Measure and Integration
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Lecture no-16
3.3 - Translation Invariance

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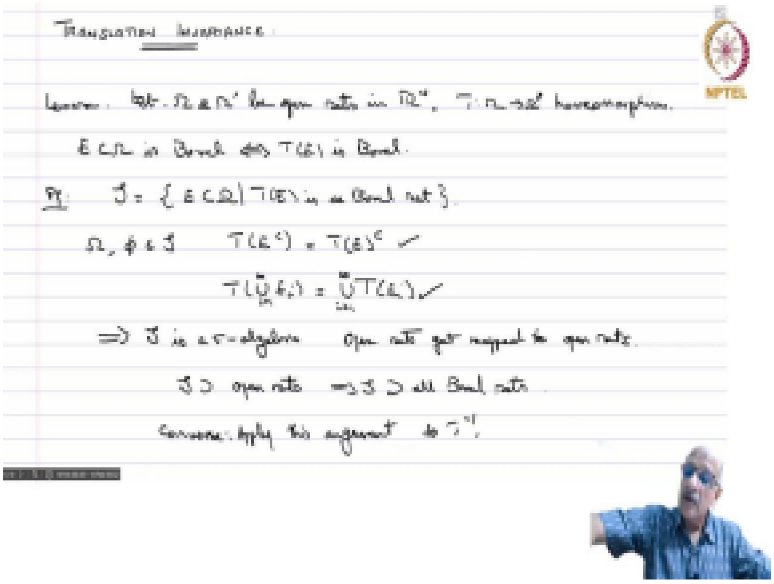

We will now discuss a very important property of the Lebesgue measure, namely Translation Invariance. So we will start with the important Lemma. So,

Lemma: Let Ω *and* Ω' be open sets in $\mathbb{R}^N$. $T: \Omega \rightarrow \Omega'$, homeomorphism. So, these are homeomorphic open sets and $E \subset \Omega$ is Borel that means, it belongs to Borel σ - algebra, if and only if $T(E)$ is Borel so, Borel sets get mapped to Borel set and nothing else can get mapped to Borel set.

Proof: we set $S = \{E \subset \Omega : T(E)\ is\ a\ Borel\ \ set\}$. So, clearly Ω and $\phi \in S$ because $T(\Omega) = \Omega'$ which is open therefore, it is Borel empty set is Borel and therefore, there is no problem. Now, if $T(E^c) = T(E)^c$.

And also $T(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} T(E_i)$

and therefore, from these two facts it follows that S is a σ algebra because this shows that is closed under countable unions and this shows that it is closed and the complementation and therefore, S is a σ algebra and now open sets get mapped to open sets because these are homeomorphism and therefore, S contains all open sets implies S contains all Borel sets. So, given any Borel set T is a Borel set. Now converse apply this argument to T^{-1} so, then automatically the converse gets true.

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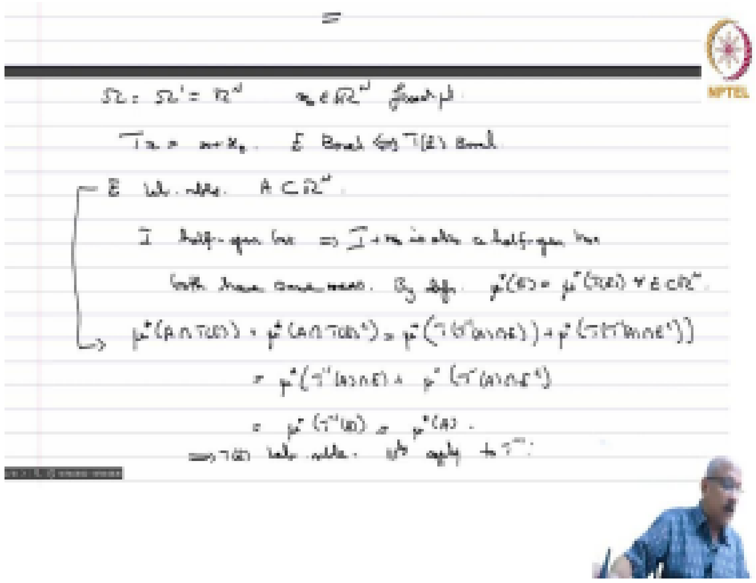

So, that proves this lemma.

