Measure and Integration Professor S Kesavan Department of Mathematics The Institute of Mathematical Sciences Lecture No - 14

3.1 - Approximation

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Prop. Let ECR" be a solder rad of finde masure. Given EDO, Ja company Pet KCE p.r. My (ELK) << $\frac{Prof}{f}$ Step 1 let $1>0$. I V open VDE $m_A(V \setminus E) < 1$ Let BLOjx) = open ball centre a radius x $\overline{B}(0,1) = \text{cloud ball}$ $R = 8M$ $V_{n} = B(\omega_{jn})\cap V$. Then $\left\{V_{n}\right\}_{n=1}^{2^{n}}$ and V_{n} V_{n} V has finitenessed $\lim_{n\to\infty} m_N(V) = m_n(V)$ \Rightarrow 3 m, $m_N(V\vee v_m)<\eta$, Then $EV_{m}CV\vee w_m$ => m, (EWm) <y, Vm bounded open set. $1/10$) $|Q|$ (0 34 Jan 1835 - 24 Jan 1839)

So, we continue with the approximation results concerning the Lebesgue measure. So, we have the following proposition.

Proposition: Let $E \subset \mathbb{R}^N$ be a measurable set of finite measure. Given $\epsilon > 0$, there exists a compact set $K \subset E$ such that $m_N(K \backslash E) < \epsilon$.

proof: We will do it in a few steps.

step 1: Let $\eta > 0$. Then there exists V open, $V \supset E$ and $m_N(V \setminus E) < \epsilon$. Let $B(0, r)$ be the open ball centered at 0 of radius r and $\overline{B(0, r)}$ equals closed ball center 0 and radius r.

So, $n \in \mathbb{N}$, and you write $V_n = B(0, r) \cap V$. Then V_n are all open, $V_n \uparrow V$. So, V has finite measure why V as finite measure and you have that $n \rightarrow \infty$ lim \rightarrow $m_{N}(V_{n}) = m_{N}(V).$

$$
\Rightarrow \exists \; m, \; m_{_N}(V \backslash V_{_m}) \; < \; \eta \; . \; \text{Then} \; E \backslash V_{_m} \subset \; V \backslash V_{_m} \Rightarrow m_{_N}(E \backslash V_{_m}) \; < \; \eta
$$

and then we are taking limits but V_m is a bounded open set.

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 $Step 2. \ \ J \ \alpha \ \ cloud \ \ \\ 2 + FCV_m \ \ m_N(V_m \backslash F) \ \ \leq \ \eta$ F bald 2 closed $(V_m (d d) =)$ F is compact. Stp 3. Then, by styps 122, given EDO 3 a trid open set W n_1 . $m_N(E/W) < 4s$. and \exists c+ and $k_1 CW \Rightarrow k_1 / (W \setminus k_1) < 9s$. $F_{no}M_{1}$, $3F_{1}$ Closed, $F_{1}CE$ at $m_{nl}(F)F_{1}$ < 43 . Set K= K, NF, => K is compact & KCE $E(k = \sqrt{(E|W)} \cup (E\cap W) \cdot F_1) \cup (W\cap F_1) \setminus K_1)$ C (EW) $U(E\backslash F_{1})$ $U(W\backslash F_{1})$ $m_N(E|k) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$

Step 2: \exists a closed set $F \subset V_m$ so that $m_N(V_m \backslash F) < \eta$. So, now F is bounded and closed (since V_m is bounded) \Rightarrow F is compact.

Step 3: Thus by steps 1 and 2, given $\epsilon > 0$, there exists a bounded open set W such that $m_N(E\backslash W) < \frac{\epsilon}{3}$ and there exists a compact set $K_1 \subset W$ such that $m_N(W\backslash K_1) < \frac{\epsilon}{3}$ $\frac{e}{3}$.

Finally, there exists F_1 closed s.t. $F_1 \subset E$ and $m_N(E \backslash F_1) < \frac{\epsilon}{3}$. Set $\frac{e}{3}$.

$$
K = K_1 \cap F_1 \Rightarrow K \text{ is closed and } K \subset E.
$$

So,now

$$
E\backslash K\,=\,(E\backslash W)\,\cup\,((E\,\cap\,W)\backslash F_{_1})\,\cup\,((W\,\cap\,F_{_1})\backslash K_{_1})\,\subset\, (E\backslash W)\,\cup\,(E\backslash F_{_1})\,\cup\,(W\backslash K_{_1}).
$$

And therefore, $m_N(E\backslash K) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.

So, the only thing you really have to check here is this particular set theoretic identity.

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£ Pen. E CIRN while m_{μ} (E) = μ = $\left\{ m_{\mu}(0) \mid 0 \leq C \leq 1 \right\}$ my is outer-regular. $m_q(E)$ < $m_{\mathbf{u}}(E) = \text{sup}\left\{m_{\mathbf{u}}(k) \mid kCE \right\}$ Kempact) MN Drees-regular.

