Measure and Integration
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Science
Lecture no -17
3.4 - Non-measurable sets

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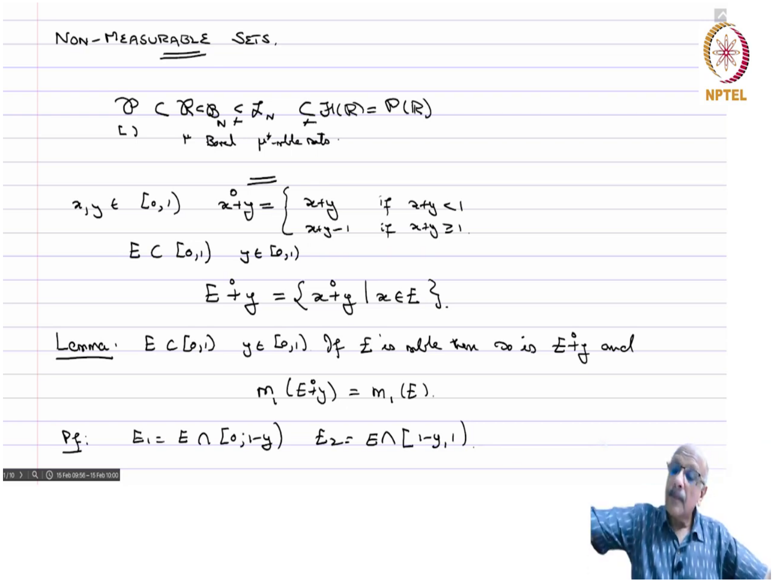
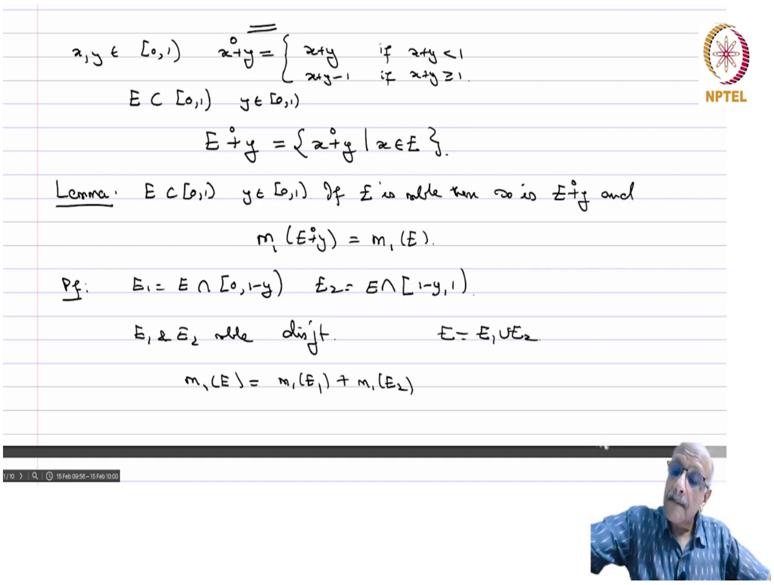




Non-Measurable Sets:

We talk a long time about non-measurable sets. So, recall how we construct the Lebesgue measure. We had P which is consisting of all finite unions of the half closed intervals and then we constructed a ring note this is just these intervals, ring is finite unions of such intervals on this we had a measure μ and then we went to the hereditary σ - ring which is

$H(\mathbb{R})$ which is nothing but the power set $P(\mathbb{R})$ of the real line $\mathbb{R}$. And then we had the μ * measurable sets which are nothing but the Lebesgue measurable sets and then we had the Borel measurable sets, which are here.

I gave you an indirect argument using cardinality that this is a strict inclusion and then we will see in the next chapter specific examples of a set. So, now, we want to show that this is also a strict inclusion, namely there exists subsets of $\mathbb{R}$ which are not Lebesgue measurable. So, that is what we want to do.

So, before we do that, let us take $x,\ y \in [0, 1)$ and you define x sum y modulo of one

$$x \overset{0}{+} y = x + y \text{ if } x + y < 1,$$

$$x \overset{0}{+} y = x + y - 1 \text{ if } x + y \geq 1.$$

So, the answers will come back into $[0, 1)$.

