Measure and Integration
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Lecture No - 14

3.1 - Approximation

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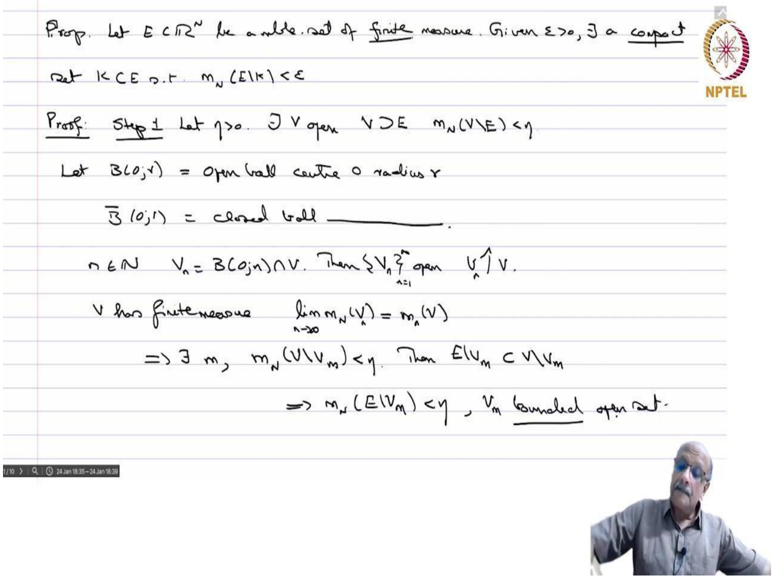

So, we continue with the approximation results concerning the Lebesgue measure. So, we have the following proposition.

Proposition: Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure. Given $\epsilon > 0$, there exists a compact set $K \subset E$ such that $m_N(K\backslash E) < \epsilon$.

proof: We will do it in a few steps.

step 1: Let $\eta > 0$. Then there exists V open, $V \supset E$ and $m_N(V\backslash E) < \epsilon$. Let $B(0, r)$ be the open ball centered at 0 of radius r and $\overline{B(0, r)}$ equals closed ball center 0 and radius r.

So, $n \in \mathbb{N}$, and you write $V_n = B(0, r) \cap V$. Then V_n are all open, $V_n \uparrow V$. So, V has finite measure why V as finite measure and you have that $\lim_{n\to\infty} m_N(V_n) = m_N(V)$.

$$\Rightarrow \exists\, m,\ m_N(V\backslash V_m) < \eta\text{. Then } E\backslash V_m \subset V\backslash V_m \Rightarrow m_N(E\backslash V_m) < \eta$$

and then we are taking limits but V_m is a bounded open set.

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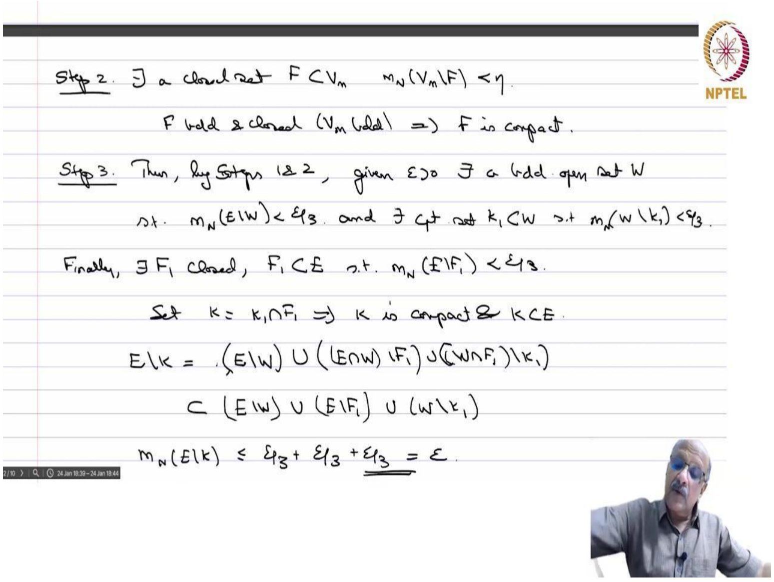

Step 2: ∃ a closed set $F \subset V_m$ so that $m_N(V_m\backslash F) < \eta$. So, now F is bounded and closed (since V_m is bounded) ⇒ F is compact.

