Measure and Integration
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Lecture No-15
3.2 - Approximation

In the previous lecture when dealing with the step functions, there were some mistakes in the indices sigma i equals 1 to n aj Kai j, et cetera. So I have corrected that in the lecture notes and therefore, you can I am sure you will make sense out of it. And I am sorry for that anyway. Now, we will continue to prove one more approximation result for which we need a very important topological result on RN. So, we will start with that.

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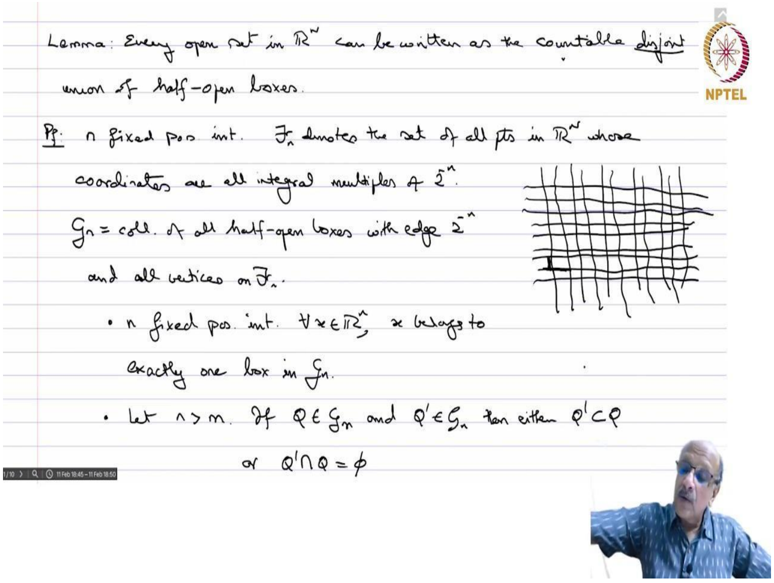

Lemma: Every open set in $\mathbb{R}^N$ can be written as the countable disjoint union of half-open boxes.

proof: Let n be a fixed positive integer so and let F_n denote the set of all points in $\mathbb{R}^N$ whose coordinates are all integral multiples of 2^{-n}. Let g_n = collection of all half open boxes with edge 2^{-n} and all vertices on F_n. So, then the following two things are obvious

- n fixed positive integer, for every $\mathbb{R}^N$, x belongs to exactly one box in g_n.
- Let $n > m$. If $Q \in g_m$ and $Q' \in g_n$, then either $Q' \subset Q$ or $Q' \cap Q = \phi$.

So, these two are all obvious observations.

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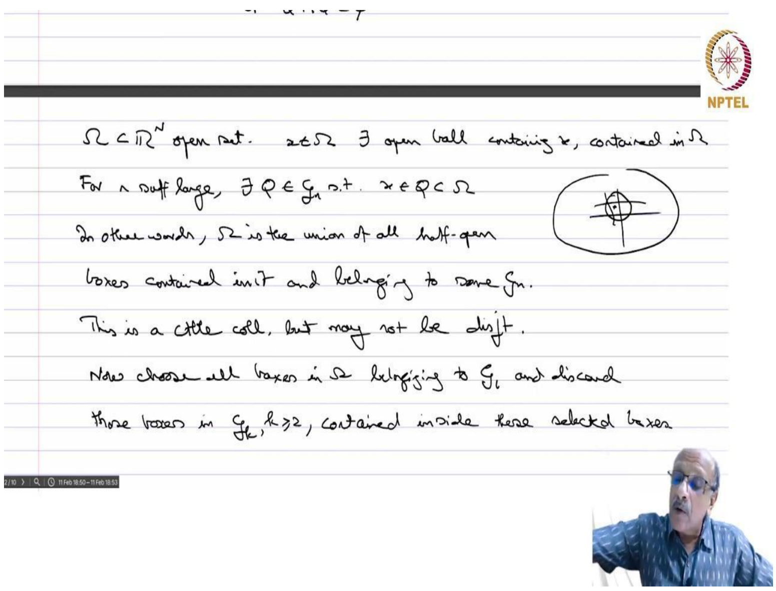

So, now let us continue. So $\Omega \subset \mathbb{R}^N$ open set. If $x \in \Omega$, then there exists an open ball containing x contained in Ω. Then for n sufficiently large there exists $Q \in g_n$ such that $x \in Q \subset \Omega$. So, in other words, Ω is the union of all half open boxes contained in it and belonging to some g_n, so, it does not matter. So, this is a countable collection but may not be disjoint. So, what you are going to do is the following. So, now choose all boxes in Ω belonging to g_1. So, they will all be disjoint anyway and discard those boxes in g_k, $k \geq 2$, contained inside these. So, you are throwing away the superfluous ones. So, now, it is easy to do what you want to do.

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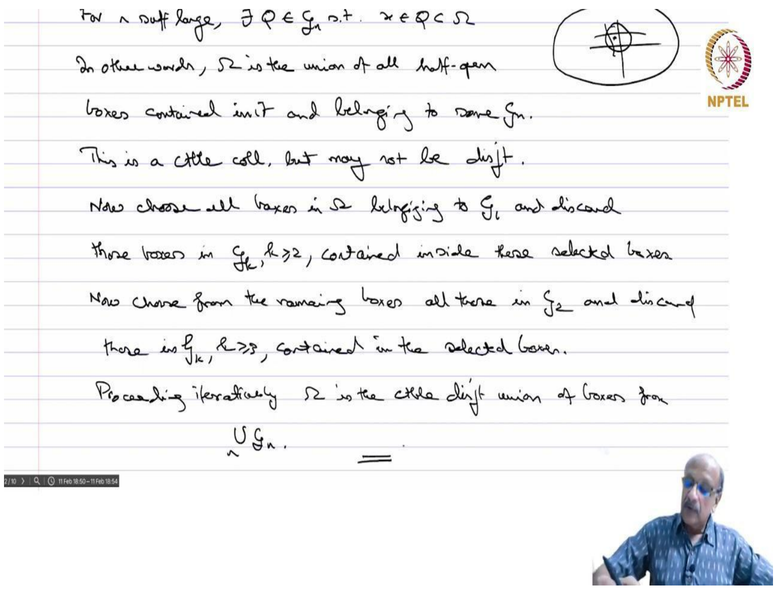