So, if $E \subset [0, 1)$, $y \in [0, 1)$. If E is measurable then so is

$$E \overset{0}{+} y = \{x \overset{0}{+} y |\ x \in E\}.$$

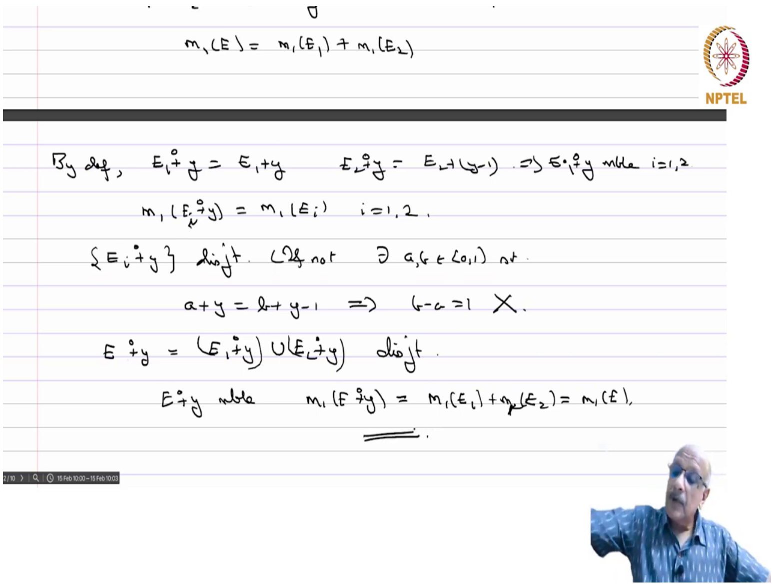
Now, we have the following lemma which is based on the translation invariance of the Lebesgue measures.

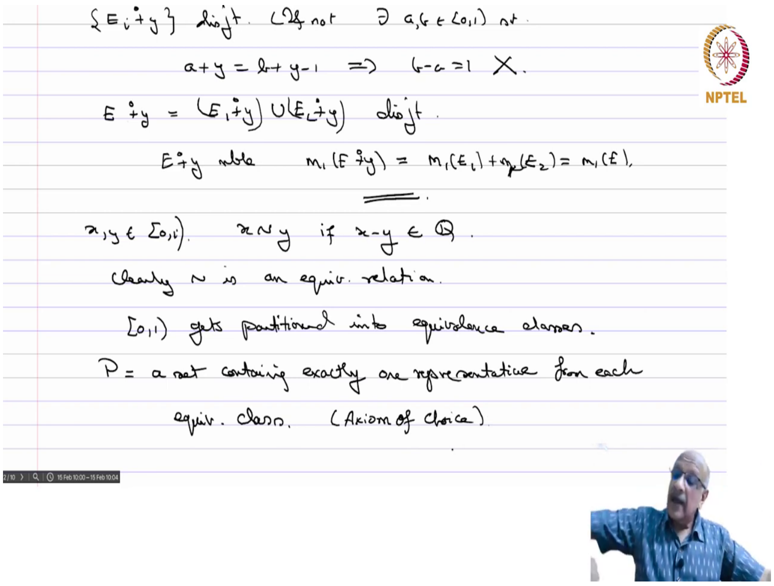
Lemma: Let $E \subset [0, 1)$ and $y \in [0, 1)$. If E is measurable then so is $E \overset{0}{+} y$ and

$$m_1(E \overset{0}{+} y) = m_1(E).$$

Proof: Let $E_1 = E \cap [0, 1 - y)$, $E_2 = E \cap [1 - y, 1)$. Then E_1 and E_2 are obviously measurable and disjoint. And

$$m_1(E) = m_1(E_1) + m_2(E_2). \text{ Because of } E = E_1 \cup E_2.$$





By definition, we have

$$E_1 +^0 y = E_1 + y$$

because you are only adding up to $1 - y$ and any element plus $1 - y$ that will be strictly less than 1 and then similarly

$$E_1 +^0 y = E_1 + (y - 1).$$

And these are translation invariants. So these implies $E_i +^0 y$ are measurable for $i = 1, 2$, and you also have

$$m_1(E_i \overset{0}{+} y) = m_1(E_i) \text{ for } i = 1, 2.$$

Now $\{E_i + y\}$ are disjoint if not there exists $a, b \in [0, 1)$ such that

$a + y = a + y - 1$ implies that mod of $b - a = 1$.

which is not possible because a and b are strictly less than 1.