Step 3: Thus by steps 1 and 2, given $\epsilon > 0$, there exists a bounded open set W such that $m_N(E\backslash W) < \frac{\epsilon}{3}$ and there exists a compact set $K_1 \subset W$ such that $m_N(W\backslash K_1) < \frac{\epsilon}{3}$.

Finally, there exists F_1 closed s.t. $F_1 \subset E$ and $m_N(E\backslash F_1) < \frac{\epsilon}{3}$. Set

$$K = K_1 \cap F_1 \Rightarrow K \text{ is closed and } K \subset E.$$

So,now

$E\backslash K = (E\backslash W) \cup ((E \cap W)\backslash F_1) \cup ((W \cap F_1)\backslash K_1) \subset (E\backslash W) \cup (E\backslash F_1) \cup (W\backslash K_1)$.

And therefore, $m_N(E\backslash K) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

So, the only thing you really have to check here is this particular set theoretic identity.

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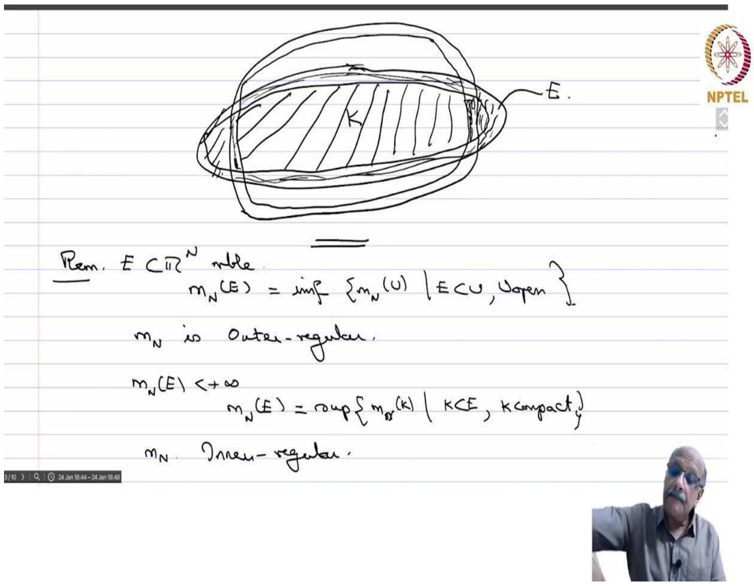

So, let us let me draw a picture for you so, that so, you have here the set E and then what you do, you found bounded open set W which was like this and then you took k1 which is a compact set and a closed set and you took a set F1 so, F1 intersection k so, this is the compact set, k which you finally got and this is a set E and therefore, if you want E minus k it consists of actually three parts which is E minus W which is here. And then you have the E intersection W minus F1 then W intersection F1 which is here minus k1.

So, if you add all that precisely E minus k. So, all the shaded new things which will give you E minus k, so that you can check it yourself and then so, this proves that approximation property.

Remark: So, you have $E \subset \mathbb{R}^N$ measurable. Then on one hand you have

$$m_N(E) = \inf\{m_N(U) : E \subset U,\ U\ open\}.$$

So, m_N is called outer regular if $m_N(E) < \infty$ and $m_N(E) = \sup\{m_N(K) : K \subset E,\ K\ compact\}$.

In fact, the supremum relation you can also prove without this restriction on mN greater than lesser than plus infinity, leave it as an exercise for you and therefore, this is called inner regularity.

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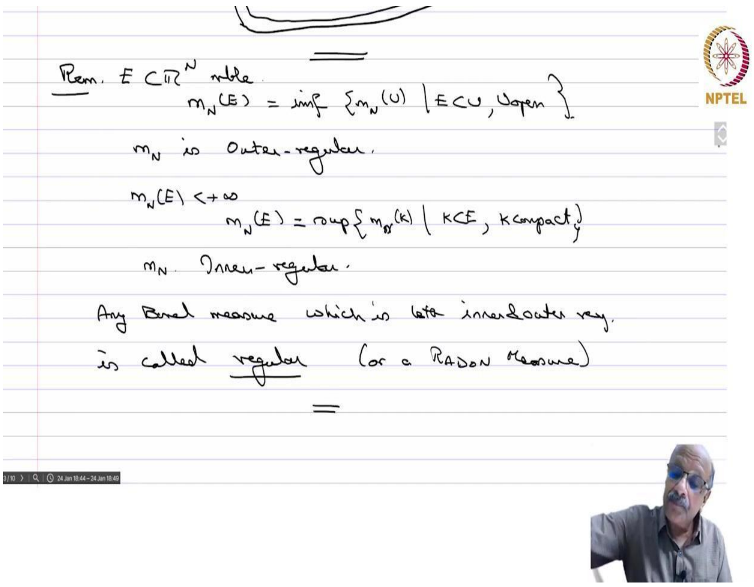

