Measure and Integration Professor S Kesavan Department of Mathematics The Institute of Mathematical Sciences Lecture No-15 3.2 - Approximation

In the previous lecture when dealing with the step functions, there were some mistakes in the indices sigma i equals 1 to n aj Kai j, et cetera. So I have corrected that in the lecture notes and therefore, you can I am sure you will make sense out of it. And I am sorry for that anyway. Now, we will continue to prove one more approximation result for which we need a very important topological result on RN. So, we will start with that.

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Lemma: Every open set in R" can be written as the countable diginit union of half-open boxes. P: n gixed poor int. I durates the set of all pts in R where coordinates are all integral multiples of 2". gn = coll. of all half-open loxes with edge 2" and all vertices on Fr. · n fixed pos. int. trenz, ~ brogs to exactly one los in fr. · let n>m. If QESm and Q'ES ton either Q'CQ $\alpha \quad Q' \cap Q = \phi$

Lemma: Every open set in \mathbb{R}^N can be written as the countable disjoint union of half-open boxes.

proof: Let n be a fixed positive integer so and let F_n denote the set of all points in \mathbb{R}^N whose coordinates are all integral multiples of 2^{-n} . Let $g_n =$ collection of all half open boxes with edge 2^{-n} and all vertices on F_n . So, then the following two things are obvious

- n fixed positive integer, for every \mathbb{R}^N , x belongs to exactly one box in g_n .
- Let n > m. If $Q \in g_m$ and $Q' \in g_n$, then either $Q' \subset Q$ or $Q' \cap Q = \phi$.

So, these two are all obvious observations.

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RCR open set. 2002 I open ball containing &, contained in R For Noufflorge, JREG. D.t. repart In other words, I is the union of all half-gen boxes contained init and beloging to some fr. This is a cittle coll, but may not be disjt. Now choose all baxes in a laborgizing to S, and discoud those boxes in G. R. 32, contained inside these selected bases

So, now let us continue. So $\Omega \subset \mathbb{R}^N$ open set. If $x \in \Omega$, then there exists an open ball containing x contained in Ω . Then for n sufficiently large there exists $Q \in g_n$ such that $x \in Q \subset \Omega$. So, in other words, Ω is the union of all half open boxes contained in it and belonging to some g_n , so, it does not matter. So, this is a countable collection but may not be disjoint. So, what you are going to do is the following. So, now choose all boxes in Ω belonging to g_1 . So, they will all be disjoint anyway and discard those boxes in g_k , $k \ge 2$, contained inside these. So, you are throwing away the superfluous ones. So, now, it is easy to do what you want to do.

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Now, choose from the remaining boxes all those in g_2 and discard those in g_k , $k \ge 3$, contained in the selected boxes. So, preceding iteratively Ω is the union countable disjoint union of boxes $\bigcup_n g_n$ and this is obvious from the two observations which we made before. So, you can always write every open set as a countable disjoint union of half open boxes, that is the moral of this Lemma.

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Peop. Let RCIRN he an open set, ECR My (E) < +00. Then 4250, 3 a set F which is a finite digit union of half open MPTEL 1 from at $m_{\mu}(E\Delta F) < E$ $E\Delta F = (E(F) \cup (F(E)).$ Pf: Let GCR of out at ECG my (G(E) < 42 Go = 200 €, =) Go open & CGo C J MJ (Go)E/ < E/2 Go open, G. S. fin mean (Since E han for mean.)

Proposition: Let $\Omega \subset \mathbb{R}^N$ be an open set, $E \subset \Omega$, and $m_N(E) < +\infty$. Then for every $\epsilon > 0$, there exists a set F, which is a finite disjoint union of half open boxes such that

$$m_{_N}(E\Delta F) < \epsilon$$

Proof: Let $G \subset \mathbb{R}^N$ open set such that $E \subset G$ and $m_N(G \setminus E) < \frac{\epsilon}{2}$. We say

$$G_0 = \Omega \cap G \Rightarrow G_0$$
 is open and $E \subset G_0 \subset \Omega$ and $m_N(G_0 \setminus E) < \frac{\epsilon}{2}$

Now, G_0 is open and hence G, G_0 all have finite measure since E has finite measure. So, there exist $\{I_j\}_{j=1}^{\infty}$ disjoint half open boxes such that $G_0 = \bigcup_{j=1}^{\infty} I_j$ and since the measures are all finite, so, $\sum_{j=1}^{\infty} m_N(I_j) < +\infty$.

So, choose a positive integer k such that $\sum_{j=k+1}^{\infty} m_N(l_j) < \frac{\epsilon}{2}$.

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$$\sum_{\substack{j=2m}{k}}^{n} m_{\mu}(\overline{z}_{j}) < \frac{\varepsilon}{2}/2.$$

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$$\sum_{\substack{j=1\\j=1}}^{n} \frac{1}{2} f_{\mu} \cdot \frac{1}{$$

Now, define $F = \bigcup_{j=1}^{k} I_j$. So, this is a finite disjoint union of half open boxes and $F \subset G_0$. So, $m_N(F \setminus E) \le m_N(G_0 \setminus E) < \frac{\epsilon}{2}$. And $m_N(E \setminus F) \le m_N(G_0 \setminus F) = \sum_{j=k+1}^{\infty} m_N(I_j) < \frac{\epsilon}{2}$. $\Rightarrow m_N(E \Delta F) < \epsilon$.

So, with this we complete the section on approximations. So, next time we will take up a very important concept called translation invariance.