So, then therefore, you have

$$E \overset{0}{+} y = E_1 \overset{0}{+} y \cup E_2 \overset{0}{+} y,$$

this is a disjoint union and therefore, $E \overset{0}{+} y$ is measurable and

$$m_1(E \overset{0}{+} y) = m_1(E_1) \cup m_1(E_2) = m_1(E).$$

So, this proves the lemma.

So, now, if $x, y \in [0, 1)$, we say that $x \sim y$ if $x - y \in \mathbb{Q}$. So, clearly $\sim$ is an equivalence relation. So, $[0, 1)$ gets partitioned into equivalence classes.

So, P equals a set containing exactly one representative from each equivalence class so, if you have $[0, 1)$ gets partition that means, the disjoint union of equivalence classes takes one representative from each equivalence. So, obviously, this is based on the axiom of choice when you have such a thing that you can find such a set is precisely the statement of the axiom of choice.

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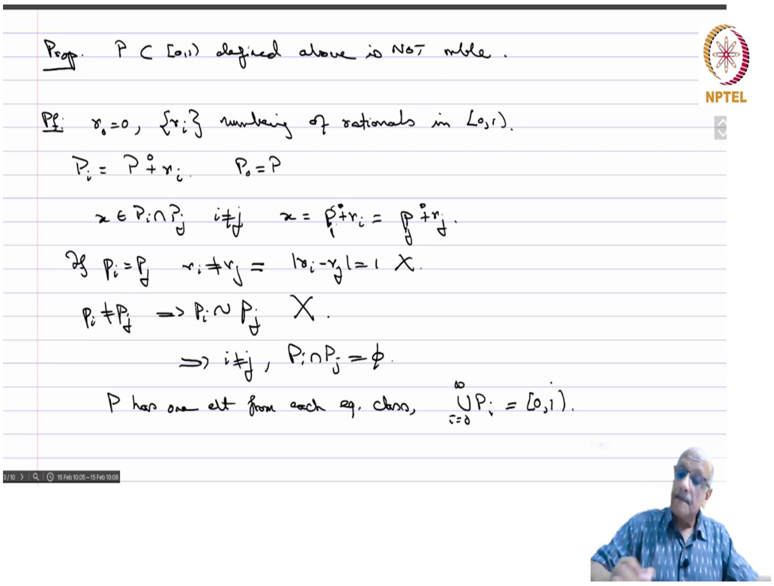

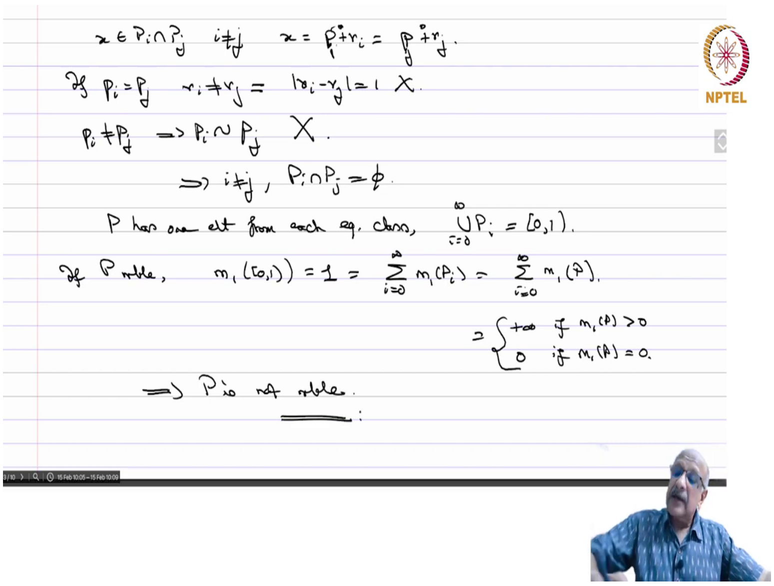

So, we have to use two properties so far, one is translation invariance of Lebesgue measure and the other is the axiom of choice. So, now proposition P contained in $[0, 1)$ defined above is not measurable. So, proof, so you said $r_0 = 0$, $\{r_i = 0\}$ numbering of rationals in $[0, 1)$, it is a countable set so, you can number it, only I put $r_0 = 0$.