So, any Borel measure that means a measure defined on all Borel sets so Lebesgue measure is an example which is both inner and outer regular is called regular it is also all RADON, RADON measure is a Borel measure which can be which is both inner and outer regular this was the just a definition.

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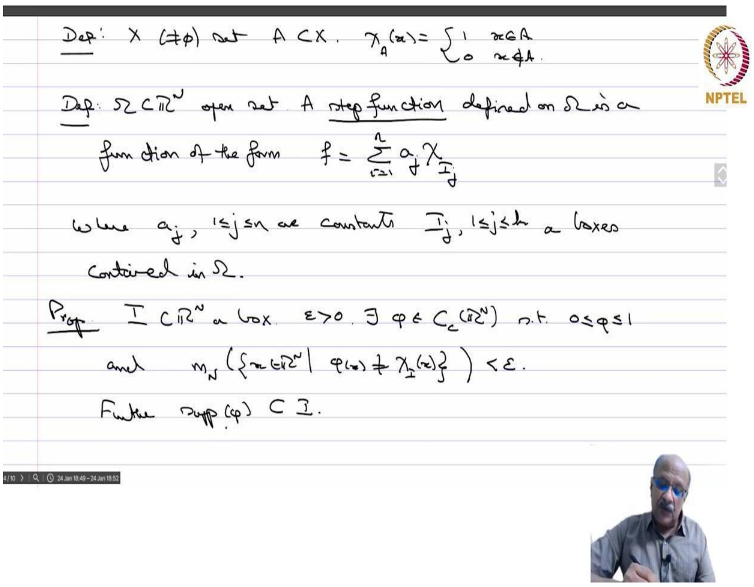

Now, we move to approximation of functions involving measurable sets.

Definition: We have $X\ (\neq \phi)$ any set, $A \subset X$. Then we know

$$\chi_A(x) = 1, \ if\ x \in A,$$

$$= 0, \ if\ x \notin A.$$

Definition: $\Omega \subset \mathbb{R}^N$ an open set. A step function defined on Ω is a function of the form

$$f = \sum_{j=1}^{n} \alpha_j \chi_{I_j}$$, where α_j, $1 \le j \le n$ are constants and I_j, $1 \le j \le n$,

are are boxes contained in Ω.

So, it is a function which is made up of boxes characteristic functions on boxes.

Proposition: $I \subset \mathbb{R}^N$ box, $\epsilon > 0$ given. Then there exists $\phi \in C_c(\mathbb{R}^N)$ $s.t.$ $0 \le \phi \le 1$ and

$$m_N(\{x \in \mathbb{R}^N : \phi(x) \ne \chi_I(x)\}) < \epsilon.$$

Further $supp(\phi) \subset I$.

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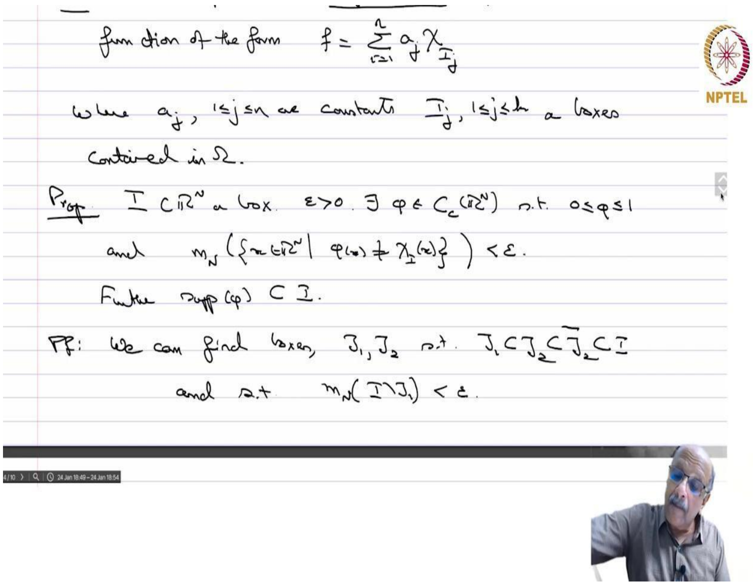

proof: We can find boxes J_1, J_2 $s.t.$ $J_1 \subset J_2 \subset \overline{J_2} \subset I$ and such that $m_N(I \setminus J_1) < \epsilon$.

