Measure and Integration
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Lecture-18
3.5 - Exercises

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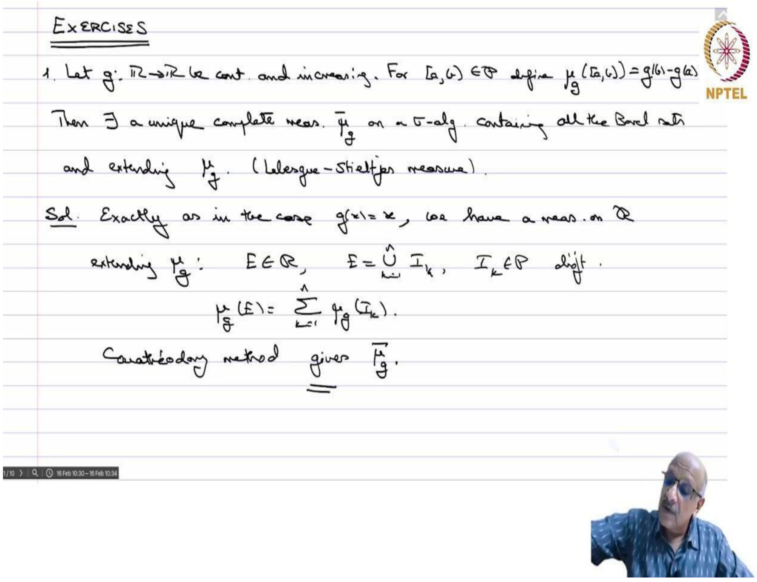

(1) : Let $g : \mathbb{R} \to \mathbb{R}$ be continuous and increasing. For $a, b \epsilon P$ define

$\mu_g([a, b)) = g(b) - g(a)$.

Then there exists a unique complete measure $\overline{\mu_g}$ on σ-algebra containing all the Borel sets and extending μ_g.

(So, this is called the Lebesgue Stieltjes measure.)

Solution: Exactly as in the case $g(x) = x$, we have a measure on $\mathbb{R}$.

extending μ_g: $E \epsilon \mathbb{R},\ E = \bigcup_{k=1}^{n} I_k,\ I_k \epsilon P$ and disjoint.

Then you define

$$\mu_g(E) = \bigcup_{k=1}^{n} \mu_g(I_k).$$

Once you do this then the caratheodory method gives $\overline{\mu}_g$.

So, you go to the hereditary σ- algebra generated by it which is turned out to be again the power set because you are dealing only with the *P* and ℝ again and therefore, and then you will get the μ-measurable sets which will give you a complete measure and we know from the abstract theory that it contains ℝ and therefore, the sigma algebra generated by ℝ namely all the Borel sets. So, this completes proof of this.

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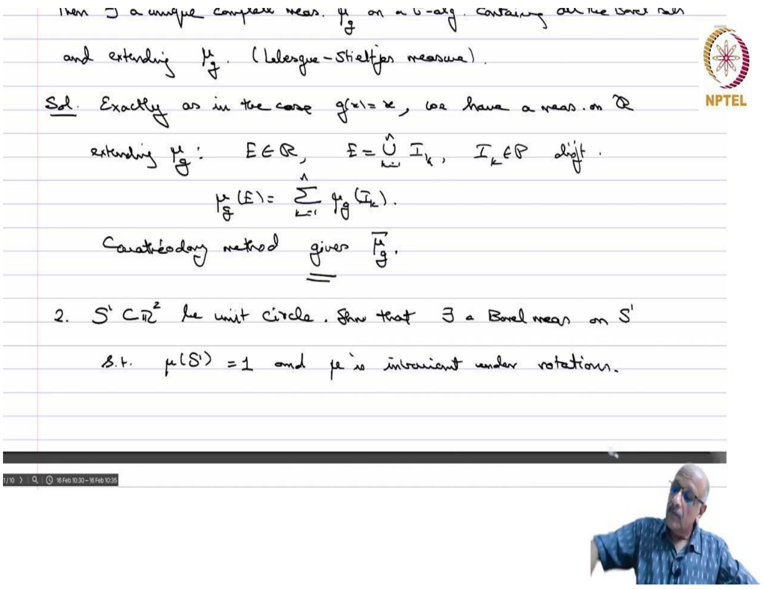