So, now as a special case let us take $\Omega = \Omega' = \mathbb{R}^N$ and $x_0 \in \mathbb{R}^N$ fixed and we take

$$T(x) = x + x_0$$

So it is just translation by x_0 and therefore, that is a homeomorphism. So,

$$E \text{ Borel by Lemma if and only if } T(E) \text{ is Borel}$$

. So, now, let E be the Lebesgue measurable and let $A \subset \mathbb{R}^N$,

]we want to show the $T(E)$ is also the Lebesgue measurable.

Now before that you take if I is half open box then $I + x_0$ is also a half open box and both have the same measure because the volume of the box does not change by translation. So, from the definition how is the definition of

$$\mu^*(E) = \mu^*(T(E)), \quad \forall\ E \subset \mathbb{R}^N.$$

Now, if you take any E as a Lebesgue measurable set and $A \subset \mathbb{R}^N$, then you have let us

$$\mu^*(A \cap T(E)) + \mu^*(A \cap T(E)^c) = \mu^*(T(T^{-1}(A) \cap E)) + \mu^*(T(T^{-1}(A) \cap E^{\ c})),$$

$$= \mu^*(T^{-1}(A) \cap E) + \mu^*(T^{-1}(A) \cap E^c),$$

$$= \mu^*(T^{-1}(A)) = \mu^*(A),$$

this is $\mu^* T^{-1}(A)$ and T^{-1} are both homomorphism and we just saw they are just translations and therefore, the same as $\mu^*(A)$. So, this implies that $T(E)$ is Lebesgue measurable, similarly, applied to T^{-1} inverse. So, then we have proved the following theorem.

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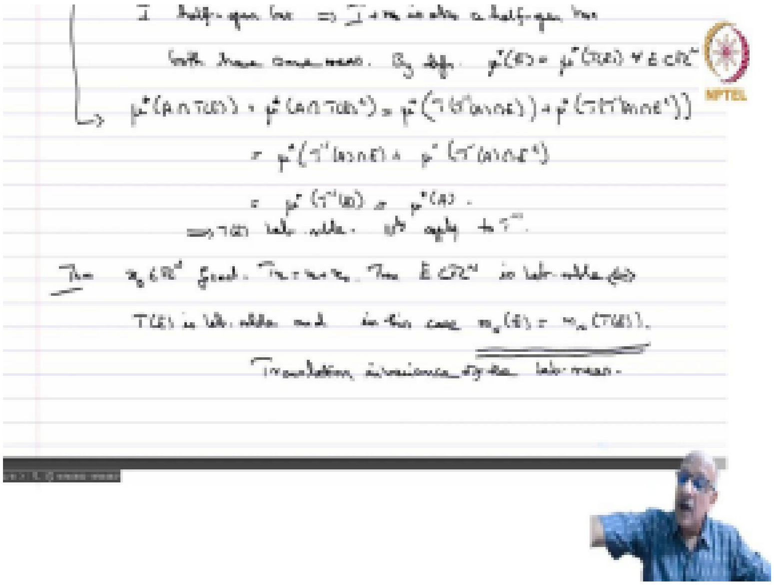

Theorem: $x_0 \in \mathbb{R}^N$ fixed $T(x) = x + x_0$, then $E \subset \mathbb{R}^N$ is Lebesgue measurable if and only if $T(E)$ is Lebesgue measurable and in this case $m_N(E) = m_N(T(E))$. So, this property is called translation invariance of the Lebesgue measure.

So, Borel measure is said to be translation invariant if the measure does not change under the translation mapping.