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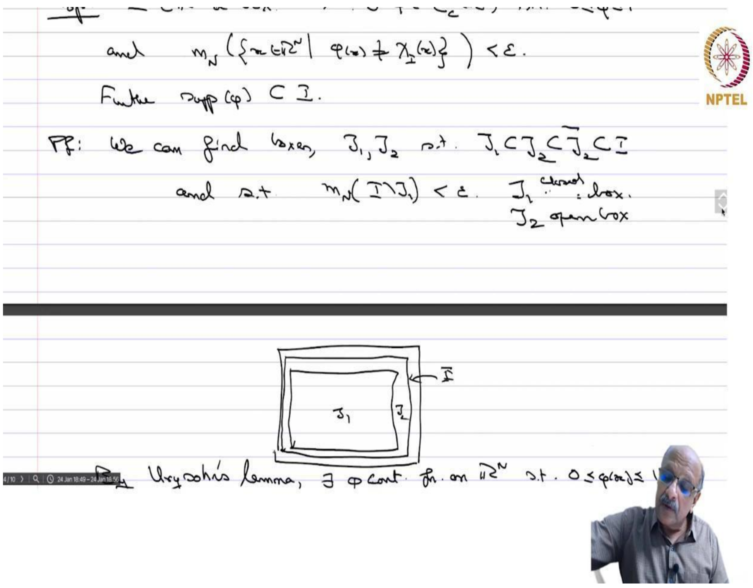

So, in two dimensions let me draw a picture here. So, this is a box I and then I make smaller boxes J_1, J_2 and then I, and then I make these boundaries sufficiently close so that they are all the whole thing difference in the area is less than epsilon so that I can do.

Now if you use Urysohn's Lemma, there exists a continuous function ϕ continuous function on $\mathbb{R}^N$ such that $0 \le \phi(x) \le 1 \; \forall x$, $\phi \equiv 1$ *on* J_1 *and* $\phi \equiv \overline{J_2}^{c}$.

So, you have two disjoint open sets. So, J_1 is an open box.

Let me write that first J1 open box. So, I have these two disjoint So, J1 should be a closed box because we are going to apply Urysohn's Lemma and let us do it correctly. So, J_1 is a closed box and J_2 is an open box.

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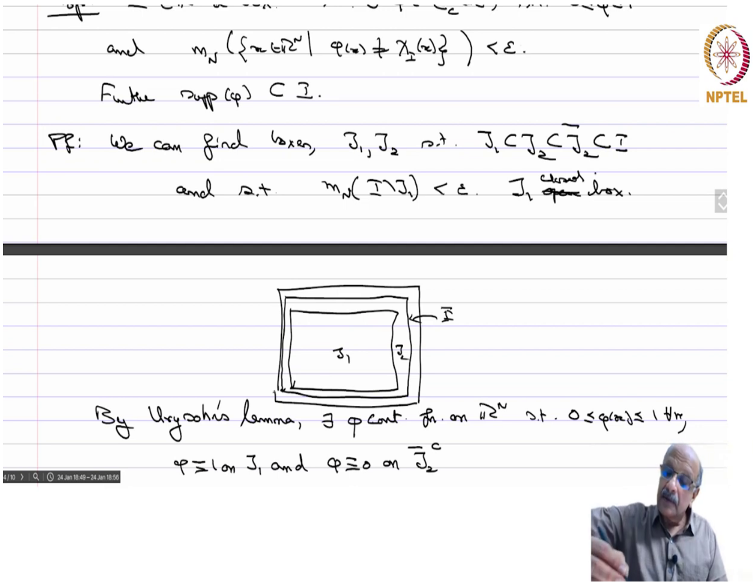