(2): Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Show that ∃ a Borel measure on S^1 (S^1 is a topological space because it inherits the topology of $\mathbb{R}^2$ and therefore, you have open sets and the sigma algebra generated by those open sets are called the Borel sets in S^1.) such that $\mu(S^1) = 1$ and μ is invariant under rotations.

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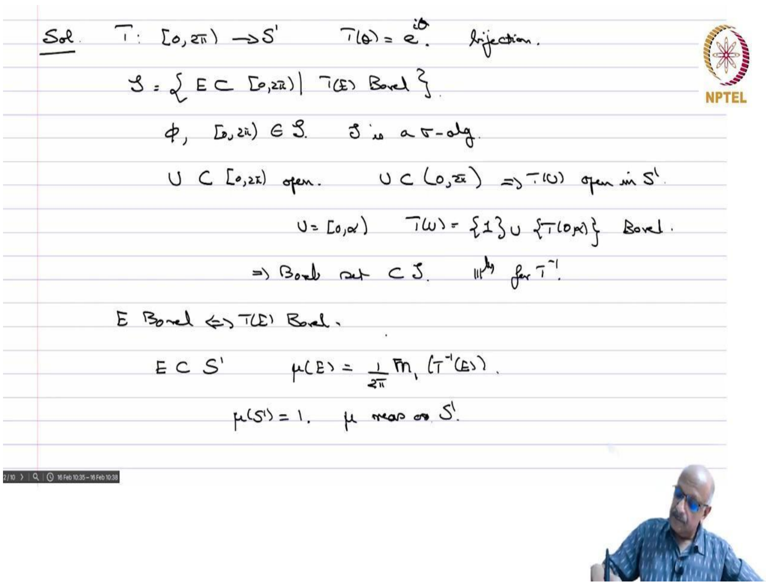

Solution: Take $T: [0, 2\pi) \rightarrow S^1$. So, $T(\theta) = e^{i\theta}$ is a bijection. So, if you define

$$S = \{E \subset [0, 2\pi) | \, T(E) = Borel\},$$

So obviously $\phi, [0, 2\pi) \in S$ and S is a σ- algebra because of this objection, so it is closed under complementation and of course, under the union, so, this σ- algebra.

Now if $U \subset [0, 2\pi)$ *open*.

So, there will be two possibilities $U \subset (0, 2\pi)$ and this implies $T(U)$ is also open in S^1 by the mapping here.

And then if you can also have you is something like $[0, \alpha)$ then

$$T(0) = \{1\} \cup \{T[0, \alpha)\} \text{ is Borel.}$$

Again and therefore, all open sets so, implies Borel sets contained in S. So, similarly for T^{-1}. therefore, E Borel if and only if $T(E)$ Borel.

Now, we will take $E \subset S^1$ and then you define

, then μ is a measure on S^1.