So, let us let me draw a picture for you so, that so, you have here the set E and then what you do, you found bounded open set W which was like this and then you took k1 which is a compact set and a closed set and you took a set F1 so, F1 intersection k so, this is the compact set, k which you finally got and this is a set E and therefore, if you want E minus k it consists of actually three parts which is E minus W which is here. And then you have the E intersection W minus F1 then W intersection F1 which is here minus k1.

So, if you add all that precisely E minus k. So, all the shaded new things which will give you E minus k, so that you can check it yourself and then so, this proves that approximation property.

Remark: So, you have $E \subset \mathbb{R}^N$ measurable. Then on one hand you have

$$
m_N(E) = \inf \{ m_N(U): E \subset U, \ U \text{ open} \}.
$$

So, m_N is called outer regular if $m_N(E) < \infty$ and $m_{N}(E) = \sup\{m_{N}(K): K \subset E, K \text{ compact}\}.$

In fact, the supremum relation you can also prove without this restriction on mN greater than lesser than plus infinity, leave it as an exercise for you and therefore, this is called inner regularity.

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Pem. E CIR^N while my(E) = inf {my(U) | ECU, Vapen } **NPTFI** \overline{c} m is Outer-regular. $m_q(E)$ < + ∞
 $m_q(E)$ = $\int_0^\infty m_q(k)$ | KCE, Kampact } MN JAREL regular. Any Bonal measure which is lett innerdoater very. is called regular (or a RADON Messure)

So, any Borel measure that means a measure defined on all Borel sets so Lebesgue measure is an example which is both inner and outer regular is called regular it is also all RADON, RADON measure is a Borel measure which can be which is both inner and outer regular this was the just a definition.

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 $Dep: X(fep)$ at $A CX.$ $\gamma_{a}(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$ Dif. 52 C.M open not A stop function defined on Shina fun dion of the form $f = \sum_{r=1}^{n} \alpha_r X_{r}$ K where a j 'sjan ar constants Ti, Isjah a laxes Contained in SL. P_{top} \top $CR^{\prime\prime}$ a \sqrt{OR} \approx 70 \exists q ϵ $C_{c}(R^{\prime\prime})$ \Rightarrow ϵ \Rightarrow ϵ 31 and $m_{1}(\tau-\epsilon\bar{\epsilon}^{2})$ $q(\epsilon)+\chi_{2}(\epsilon)\frac{2}{\epsilon})<\epsilon$.

Now, we move to approximation of functions involving measurable sets.

Definition: We have X (\neq ϕ) any set, $A \subset X$. Then we know

$$
\chi_A(x) = 1, \text{ if } x \in A,
$$

= 0, if $x \notin A$.

Definition: $\Omega \subset \mathbb{R}^N$ an open set. A step function defined on Ω is a function of the form

$$
f = \sum_{j=1}^{n} \alpha_j \chi_{I_j},
$$
 where α_j , $1 \le j \le n$ are constants and I_j , $1 \le j \le n$,

are are boxes contained in Ω.

So, it is a function which is made up of boxes characteristic functions on boxes.

Proposition: $I \subset \mathbb{R}^N$ box, $\epsilon > 0$ given. Then there exists $\phi \in C_c(\mathbb{R}^N)$ *s. t.* $0 \le \phi \le 1$ and

$$
m_{N}(\{x \in \mathbb{R}^{N} : \phi(x) \neq \chi_{I}(x)\}) < \epsilon.
$$

Further $supp(\phi) \subset I$.

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 $\frac{1}{\text{fwn dim of the }f^{\text{tan}}}$ $f = \frac{1}{\sqrt{2}}$ of $\frac{1}{\sqrt{2}}$ where a j, 15/5 are combats Tj, 15/52 a laxes Contained in 2. R_{top} \top $CR^{\prime\prime}$ a \sqrt{PR} $R>0$ \exists $\phi \in C_{c}(\overline{R}^{\prime\prime})$ R . \circ $\leq \phi \leq 1$ $\hat{\mathbf{v}}$ and m_{ν} ($\{\approx \epsilon \bar{\epsilon} \tilde{\epsilon}^{\nu} \mid \epsilon(\omega) \neq \gamma_{\mu}(\omega) \}$) < ϵ . F_{unkl} $\neg_{unpl}(q)$ C_1 . PF: We can find laxen 3, J2 At JCJ CJ_CJ $3 > (CTS)_{M}$ + a lamp

proof: We can find boxes J_1 , J_2 *s. t.* $J_1 \subset J_2 \subset J_2 \subset I$ and such that $m_N(I \setminus J_1) < \epsilon$.

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So, in two dimensions let me draw a picture here. So, this is a box I and then I make smaller boxes J_1 , J_2 and then I, and then I make these boundaries sufficiently close so that they are all the whole thing difference in the area is less than epsilon so that I can do.