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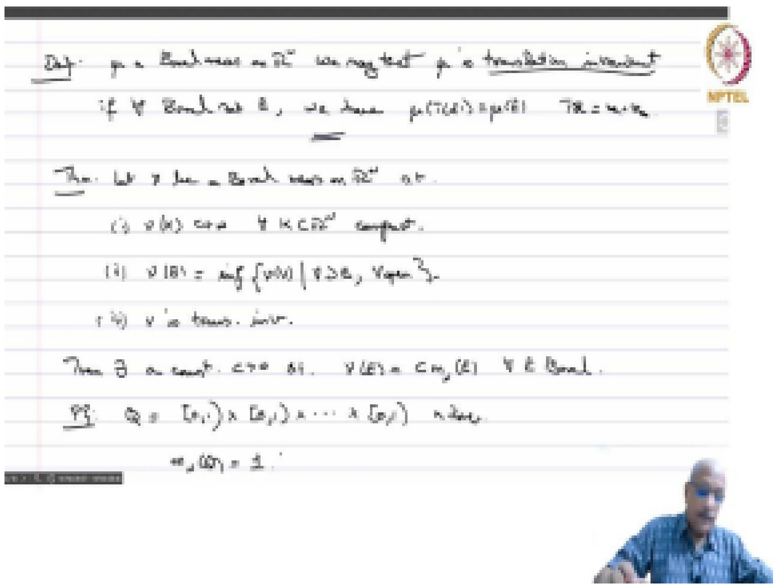

Definition: μ a Borel measure on $\mathbb{R}^N$ we say that μ is translation invariant if for every Borel set E, we have $\mu(T(E)) = \mu(E)$; $T(x) = x + x_0$.

So, now, the Lebesgue measure is translation invariant what are the other properties we have seen important properties Lebesgue measure is finite on compact sets Lebesgue measure is outer regular, namely it is determined entirely it is the infimum of μ of $\mu^*(U)$, $E \subset U$ open set containing any set E.

So now it happens these properties completely characterize Lebesgue measure. Any other measure which has the same properties will just be a constant multiple look Lebesgue measure if you take a measure multiply it by a positive constant you get a new measure, but essentially the measurable sets are the same and therefore, this so, we have the following important theorem.

Theorem: Let ν be a Borel measure on $\mathbb{R}^N$ such that

(i) $\nu(K) < \infty, \quad \forall\ K \subset \mathbb{R}^N\ compact,$

(ii) $\nu(E) = inf\{\nu(V) \ : \ V \supset E,\ V\ is\ open\}$,

(iii) ν is translation invariant.

Then there exists a constant $C > 0$ such that $\nu(E) = Cm_N(E), \quad \forall\, E\ Borel$..

Proof:

you take $Q = [0, 1) \times [0, 1) \times ... \times [0, 1)$ n times So, what is

$m_N(Q) = 1.$

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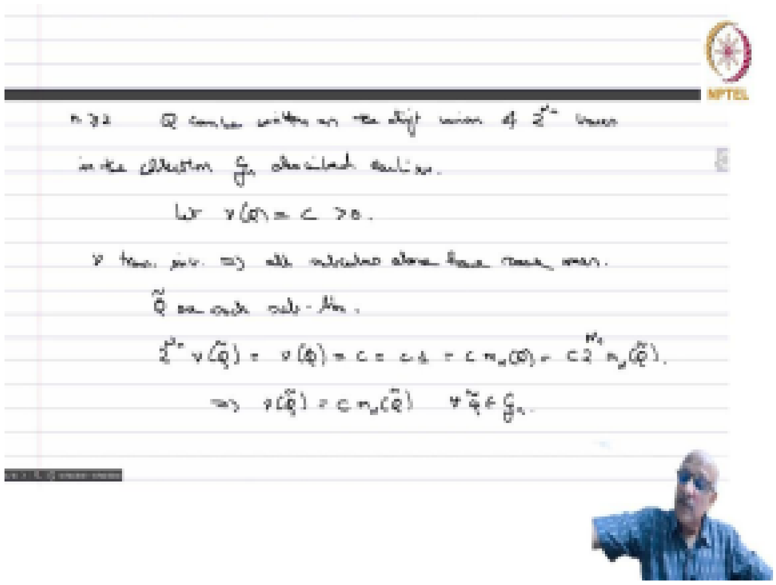

So, now, let $n \geq 2$ now, we take this cube and subdivided it well by drawing lines parallel to the coordinate axis into cubes of size 2 power minus n. So, n bigger
than 2 then we can write Q can be written as the disjoint union of 2 power n small n boxes in the collection g_n described earlier.