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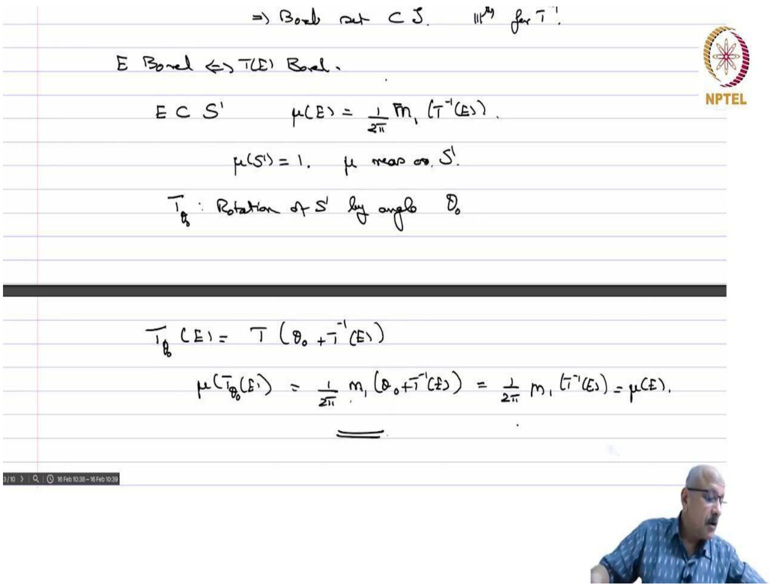

Now, we want to show that rotation invariance. So, you have led T_{θ_0} is rotation of S^1 by an θ_0. So, you take this circle and you rotate it you get back the circle again and so, you have this now,

$$T_{\theta_0}(E)=T(\theta_0+T^{-1}(E)),$$

and therefore, $T_{\theta_0}(E) = \frac{1}{2\pi} m_1(\theta_0 + T^{-1}(E)) = \frac{1}{2\pi} m_1(T^{-1}(E)) = \mu(E\ ).$

Therefore, it is invariant under rotations.

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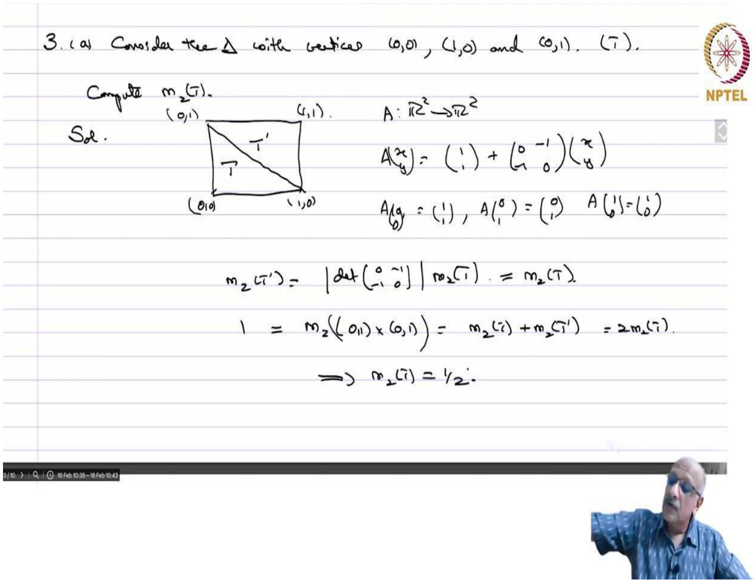

(3): Consider the triangle with vertices (0, 0), (1, 0) and (0, 1) and call this triangle T. Compute $m_2(T)$.

{So, we know what it is because we know that essentially this area but we have to prove that and therefore we have it is half base into height so, the answer should be half we know this. So, let us try to show it exactly. }

__Solution:__ T is a triangle with vertices (0, 0) (1, 0), (0, 1). Now, you look at

$A: \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathrm{A}(x, y) = (1\ 1) + (0\ \ -1; -1\ \ 0)(x\ y)$$

I am writing just a reflection along this line, what are you doing here this is nothing but the reflection along this diagonal here.

$$A(0, 0) = (1\ 1), A(0, 1) = (0\ 1), A(1, 0) = (1\ 0).$$

So, if you call the upper triangle T', then

$m_2(T')$ is nothing but by translation in radians, this (1, 1) does not matter.

So, $m_2(T') = |det(0, -1; -1\ 0)| m_2(T) = m_2(T)$.