So, I said $P_i = P + r_i$. So, then P_0 is the same as P because it is 0 and if x belongs to $P_i \cap P_j$, where i≠ j then $x = P_i +^0 r = x = P_j +^0 r_j$.

If $P_i = P_j$ then $r_i \neq r_j$ and therefore, this equality in place and $|r_i - r_j| = 1$ and that is not possible, it is a contradiction.

So, this means $P_i \neq P_j$, but then $P_i + r_i = P_j + r_j$. This implies that $P_i \sim P_j$ again not possible because these are distinct elements from distinct equivalence classes and therefore this is also not possible.

So, we get that if $i \neq j$, $P_i \cap P_j = \phi$.

Now, because P has one element from each equivalence class and we have taken the numbering of all the rationals therefore, we have that

$$\bigcup_{i=0}^{\infty} P_i = [0, 1).$$

So, if P is measurable,

$$m_1([0, 1)) = 1 = \sum_{i=0}^{\infty} m_1(P_i) = \sum_{i=0}^{\infty} m_1(P) = \begin{cases} +\infty & if\ m_1(P) > 0 \\ 0 & if\ m_1(P) = 0 \end{cases}.$$

So, if this has to converge, then either all of them have to be 0. So, this will be equal to $+\infty$ if $m_1(P) > 0$ and 0 if $m_1(P) = 0$. So, either ∞ or 0 it cannot be equal to one therefore this is not possible and therefore, you have that P is not measurable. So, we have explicitly constructed a subset of $[0, 1)$ which is not measured.

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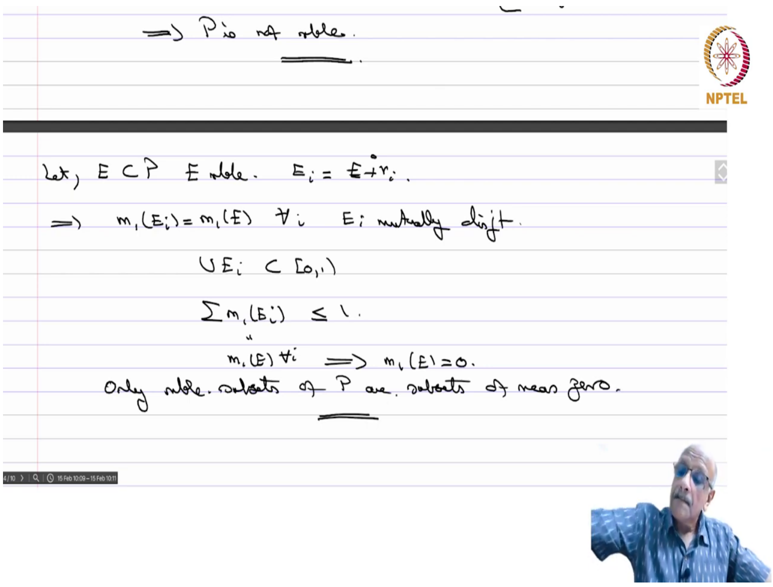