So, what is this g_n? It is the set of all cubes of size 1 by 2 power n and whose vertices were at the integral lattice points with this mesh of 1 by 2 and therefore, you can take the unique cube and simply subdivide it into two each direction into 2 power n parts and therefore, you have 2 power n times n cubes each of them is of the same size and they are all.

So, so, let nu of Q equal to C which is strictly positive then nu is translation invariant implies all sub cubes above have same measure. So, let us take $Q^{\sim}$ one such sub cube then what do you know that we know that

$$2^{N_n}\tilde{Q} = \nu(Q) = Cm_N(\tilde{Q}). \quad \forall \tilde{Q} \in g_n.$$

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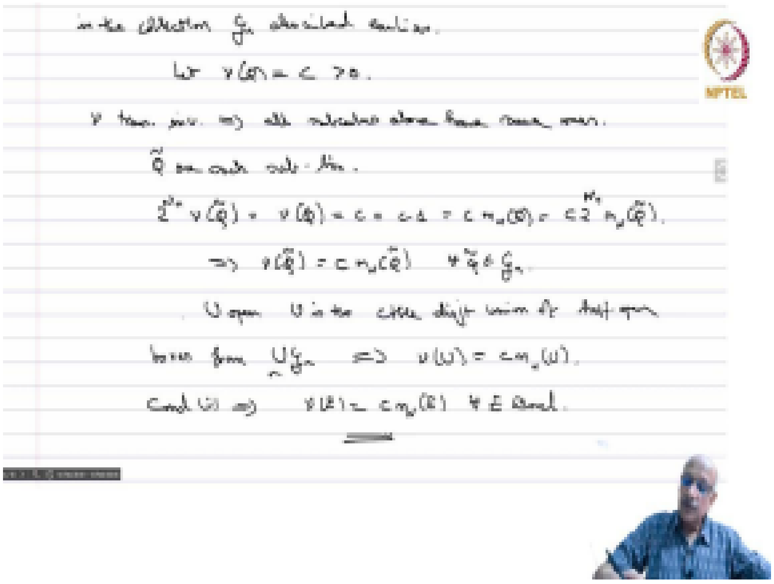

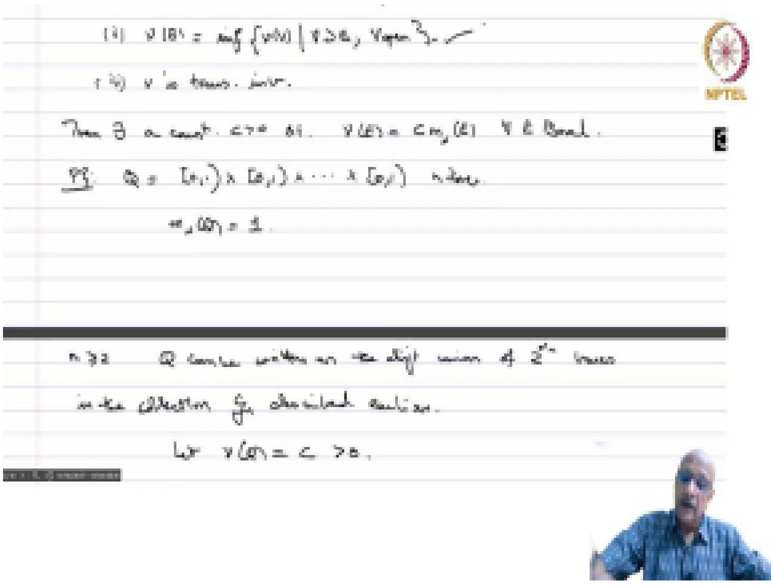

But then we prove this lemma that every open set is a countable disjoint union of elements from the union gn and therefore, if

U open, U is the countable disjoint union of half open boxes from union gn over n and that implies that you also have the

$$\nu(U) = Cm_N(U).$$

Then condition two implies because it is starting, but the infimum of all open sets gives you the measure of saying here. And therefore, this condition 3 implies that

$$\nu(E) = Cm_N(E), \quad \forall\, E\ Borel$$

, so, this proves the theorem.