Now, choose from the remaining boxes all those in g_2 and discard those in g_k, $k \geq 3$, contained in the selected boxes. So, preceding iteratively Ω is the union countable disjoint union of boxes $\cup_n g_n$ and this is obvious from the two observations which we made before. So, you can always write every open set as a countable disjoint union of half open boxes, that is the moral of this Lemma.

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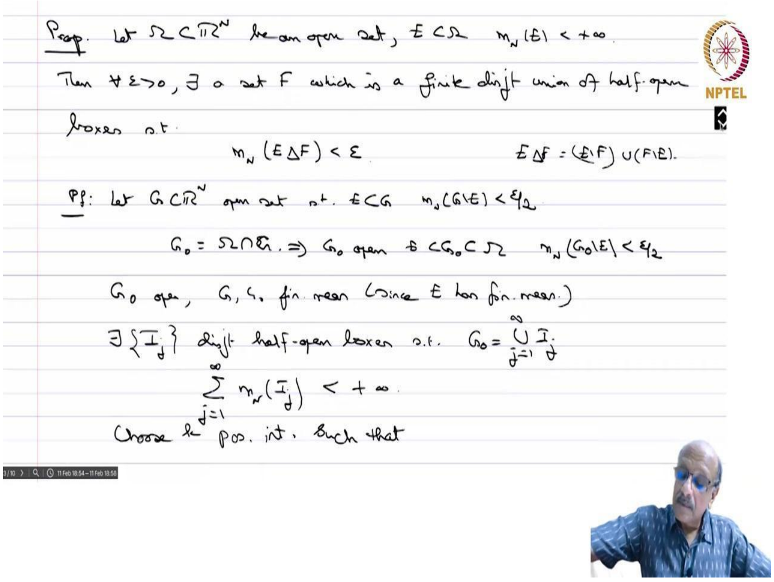

Proposition: Let $\Omega \subset \mathbb{R}^N$ be an open set, $E \subset \Omega$, and $m_N(E) <+ \infty$. Then for every $\epsilon > 0$, there exists a set F, which is a finite disjoint union of half open boxes such that

$$m_N(E \Delta F) < \epsilon.$$

Proof: Let $G \subset \mathbb{R}^N$ open set such that $E \subset G$ and $m_N(G \backslash E) < \frac{\epsilon}{2}$. We say

$$G_0 = \Omega \cap G \Rightarrow G_0 \text{ is open and } E \subset G_0 \subset \Omega \text{ and } m_N(G_0 \backslash E) < \frac{\epsilon}{2}$$

Now, G_0 is open and hence G, G_0 all have finite measure since E has finite measure. So, there exist $\{I_j\}_{j=1}^{\infty}$ disjoint half open boxes such that $G_0 = \cup_{j=1}^{\infty} I_j$ and since the measures are all finite, so, $\sum_{j=1}^{\infty} m_N(I_j) <+ \infty$.

So, choose a positive integer k such that $\sum_{j=k+1}^{\infty} m_N(I_j) < \frac{\epsilon}{2}$.

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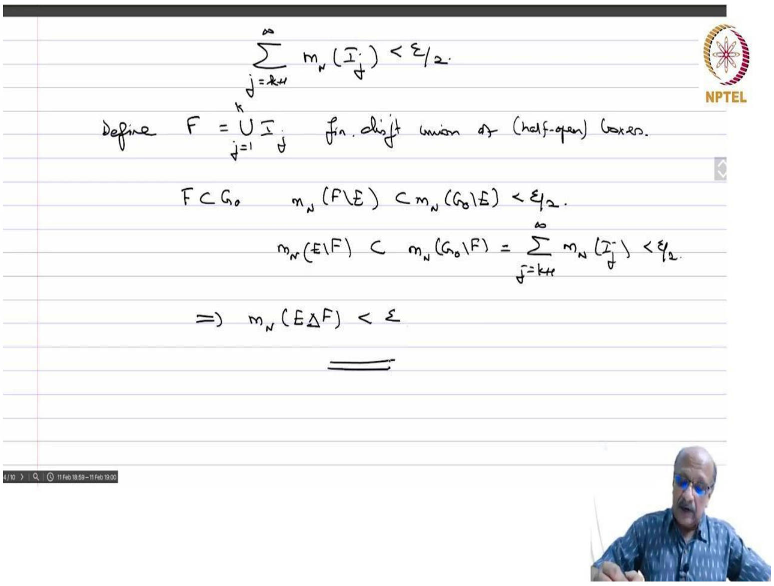

Now, define $F = \cup_{j=1}^{k} I_j$. So, this is a finite disjoint union of half open boxes and $F \subset G_0$.

So, $m_N(F\backslash E) \leq m_N(G_0\backslash E) < \frac{\epsilon}{2}$. And $m_N(E\backslash F) \leq m_N(G_0\backslash F) = \sum_{j=k+1}^{\infty} m_N(I_j) < \frac{\epsilon}{2}$.

$$\Rightarrow m_N(E\Delta F) < \epsilon.$$

So, with this we complete the section on approximations. So, next time we will take up a very important concept called translation invariance.