Now if you use Urysohn's Lemma, there exists a continuous function φ continuous function on \mathbb{R}^N such that $0 \leq \phi(x) \leq 1 \forall x, \phi \equiv 1 \text{ on } J_1$ and $\phi \equiv \overline{J_2}$ $\frac{c}{\cdot}$

So, you have two disjoint open sets. So, J_1 is an open box.

Let me write that first J1 open box. So, I have these two disjoint So, J1 should be a closed box because we are going to apply Urysohn's Lemma and let us do it correctly. So, J_1 is a closed box and J_2 is an open box.

> and m_{μ} ($\left\{ -\frac{1}{2}e^{i\theta} + \frac{1}{2}e^{i\theta} \right\}$) < E. FF: We can find laxen 3, J2 At JCJ_CJ_CI and at Mr (IV3) < c. J, aparel dox. Ŧ By Urysohis lemma, 3 pent from PM st. Ospings 1 th $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ $\ddot{}$ By Urysoli's lemme, a pent of on R st. Ospinis 1 th 95 lon 3 , and 950 on 3^{6} $7,03$ and
 $3,35$ closes \Rightarrow supp $C\overline{3}$, $C\overline{2}$, $\overline{3}$, and \Rightarrow $\varphi \in C_c(\overline{n})$ \circ \circ $\varphi \in \mathbb{Z}$ $\{x \mid \varphi(x) = \lambda_{x}(x) \} \subset I \setminus J_{1} \qquad m_{A}(\Gamma \setminus J_{1}) < E.$

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So, this is the J_2 complement so, you have J_2 complement is a closed set J_1 is a closed set and J1 intersection J2 complement is empty and J1, J2 complement or closed. Therefore, by Urysohn's Lemma I can construct such a function which is like this. So, this implies that

$$
\Rightarrow supp(\phi) \subset \overline{J_2} \subset I, \overline{J_2} \; cpt \; \Rightarrow \; \phi \in C_{\mathcal{C}}(\mathbb{R}^N), \; supp(\phi) \subset I.
$$

Now, what about the set $\{x: \phi(x) \neq \chi_l(x)\} \subset I \setminus I_1$ and then $m_N(I \setminus I_1) < \epsilon$.

So, that proves the theorem. So, it is just a simple application of Urysohn's Lemma. (Refer Slide Time: 23:11)

Corollary: $\Omega \subset \mathbb{R}^N$ open set and $f: \Omega \to \mathbb{R}$ a step function, $\epsilon > 0$. Then there exists $\Phi \in C_c(\Omega)$ s.t. $m_N(\{x \in \Omega : f(x) \neq \Phi(x)\}) < \epsilon$ and $\max_{x \in \Omega} |\Phi(x)| \leq \max_{x \in \Omega} |f(x)|$.

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$$
\frac{P_{rad}f_{i}}{f_{i}} = \sum_{i=1}^{n} a_{i}^{T} \frac{Y_{T_{i}}}{Y_{T_{i}}}
$$
\n11.066. T_{i}^{S} and all $di\frac{1}{2}h$.

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proof: So, let us take $f = \sum_{\alpha, \chi_i}$. Without loss of generality, let I are all disjoint. $i=1$ k $\sum_{i=1}^{\infty} \alpha_i \chi_{I_{i}}$ I_{j}

So, there exists functions $\phi_j \in C(\mathbb{R}^n)$, $0 \le \phi_j \le 1$, $supp(\phi_j) \subset I_j$ and $\mathcal C$ (\mathbb{R}^N) , $0 \le \phi_j \le 1$, $supp(\phi_j) \subset I_j$

$$
m_{N}(\{x \in \mathbb{R}^{N} : \phi_{j}(x) \neq \chi_{j}(x)\}) < \frac{\epsilon}{2}.
$$

Now, define $\phi = \sum \alpha_i \phi_i$. So $j=1$ k $\sum_i \alpha_i \phi_i$.

 ${x \in \Omega: \phi(x) \neq f(x)} \subset U_{i=1}$ ${}^{k}(x \in \Omega: \phi_{j}(x) \neq \chi_{l_{j}})$ (x) } ⊂ $U_{i=1}$ ¹ ${}^{k}\{x \in \mathbb{R}^{N}: \phi_{j}(x) \neq \chi_{l_{j}}}$ $(x)\}$ \Rightarrow m_N ({ $x \in \Omega$: $\phi(x) \neq f(x)$ }) < ϵ .

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So, we have found a continuous function phi with these properties: $supp(\phi_j) \subset I_j$, all disjoint \Rightarrow $\max_{x \in \Omega} |\phi(x)| \leq \max_{1 \leq j \leq k} |\alpha_j| = \max_{x \in \Omega} |f(x)|$.

Finally, ϕ has compact support because support ϕ_j is in Ij, Ij is a finite box. So, inside you have a closed set which is closed, bounded therefore, it is compact and compact support contained in $\bigcup_{j=1}^n$ ${}^{k}I_{j} \subset \Omega \Rightarrow \Phi \in C_{c}(\Omega).$

So, that completes the proof of this step. So, we will continue afterwards.