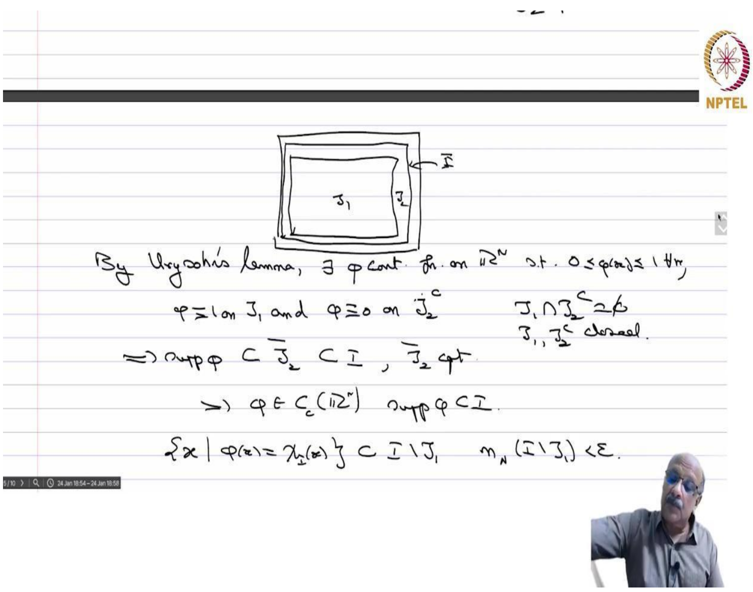

So, this is the J_2 complement so, you have J_2 complement is a closed set J_1 is a closed set and J1 intersection J2 complement is empty and J1, J2 complement or closed. Therefore, by Urysohn's Lemma I can construct such a function which is like this. So, this implies that

$$\Rightarrow supp(\phi) \subset \overline{J_2} \subset I, \overline{J_2}\ cpt \ \Rightarrow \ \phi \in C_c(\mathbb{R}^N),\ supp(\phi) \subset I.$$

Now, what about the set $\{x\colon\ \phi(x) \neq \chi_I(x)\} \subset I \setminus J_1$ and then $m_N(I \setminus J_1) < \epsilon$.

So, that proves the theorem. So, it is just a simple application of Urysohn's Lemma.

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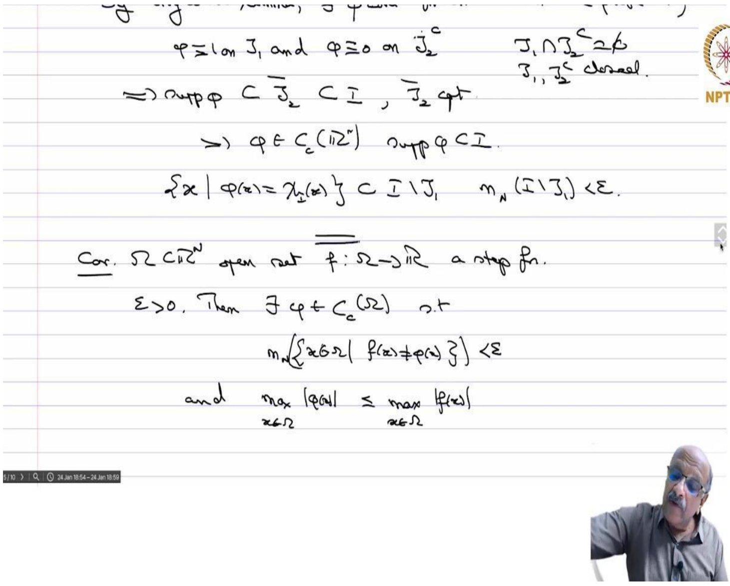

Corollary: $\Omega \subset \mathbb{R}^N$ open set and $f\colon \Omega \to \mathbb{R}$ a step function, $\epsilon > 0$. Then there exists $\phi \in C_c(\Omega)$ s.t. $m_N(\{x \in \Omega\colon\ f(x) \neq \phi(x)\}) < \epsilon$ and $\max\limits_{x\in\Omega} |\phi(x)| \leq \max\limits_{x\in\Omega} |f(x)|$.