And then the single line is one dimensional. So, it is measured as 0 and therefore, we have

$$m_2((0, 1) \times (0, 1)) = m_2(T) + m_2(T') = 2m_2(T).$$

$$\Rightarrow m_2(T) = \frac{1}{2}.$$

So, this is a formal proof of this.

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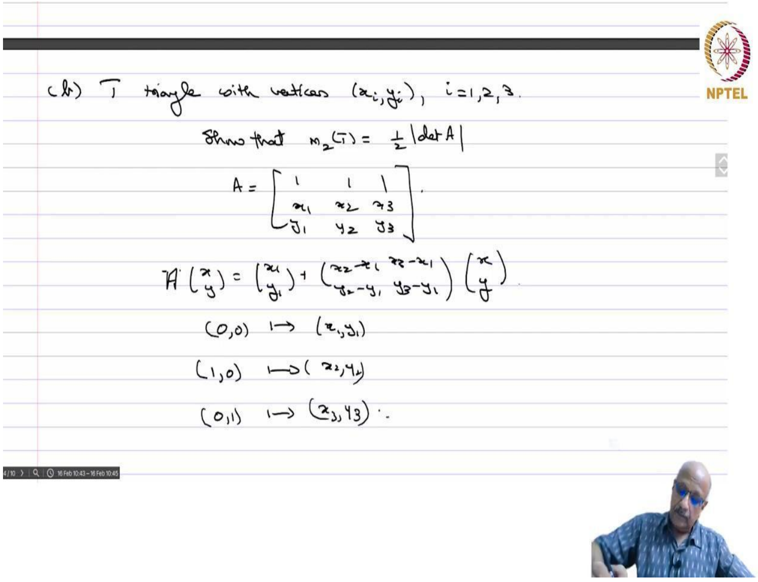

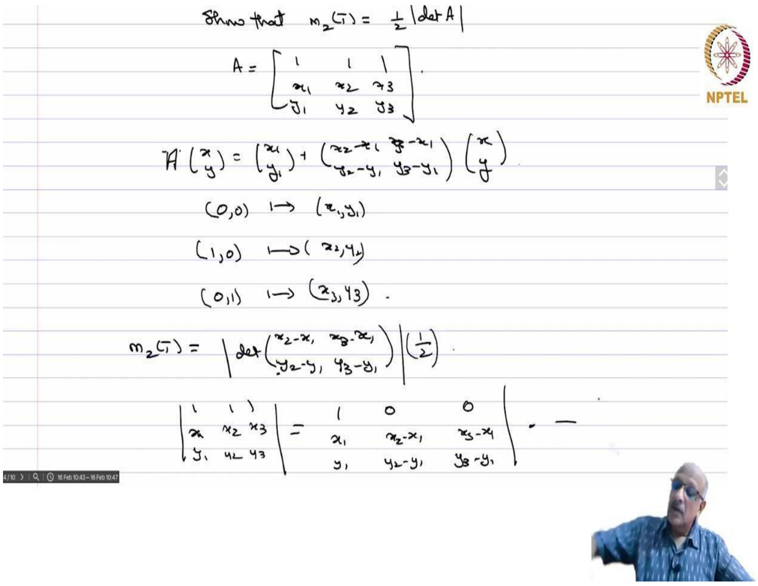

(b): T triangle with vertices $(x_i, y_i), i = 1, 2, 3$. Show that $m_2(T) = \frac{1}{2}|det(A)|$,

where A =[1 1 1; x1 x2 x3; y1 y2 y3].

So, this formula you would have been taught when you were doing analytic geometry given the three vertices how to find the area of the triangle. So, we are going to form a proof of this formula here. Again, we use the same trick as the previous exercise part.

So, we define

$$A\,(x\ y) = (x_1\, y_1) + [x_2 - x_1\ \ x_3 - x_1;\ y_2 - y_1\ \ y_3 - y_1](x\, y).$$

Then $(0, 0) \rightarrow (x_1\, y_1)$,

$(1,\ 0) \rightarrow (x_2\, y_2)$,

and $(0,\ 1) \rightarrow (x_3\, y_3)$.

