathcal{S}$.

$(a,b) \subset \mathbb{R}$ $f^{-1}(a,b) = f^{-1}(-\infty, b) \cap f^{-1}(a, +\infty) \in \mathcal{S}$.

Every open set $U \subset \mathbb{R}$ is a ctble union of intervals.

Eg. 2 $E \subset [0,1)$ E not Leb mble. $E \notin \mathcal{L}_1$.

Define $f(x) = \begin{cases} x & \text{if } x \in E. \\ -x & \text{if } x \in [0,1) \setminus E. \\ -2 & x \notin [0,1). \end{cases}$

$$\begin{aligned} f^{-1}(\{\alpha\}) &= \mathbb{R} \setminus [0,1) \quad \text{if } \alpha = -2. && \in \mathcal{L}_1 \\ &= \{-\alpha\} \quad \text{if } -\alpha \in [0,1) \setminus E. && \in \mathcal{L}_1 \\ &= \{\alpha\} \quad \text{if } \alpha \in E. && \in \mathcal{L}_1 \\ &= \phi \quad \text{otherwise}. && \in \mathcal{L}_1. \end{aligned}$$

$$f^{-1}((0, \infty]) = E \notin \mathcal{L}_1 \Rightarrow f \text{ not (Leb) mble}.$$

So, now, we give some examples, so

Example: let us take $X = \mathbb{R}^N$, then $f: X \to \mathbb{R}$ continuous, continuous and real valued function, I am not taking extended real valued functions, then f is both Lebesgue and Borel measurable because

$$f^{-1}((\alpha, +\infty]) = f^{-1}((\alpha, +\infty)) = open \in Borel\, set$$

So, this is equal to open and therefore, which belongs to Borel and Lebesgue sigma g plus. So, this is a really easy example.

So, example two, so, here we give a counterexample. So, here we proved in this corollary, that f^{-1} of every singleton is measurable if S is measurable, the converse is not true. So, f^{-1} of singletons all singletons can be measurable, but still the function may fail to be measurable. So, let us take the following example.

Let us take $E \subset [0, 1)$, not measurable. So, E does not belong to L_1. Now, you define

$$f(x) = x \;\; if \;\; x \in E$$

$$= -x \;\; if \;\; x \in [0, 1)\backslash E$$

$$= -2 \;\; if \;\; x \notin [0, 1).$$

So, then let us compute what is the

$$f^{-1}(\{\alpha\}) = \mathbb{R}\backslash[0.1) \;\; if\, \alpha = -2$$

$$= \{-\alpha\} \;\; if \;\; \alpha \in [0, 1)\backslash E$$

$$= -2 \;\; if \;\; \alpha \in E$$

$$= \Phi \;, \; otherwise.$$

so, you can really easily verify all this. So, all of these belong to S. So, all of these, so, every $f^{-1}(\{\alpha\})$ is in this. Now if you take

$$f^{-1}((0, +\infty]) = E \notin L_1$$

which are the various only place where it takes positive values is on E and this is equal to E but this does not belong to L_1 implies f not Lebesgue measure.

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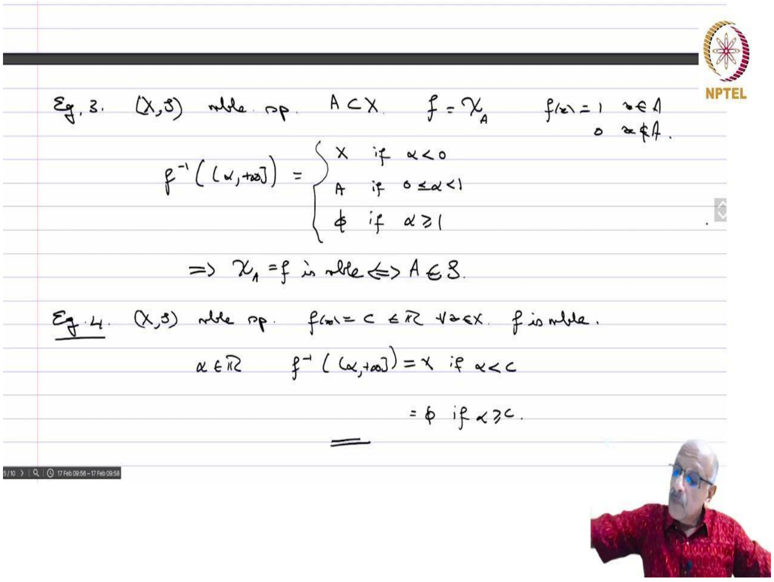

So, measurability of singletons is not a test then example again.