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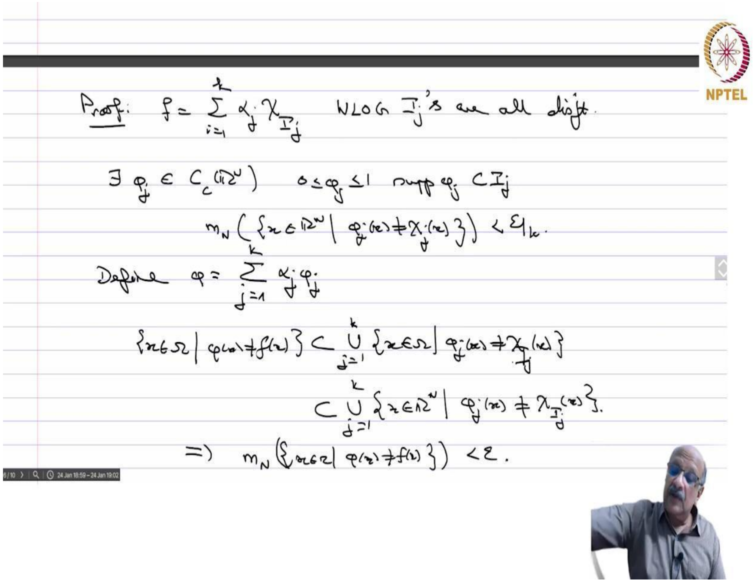

proof: So, let us take $f = \sum_{i=1}^{k} \alpha_j \chi_{I_j}$. Without loss of generality, let I_j are all disjoint.

So, there exists functions $\phi_j \in C_c(\mathbb{R}^N)$, $0 \le \phi_j \le 1$, $supp(\phi_j) \subset I_j$ and

$$m_N(\{x \in \mathbb{R}^N : \phi_j(x) \neq \chi_j(x)\}) < \frac{\epsilon}{2}.$$

Now, define $\phi = \sum_{j=1}^{k} \alpha_j \phi_j$. So

$$\{x \in \Omega : \phi(x) \neq f(x)\} \subset \cup_{i=1}^{k} \{x \in \Omega : \phi_j(x) \neq \chi_{I_j}(x)\} \subset \cup_{i=1}^{k} \{x \in \mathbb{R}^N : \phi_j(x) \neq \chi_{I_j}(x)\}$$

$$\Rightarrow m_N(\{x \in \Omega : \phi(x) \neq f(x)\}) < \epsilon.$$

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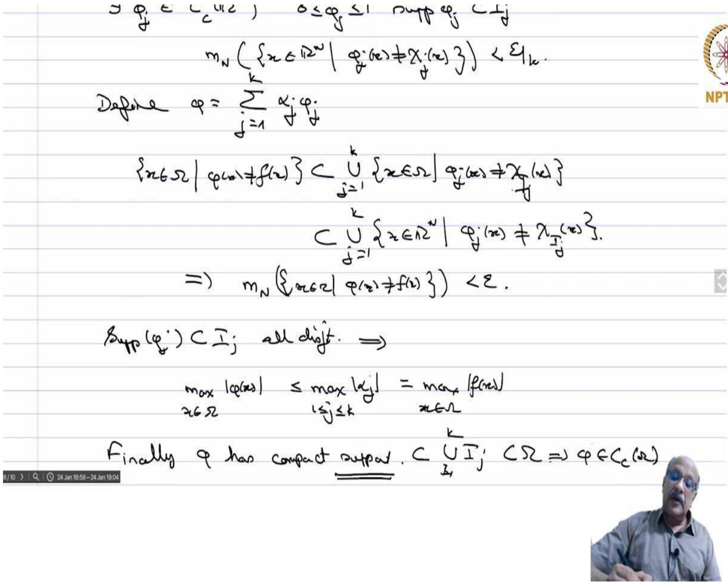

So, we have found a continuous function phi with these properties: $supp(\phi_j) \subset I_j$, $all\ disjoint\ \Rightarrow \max_{x\in\Omega} |\phi(x)| \leq \max_{1\leq j\leq k} |\alpha_j| = \max_{x\in\Omega} |f(x)|$.

Finally, ϕ has compact support because support ϕ_j is in Ij, Ij is a finite box. So, inside you have a closed set which is closed, bounded therefore, it is compact and compact support contained in $\cup_{j=1}^{k} I_j \subset \Omega \Rightarrow \phi \in C_c(\Omega)$.

So, that completes the proof of this step. So, we will continue afterwards.