So, $m_2(T) = \frac{1}{2}|det([x_2 - x_1\ x_3 - x_1;\ y_2 - y_1\ y_3 - y_1])|$.

Now, if you look at the determinant

$det([\,1\ \ 1\ \ 1\,;\ x_1\ x_2\ x_3;\ y_1\ y_2\ y_3]) = det([\,1\ \ 0\ \ 0\,;\ x_1\ x_2 - x_1\ x_3 - x_1;\ y_1\ y_2 - y_1\ y_3 - y_1])$

Therefore, hence the result, this proves.

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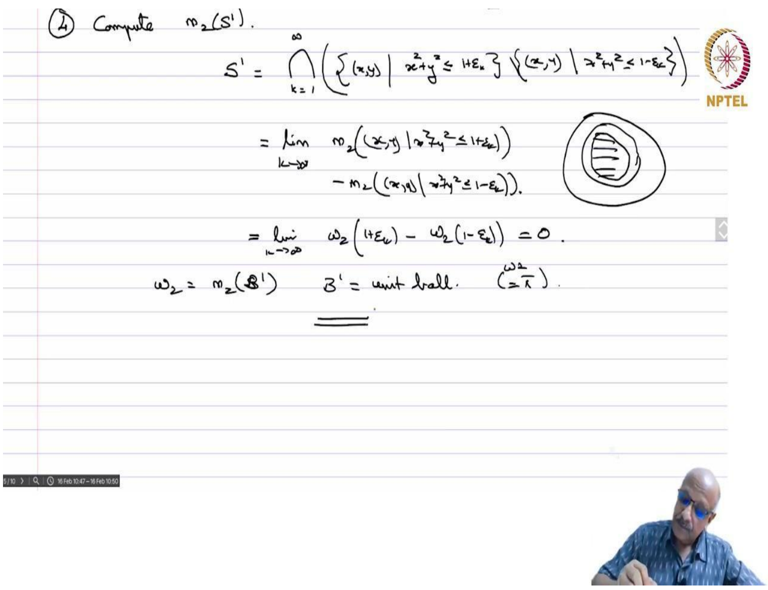

(4): Compute $m_2(S^1)$.

So, you have S^1 is unit circle and then we want to compute its Lebesgue measure in this. So, now, you take

$$S^1 = \bigcap_{k=1}^{\infty} (\{(x, y)|x^2 + y^2 \leq 1 + \varepsilon_k\}\backslash(x, y)|x^2 + y^2 \leq 1 - \varepsilon_k\})$$

(So, you take the unit circle and then you take a circle of slightly smaller radius and slightly bigger radius and then you remove this and therefore the intersection of all these and therefore, that is equal to now you have a set of decreasing sequences sets and then the intersection is S^1.)

$$= \lim_{k\to\infty} m_2(\{(x,y) | x^2 + y^2 \le 1 + \varepsilon_k\}) - m_2(\{(x,y) | x^2 + y^2 \le 1 - \varepsilon_k\})$$

$$= \lim_{k\to\infty} \omega_2(1+\varepsilon_k) - \omega_2(1-\varepsilon_k)) = 0.$$

where $\omega_2 = m_2(B^1)$, B^1= unit ball. and we know of course, that is equal to $\boldsymbol{\pi}$.

This is in the assignment you will prove that if you are given the measure of the unit ball then the measure of the ball of radius r is nothing but $\omega_2 r^2$. So, this is what we have.