Example 3: Let (X, S) measurable space and let $A \subset X$. Then you take

$$f^{-1}(\ (\alpha, +\ \infty]) =\ X\ \ if\ \ \alpha < 0$$

$$= A\ \ if\ \ 0 \leq \alpha < 1$$

$$= \Phi\ \ if\ \ \alpha \geq 1.$$

and therefore, the same place $\chi_A = f$ is measurable if and only if $A \in S$. So, whenever we deal with simple functions we will deal with simple measurable functions. That means, we are talking about characteristic functions of measurable sets.

Finally,

Example 4: (X, S) measurable space and $f(x) = C \in \mathbb{R}$, $\forall x \in X$ then f is measurable because, if

$$\alpha \in \mathbb{R},\ \ f^{-1}((\alpha, +\ \infty]) = X\ if\ \ \alpha < C$$

$$= \Phi\ \ if\ \ \alpha \geq C.$$

So, we have some examples of it.

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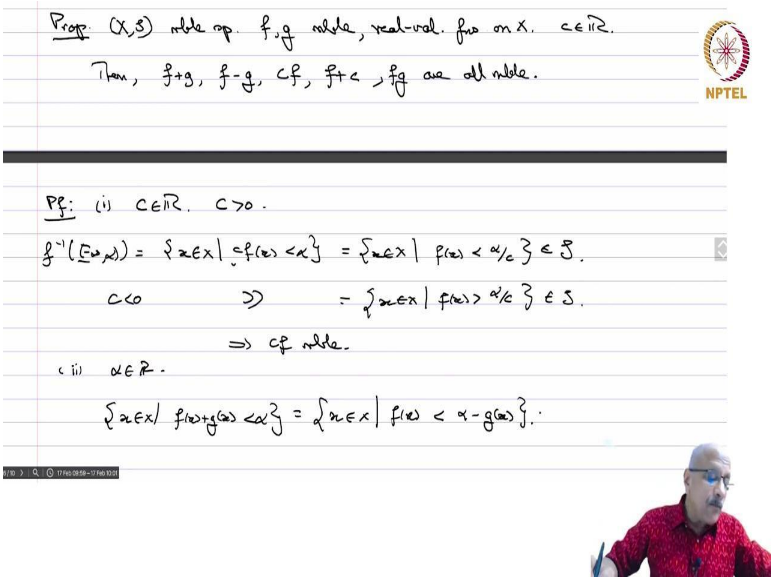

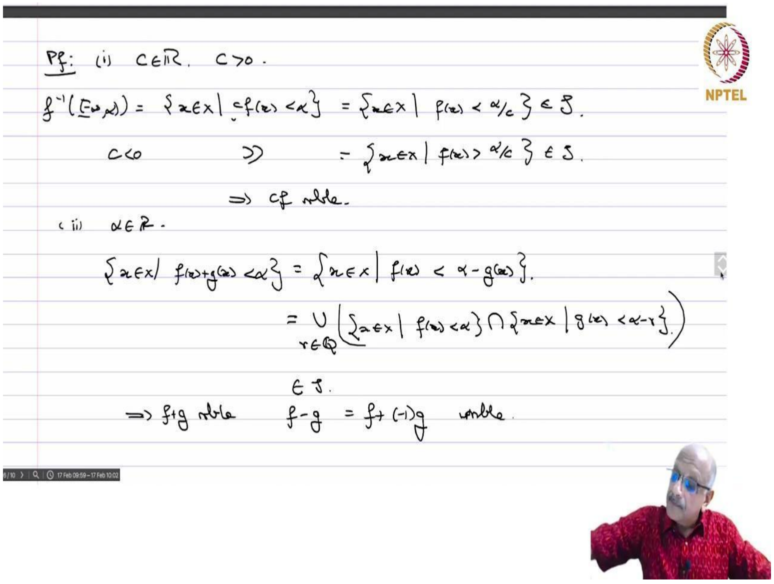

So, now we will prove a proposition which is useful. So,

Proposition: Let (X, S) measurable space f, g measurable real value not extended real valued, real valued functions on X, $C \in \mathbb{R}$, then $f + g$, $f - g$, Cf, $f + C$, fg are all measurable.