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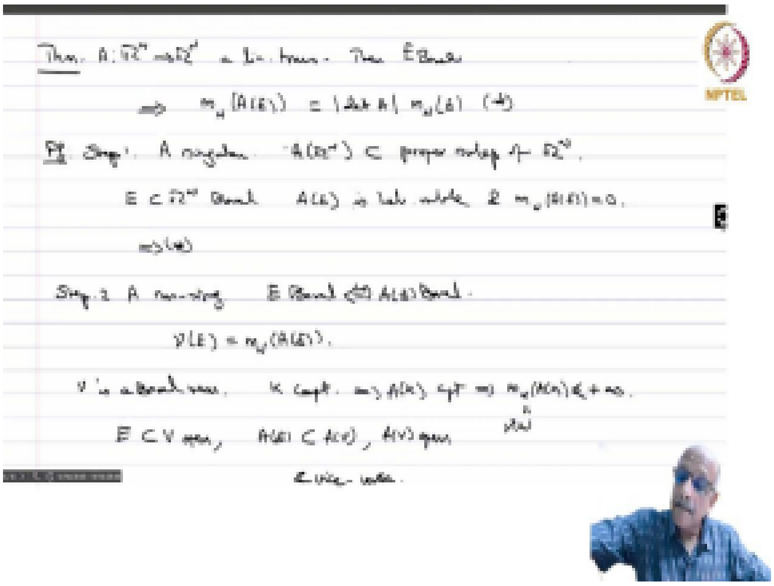

So, finiteness on compact sets, translation invariance and outer regularity. So, this determine completely the Lebesgue measure. So, let us apply this to very nice results theorem,

Theorem: $A: \mathbb{R}^N \rightarrow \mathbb{R}^N$ a linear transformation then E Borel

$$\Rightarrow \ m_N(A(E)) = |det\, A|\, m_N(E).$$

So, this is the meaning. So, if you have a linear transformation you have a matrix so, you have a determinant and the meaning of the determinant is that if you apply the linear transformation then given any Borel set the measure gets multiplied by the modulus of the determinant. So, this is the geometric meaning of the determinant of a matrix.

Proof:

So, **step 1** let us assume A singular then $A(\mathbb{R}^N)$ is contained in a proper subspace of $\mathbb{R}^N$. Let me not write here $A(E)$ Borel then E Borel implies this. So, now if E is any Borel set in $\mathbb{R}^N$

then what is $A(E)$, $A(E)$ is contained in a proper subspace implies E is Lebesgue measurable by the completeness and $m_N(A(E)) = 0$ because it is in a proper subspace we know that the measure of a proper subspace $\mathbb{R}^1$ and $\mathbb{R}^2$, etc we did this calculation always in a proper subspace and measure is 0. Therefore, implies star because $m_N(A(E)) = 0$.

So in this case, Borel may be a Lebesgue measurable set. I am not saying it is a Borel measurable set that is why I erased that portion.

Step 2, so, now we assume the A is non singular. So then it becomes a homeomorphism therefore, the E Borel implies if and only if $A(E)$ a Borel,

so E Borel by Lemma $\Leftrightarrow$ if $A(E)$ is Borel.

And you define $\nu(E) = m_N(A(E))$

then ν is Borel measure. If K is compact, continuous image of compact set is compact. So $A(K)$ is compact implies

$$m_N(A(K)) <+ \infty.$$

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Then, if $E \subset V$ open, then $A(E) \subset A(V)$ and $A(V)$ is also and vice versa.