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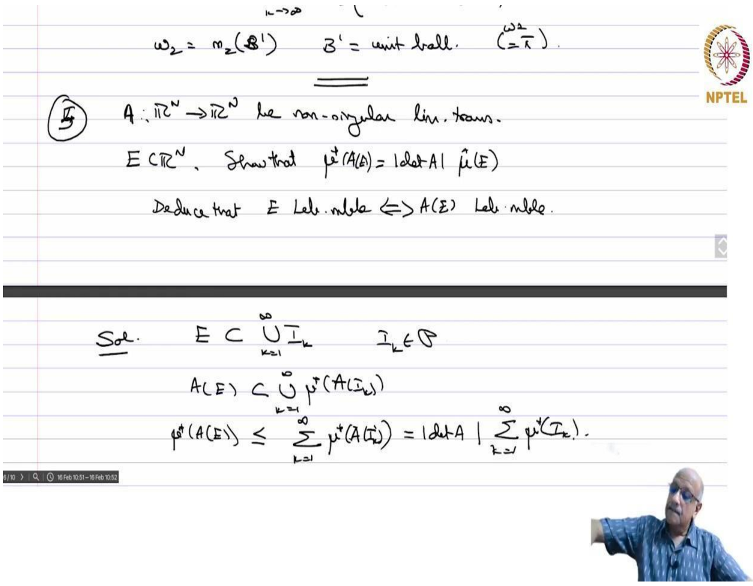

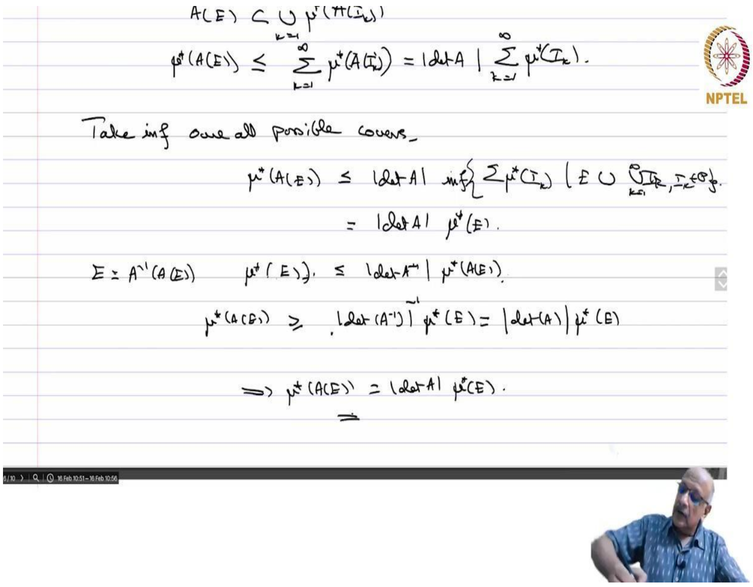

(5): Let $A: \mathbb{R}^N \to \mathbb{R}^N$ be a non-singular linear transformation. E contained in $\mathbb{R}^N$. Show that

$$\mu^*(A(E)) = |detA|\,\mu^*(E),$$

Then deduce that E Lebesgue measurable if and only if $A(E)$ is Lebesgue measure.

(We did this only for Borel sets earlier and now we can use this thing to show this.)

Solution: $E \subset \bigcup_{k=1}^{\infty} I_k$ and $I_k \epsilon P$.

So, $A(E) \subset \bigcup_{k=1}^{\infty} \mu^*(A(I_k))$.

So, $\mu^*(A(E)) \subset \sum_{k=1}^{\infty} \mu^*(A(I_k)) = |det(A)| \sum_{k=1}^{\infty} \mu^*(I_k)$.

So, if you take over all possible covers, you get

$$\mu^*(A(E)) \leq |det(A)|\, inf\{\sum_{k=1}^{\infty} \mu^*(I_k) | E \cup \bigcup_{k=1}^{\infty} I_k,\ I_k \epsilon P\}$$

$$= |det(A)|\mu^*(E).$$

$$E = A^{-1}(A(E)),\ \mu^*(E) \leq |det(A^{-1})|\mu^*(A(E)).$$

$$\Rightarrow \mu^*(A(E)) \geq |det(A^{-1})|^{-1}\mu^*(E) = |det(A)|\mu^*(E).$$

So now it is now straightforward for Lebesgue measurability.

