Measure and Integration Professor S Kesavan Department of Mathematics The Institute of Mathematical Sciences Lecture No-15 3.2 - Approximation

In the previous lecture when dealing with the step functions, there were some mistakes in the indices sigma i equals 1 to n aj Kai j, et cetera. So I have corrected that in the lecture notes and therefore, you can I am sure you will make sense out of it. And I am sorry for that anyway. Now, we will continue to prove one more approximation result for which we need a very important topological result on RN. So, we will start with that.

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Lemma: Every open sut in R" can be written as the countable digint mon of half-open boxer. P: n fixed pop int. F. Anotes the set of all the in R whose coordinates are all integral multiples of 2. In = coll. of all thatf-gen loxes with edge 2" and all vertices on F. · n fixed pos. int. VxERS x belogs to exactly one los in fr. . Let $n > m$. If $Q \in \mathcal{G}_n$ and $Q' \in \mathcal{G}_n$ in either $Q' \subset Q$

Lemma: Every open set in \mathbb{R}^N can be written as the countable disjoint union of half-open boxes.

proof: Let n be a fixed positive integer so and let F_n denote the set of all points in \mathbb{R}^N whose coordinates are all integral multiples of 2^{-n} . Let $g_n =$ collection of all half open boxes with edge 2^{-n} and all vertices on F_n . So, then the following two things are obvious

- n fixed positive integer, for every \mathbb{R}^N , x belongs to exactly one box in g_n .
- Let $n > m$. If $Q \in g_m$ and $Q' \in g_{n'}$, then either $Q' \subset Q$ or $Q' \cap Q = \Phi$.

So, these two are all obvious observations.

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RCR" open put. 2652 3 open ball entering 2, contained in A For a suff longe, $\frac{1}{2}$ QE g, s.t. xe QC s2 In other words, 52 is the union of all holf-gen bases contained init and beloging to some for. This is a cittle coll, but may not be digit. Now chose all baxos in a letroging to G, and discord there boxes in G. 232, contained inside these selected bases

So, now let us continue. So $\Omega \subset \mathbb{R}^N$ open set. If $x \in \Omega$, then there exists an open ball containing x contained in Ω . Then for n sufficiently large there exists $Q \in g_n$ such that $x \in Q \subset \Omega$. So, in other words, Ω is the union of all half open boxes contained in it and belonging to some g_n , so, it does not matter. So, this is a countable collection but may not be disjoint. So, what you are going to do is the following. So, now choose all boxes in Ω belonging to g_1 . So, they will all be disjoint anyway and discard those boxes in g_k , $k \ge 2$, contained inside these. So, you are throwing away the superfluous ones. So, now, it is easy to do what you want to do.

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Now, choose from the remaining boxes all those in g_2 and discard those in $g_{k'}$, $k \geq 3$, contained in the selected boxes. So, preceding iteratively Ω is the union countable disjoint union of boxes $\cup_{n} g_{n}$ and this is obvious from the two observations which we made before. So, you can always write every open set as a countable disjoint union of half open boxes, that is the moral of this Lemma.

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 $\begin{picture}(160,10) \put(0,0) {\put(0,0){\line(1,0){155}} \put(0,0){\line(1,0){155}} \put($ $M_N(E \Delta F) < \epsilon \qquad \qquad E \Delta F \cdot (E \cdot F) \cup (F \cdot E).$ Pf : let GCR^V apm set at $f \in CR$ $m_0(S \setminus E) < 40$ $G_{0} = \Omega \cap \overline{G_{1}}$ = G_{0} apen θ $CG_{0}C$ D_{2} m_{1} $(G_{0})E$ K G_{2} Go open, G, S, fin mean Coince & han for mean) $\exists \{ \frac{1}{4} \}$ $\exists \{ \frac{1}{4} \}$ $\exists \pi/4$ $\forall n \in \mathbb{Z}$ or $\pi/4$ $\forall n \in \mathbb{Z}$ or $\pi/4$ $\leq \pi/4$ or $\exists n/4$
 $\exists n/4 \in \mathbb{Z}$ or $\pi/4$ or $\exists n/4$ or $\exists n$

Proposition: Let $\Omega \subset \mathbb{R}^N$ be an open set, $E \subset \Omega$, and $m_N(E) \leq +\infty$. Then for every $\epsilon > 0$, there exists a set F, which is a finite disjoint union of half open boxes such that

$$
m_{N}(E\Delta F) < \epsilon.
$$

Proof: Let $G \subset \mathbb{R}^N$ open set such that $E \subset G$ and $m_N(G \backslash E) < \frac{\epsilon}{2}$. We say 2

$$
G_0 = \Omega \cap G \Rightarrow G_0 \text{ is open and } E \subset G_0 \subset \Omega \text{ and } m_N(G_0 \backslash E) < \frac{\epsilon}{2}
$$

Now, G_0 is open and hence G, G_0 all have finite measure since E has finite measure. So, there exist $\{I_j\}_{j=1}^{\infty}$ disjoint half open boxes such that $G_0 = \bigcup_{j=1}^{\infty} I_j$ and since the measures are all \int_{i}^{∞} finite, so, $j=1$ ∞ $\sum_{j} m_{N}(I_{j}) < +\infty$.

So, choose a positive integer k such that $j=k+1$ ∞ $\sum_{j=1}^{\infty} m_N(I_j) < \frac{\epsilon}{2}$ $rac{\epsilon}{2}$.

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\sum_{j=k}^{m} m_{\mu} (\underline{T}_{j}) < \frac{2}{\mu}
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\sum_{k} m_{\mu} (\underline{T}_{j}) < \frac{2}{\mu}
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F \subset G_{0} \qquad m_{\mu} (FE) \subset m_{\mu} (G_{0})E < \frac{2}{\mu}
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m_{\mu} (EF) \subset m_{\mu} (G_{0})F = \sum_{j=k}^{m} m_{\mu} (\underline{T}_{j}) < \frac{6}{\mu}
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\sum_{k=1}^{m} m_{\mu} (\underline{T}_{k}) = \sum_{k=1}^{m} m_{\mu} (\underline{T}_{k})
$$

Now, define $F = \bigcup_{j=1}^{n} I_j$. So, this is a finite disjoint union of half open boxes and $F \subset G_0$. ${}^{k}I_{j}$. So, this is a finite disjoint union of half open boxes and $F \subset G_{0}$. So, $m_N(F \ E) \le m_N(G_0 \ E) < \frac{\epsilon}{2}$. And $\frac{E}{2}$. And $m_N(E \backslash F) \leq m_N(G_0 \backslash F) =$ $j=k+1$ ∞ $\sum_{j=1}^{\infty} m_N(I_j) < \frac{\epsilon}{2}$ $rac{\epsilon}{2}$. \Rightarrow $m_N(E\Delta F) < \epsilon$.

So, with this we complete the section on approximations. So, next time we will take up a very important concept called translation invariance.