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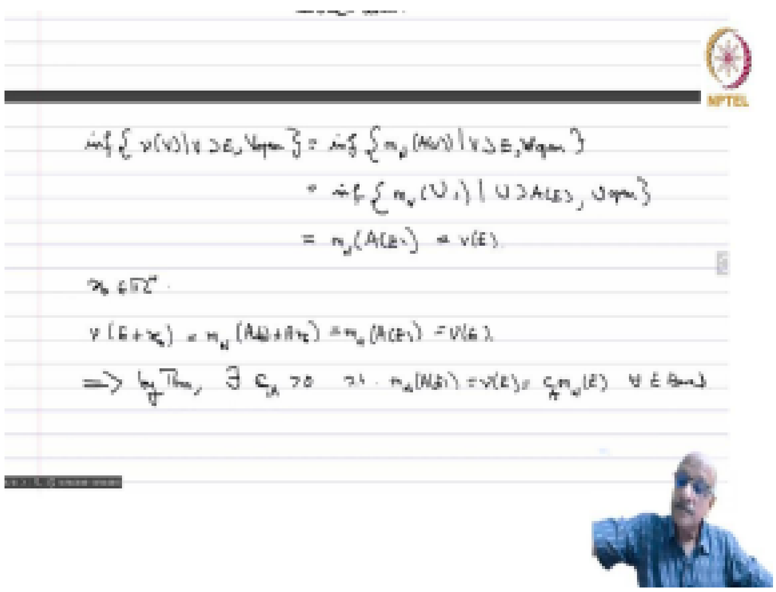

Therefore, you have, let us compute

$$inf\{\nu(V) : V \supset E, V\, open\} = inf\left\{m_N(A(V)) : V \supset E, V\, open\right\},$$

$$= inf\left\{m_N(U) : U \supset A(E), U\, open\right\}$$

$$= m_N(A(E)) = \nu(E).$$

Therefore, condition two is also satisfied $\nu(E)$ is the infimum of ν.

Finally, translation invariance. So, $x_0 \in \mathbb{R}^N$

and let us take

$$\nu(E + x_0) = m_N(A(E) + Ax_0) = m_N(A(E)) = \nu(E).$$

Therefore, ν is also translation invariant therefore, this implies by theorem, there exists a $C_A > 0$ such that $m_N(A(E)) = \nu(E) = C_A m_N(E)$ for every E Borel.

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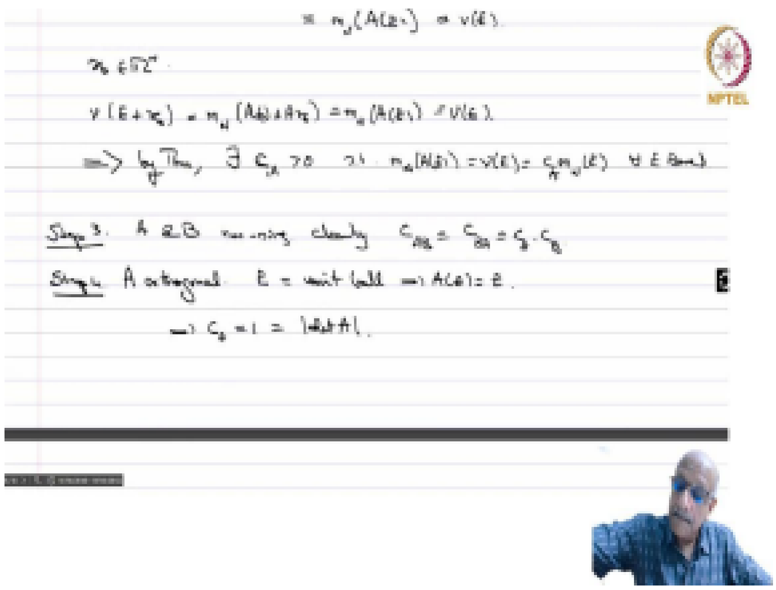
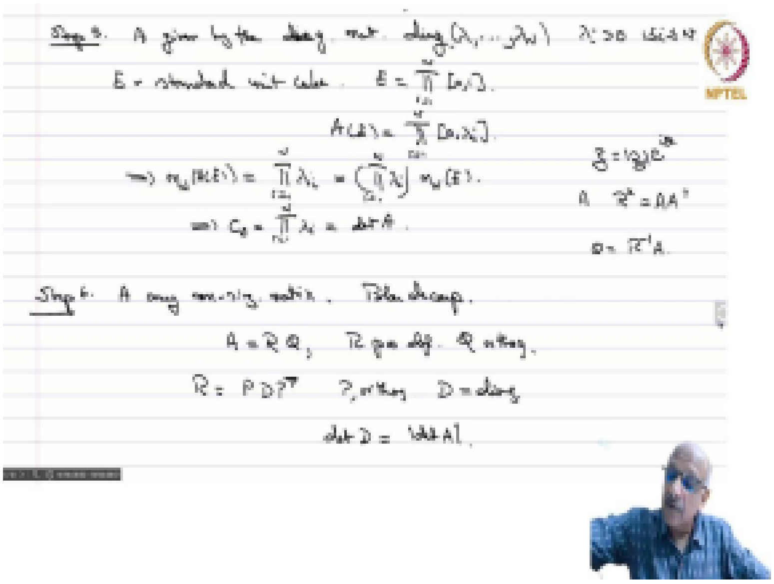