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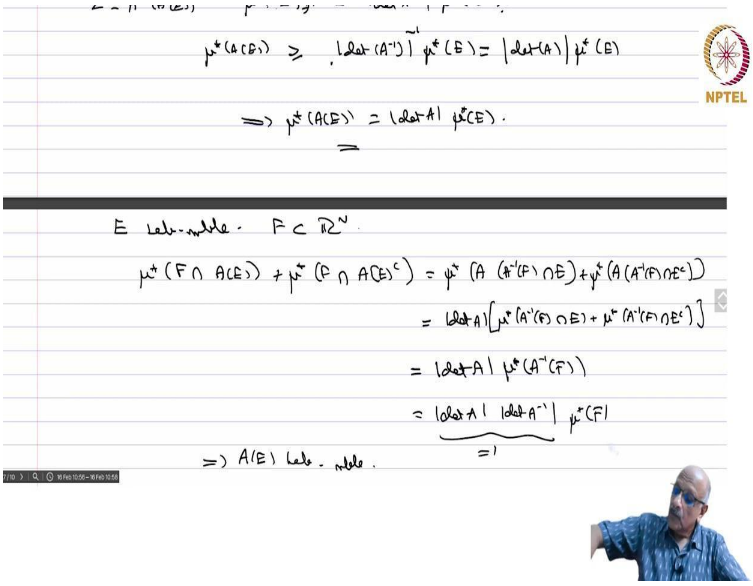

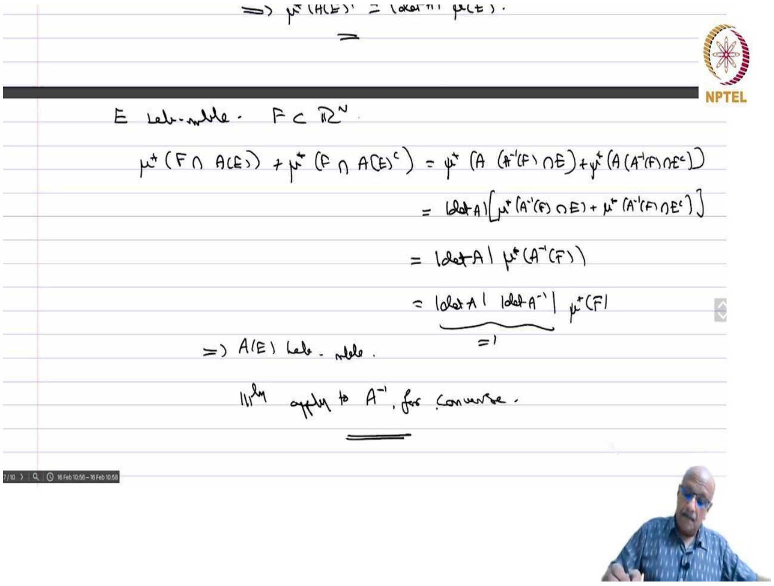

So, let us just take E Lebesgue measurable and E $\subset \mathbb{R}^N$. So, let us take

$$\mu^*(F \cap A(E)) + \mu^*(F \cap A(E)^c) = \mu^*(A(A^{-1}(F) \cap E) + \mu^*(A(A^{-1}(F) \cap E^c))$$

$$=|det(A)| [\mu^*(A^{-1}(F) \cap E + \mu^*(A^{-1}(F) \cap E^c)]$$

$$=|det(A)| \mu^*(A^{-1}(F)$$

$$=|det(A)| |det(A^{-1})| \mu^*(F)$$

$\Rightarrow A(E)$ is Lebesgue measurable.

So, now, similarly apply A^{-1} for converse.