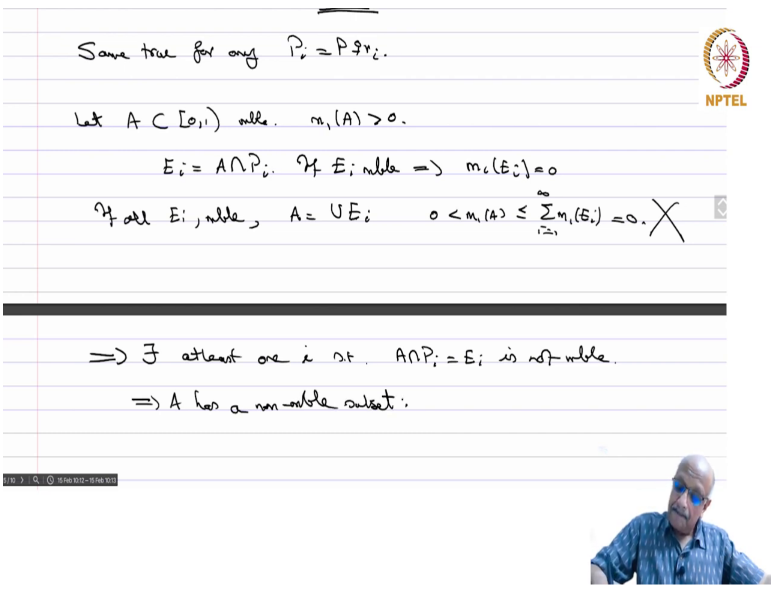

So, now, let $E \subset P$, E measurable and you said $E_i = E + r_i$.

$\Rightarrow m_1(E_i) = (E\ )$ for all i, E_i are mutually distinct by the same argument.

Now, you have

$$\cup\, E_i \subset [0, 1),$$

and therefore, you have

$\sum m_i(E_i\ ) \leq 1$ and this is equal to $m\ (E)$ for all i and therefore, this is possible only if $m_1(E) = 0$.

So, only measurable subsets of P are subsets of measure 0. Same true for any $P_i = P + r_i$. So, say they will also have all these.

Let A $\subset [0, 1)$ measurable and $m_1(P) > 0$. Now, you said $E_i = A \cap P_i$. If E_i is measurable this implies that $m_1(E_i) = 0$ because E_i is a subset of P_i and we have seen the only measurable subsets of P or any P_i are only sets of measure 0 and so, if all E_i are measurable then

$$A = \cup E_i$$

and therefore, $0 < m_1(A) \leq \sum_{i=0}^{\infty} m_1(E_i) = 0$.

So, you have another contradiction therefore, there exists at least one i such that $E_i = A \cap P_i$ is not measurable. So, implies A has a non-measurable subset. Now, you can do this in any interval.

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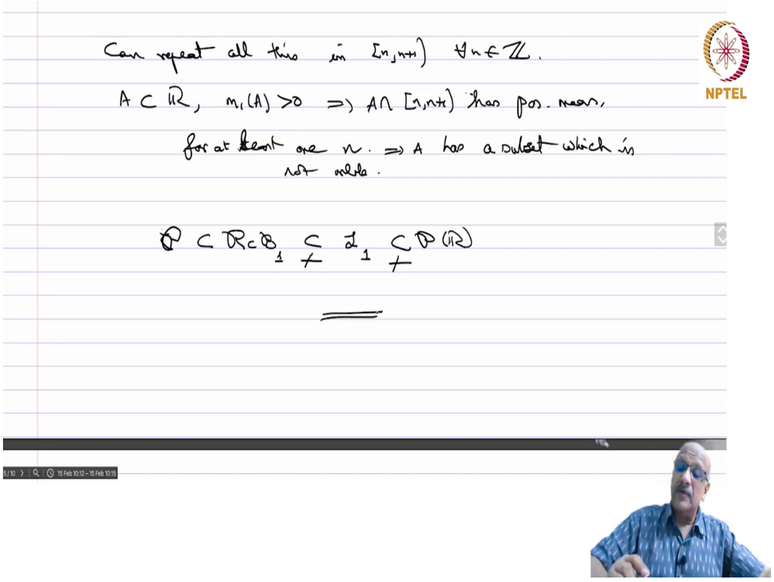

So, you can repeat all this in $[n, n + 1)$ for all $n \epsilon \mathbb{Z}$.

If A is contained in R, $m_1(A) > 0$ implies $A \cap [n, n + 1)$ has to be positive measure for at least one year and implies A has a subset which is not measurable.

So, every subset of R of positive measure has a non measurable subset. So, there are plenty of non measurable subsets and therefore, you have strict inclusion.

So, again let me recall for you, so, you have P the set of all intervals then you have R the ring and then you have the power set of the real line which headed three σ- ring and then you have L_1 which is Lebesgue measurable and then you have B_1 is the caratheodory construction. So, we have shown that this is not true and this also is strictly a thing that we will reinforce with a specific example later on right now. So, with this I will conclude this chapter on the Lebesgue measure. So, before proceeding further we will do some exercises next time.