Step 3, so, if A and B nonsingular clearly

$$C_{AB} = C_{BA} = C_A C_B.$$

because the measure of AB of E is nothing but the CB times CA times measure of BE which is CA into CB times measure of E and therefore, C of AB is same as CA times CB.

Step 4, let us take A orthogonal so, if B is orthogonal and you should take $E =$ unit ball this implies $A(E) = E$ because and therefore, $C_A = 1 = |det\, A|$. (Refer Slide Time: 23:43)



Step 5, let us take A given by the diagonal matrix

$$A = diag(\lambda_1, \ldots, \lambda_N);\ \lambda_i > 0,\ i \leq i \leq N.$$

Then you take

$$E = standard\, unit\, cube;\quad E = \prod_{i=1}^{N} [0, 1];$$

$$A(E) = \prod_{i=1}^{N} [0, \lambda_i]$$

$$\Rightarrow m_N(A(E)) = \prod_{i=1}^{N} \lambda_i = \left(\prod_{i=1}^{N} \lambda_i\right) m_N(E).$$

$$\Rightarrow C_A = \prod_{i=1}^{N} \lambda_i = det\, A$$

Step 6, so, A any non singular matrix we have the polar decomposition $A = RQ$ can be, R positive definite, Q orthogonal. This is same as any complex number can be written as mod Z into $e^{i\theta}$ or $e^{i\theta}|z|$. Similarly, you take EA and you find

$$R = PDP^T$$

R such that $R = A^TA$ or A^T and then R, the square root of this the so, this matrix R you can find such like this because this positive definite matrix and therefore, you have and you take $Q = R^{-1}A$.

So, this is called a polar decomposition, it is the same as writing a complex number in terms of its modulus and its argument. So, R itself is positive definite therefore, you can write this as

$$R = PDP^T$$

Porthogonal and then $D = diagonal$ is the determinant of R and that is nothing but the $|det(A).|$.

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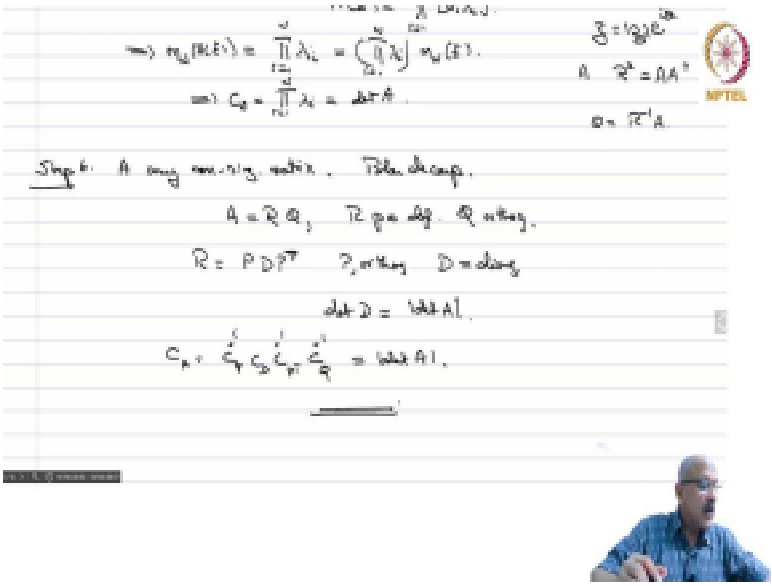

And now, you have $C_A = C_p C_D C_{P^T} = |det(A)|$

and that proves this theorem. So, the determinant, this is the geometric meaning of the determinant and we have using this fact that the Lebesgue measure being translation invariant, regular and finite on compact sets determines the measure itself.