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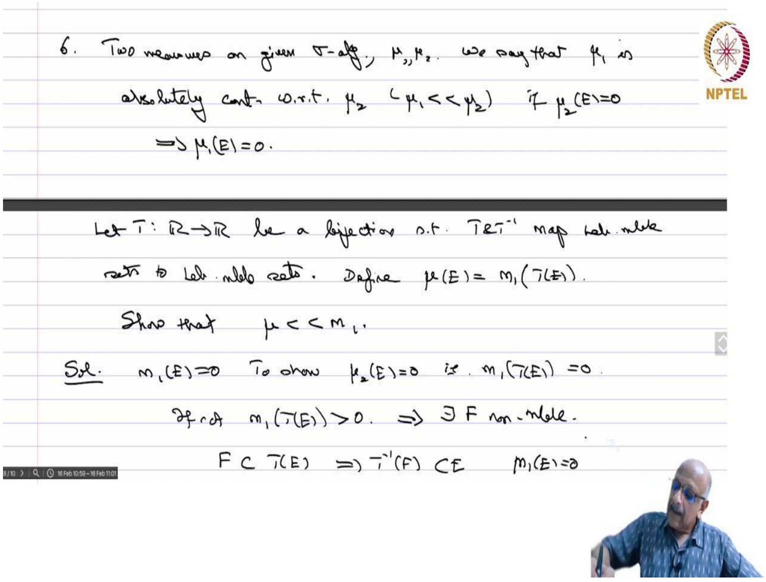

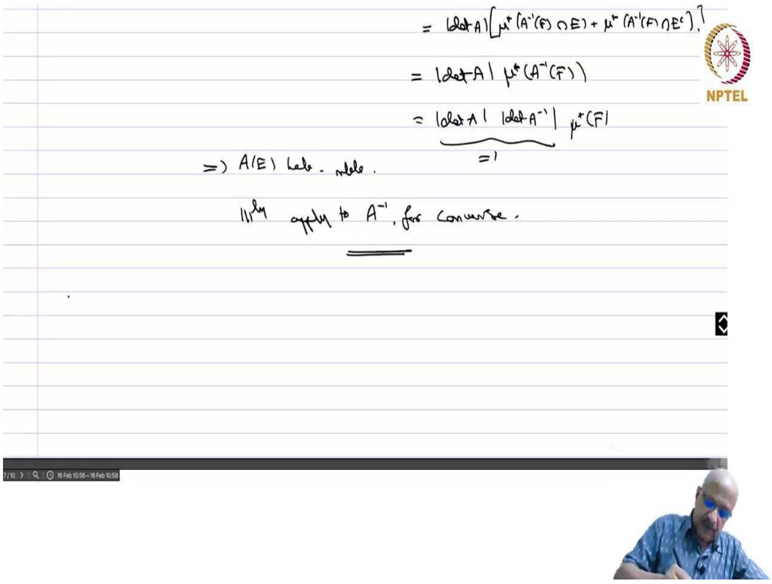

(6). If you have two measures on a given σ-algebra μ_1, μ_2, we say that μ_1 is absolutely continuous with respect to μ_2 and you write $\mu_1 << \mu_2$ this notation

$$\text{if } \mu_2(E = 0) \Rightarrow \mu_1(E) = 0.$$

Let $A: \mathbb{R} \to \mathbb{R}$ be a bijection such that T and T^{-1} map Lebesgue measurable sets to Lebesgue measurable sets define

$$\mu\ (E) = m_1(T(E)).$$

Show that $\mu \ll m_1$.

Solution: $m_1(E) = 0$ to show

$$\mu_2(E) = 0 \text{ that is } m_1(T(E)) = 0.$$

If not $m_1(T(E)) > 0$, then we know that there exists F non measurable $F \subset T(E)$.

This implies $T^{-1}(E) \subset E$ but $m_1(E) = 0$ and Lebesgue measure is complete and therefore this implies $T^{-1}(E)$ measurable implies F has to be measurable and that is a contradiction and therefore you have that $m_1(T(E)) = 0$ and therefore you have absolute continuity.