Proof, one, let us $C \in \mathbb{R}$, $C > 0$. So,

$$f^{-1}([-\infty, \alpha)) = \{x \in X : Cf(x) < \alpha\} = \left\{x \in X : f(x) < \frac{\alpha}{C}\right\}$$

and that of course is measureable because f is measurable. Similarly, if

$f^{-1}([-\infty,\alpha)) = \{x \in X : Cf(x) < \alpha\} = \left\{x \in X : f(x) > \frac{\alpha}{C}\right\}$

So, for every alpha this is measurable therefore, by the characterization we have so, this implies that cf measurable.

(ii) let $\alpha \in \mathbb{R}$, ,

$$\{x \in X : f(x) + g(x) < \alpha\} = \{x \in X : f(x) < \alpha - g(x)\}$$

that means, you can put in a rational between $f(x)$ and alpha minus $g(x)$ and therefore,

$$= \bigcup_{r \in \mathbb{Q}} (\{x \in X : f(x) < r\} \cap \{x \in X : g(x) < \alpha - r\})$$

so, each of this is in S and $\mathbb{Q}$ is countable therefore, this belongs to S.

So, $f + g$ is measurable, $f - g$ is nothing but$f + (-g)$, f is measurable $-g$ is measurable by the first argument and therefore, this measurable.

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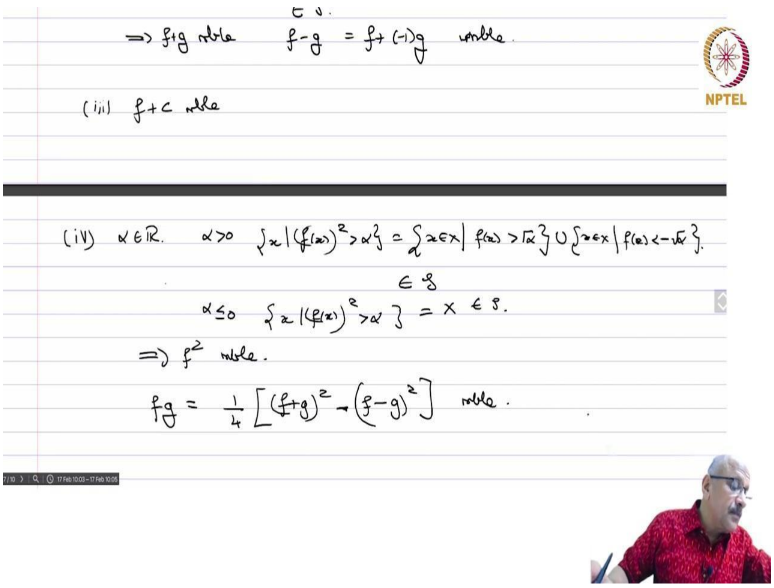

(iii), $f + C$ measurable because constant functions are measurable, f is measurable sum of measurable functions is measurable which we have already proved. So, finally, we want to prove the product.

(iv) So, let $\alpha \in \mathbb{R}$, $\alpha > 0$,

$$\left\{x \in X : f(x)^2 > \alpha\right\} = \left\{x \in X : f(x) > \sqrt{\alpha}\right\} \cup \left\{x \in X : f(x) < -\sqrt{\alpha}\right\}$$

and therefore, this implies, so this will belong to S. If $\alpha \leq 0$,

$$\{x \in X : f(x)^2 > 0\} = X$$

belongs to S and therefore, from this we deduce. So, this implies it f^2 is measurable.

Now, $fg = \frac{1}{4}\left[(f + g)^2 - (f - g)^2\right]$.

Now $f + g$ is measurable so, the square is measurable. $f + g$ is measurable, so the square is measurable. The difference of measurable functions is measurable multiplying by a constant 1 by 4 is measurable, so, this is measurable. So, this proves.

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Remark: About proposition holds whenever given functions are well defined when f, g are extended real value. Functions if $f + g$ not defined on at x where $f(x) = +\infty$ and $g(x) = -\infty$, you cannot define f plus g. So, whenever the function is well defined then the previous proofs will all go through other ways. So, we will continue with the properties of measurable functions next.