Measure and Integration
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Lecture-13
2.8-Approximation

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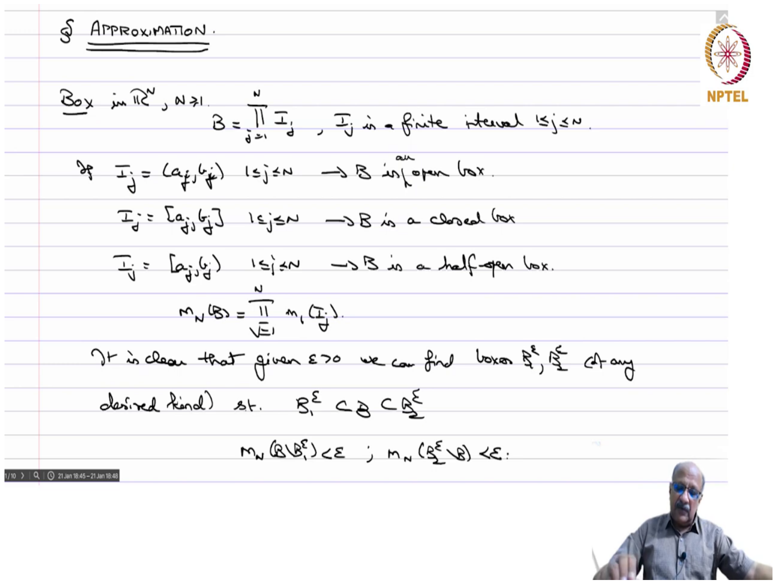

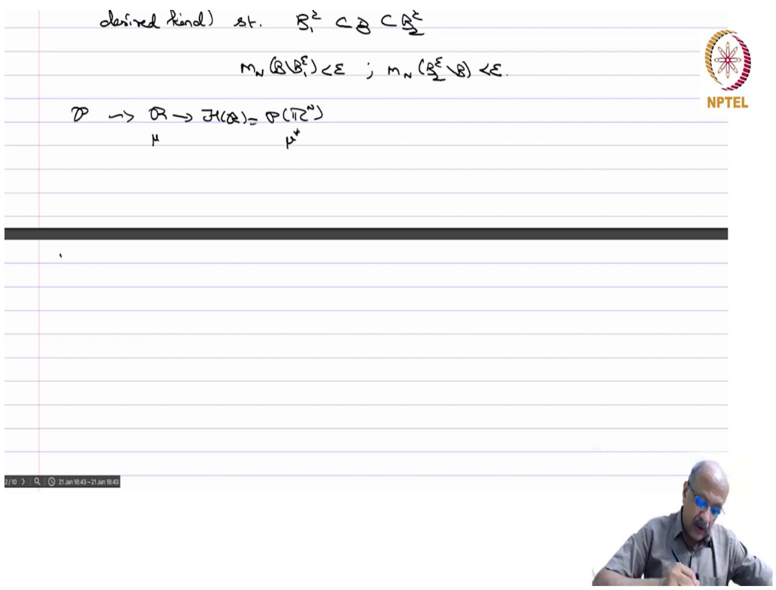

Next section in the study of the Lebesgue Measure is approximation. So, in this section we will present various results on approximation of measurable sets by topological sets in the context of the Lebesgue Measure. So, by a box in $\mathbb{R}^N$, we mean $N \geq 1$, we will mean a set

$$B = \Pi_{j=1}^{N} I_j,\ I_j \text{ is a finite interval for } 1 \le j \le N.$$

If $I_j = (a_j, b_j),\ 1 \le j \le N, \rightarrow B$ is an open box.

If $I_j = [a_j, b_j],\ 1 \le j \le N, \rightarrow B$ is a closed box.

If $I_j = [a_j, b_j),\ 1 \le j \le N, \rightarrow B$ is a half-open box.

So, if B is a box, then $m_N(B) = \Pi_{j=1}^{N} m_1(I_j)$. Also it is clear that given $\epsilon > 0$, we can find boxes $B_1^{\epsilon},\ B_2^{\epsilon}$ of any desired kind open closed or half open et cetera such that $B_1^{\epsilon} \subset B \subset B_2^{\epsilon}$

and $m_N(B \backslash B_1^{\epsilon}) < \epsilon,\ m_N(B_2^{\epsilon} \backslash B) < \epsilon$.

So, you just have to move the boundaries a little bit for the box and then you will have whatever you want. So, now, recall that how we Lebesgue Measure come you had this ring are from these boxes, semi open boxes, half open boxes, P is half open boxes then we had the ring and then from the ring we went to the head 3 sigma ring generated by this which is nothing but the power set of $\mathbb{R}^N$.

And there, so, from the measure here we had the natural outer measure from which we got the measurable sets which gave you the Lebesgue Measure. So, we will use this notation μ * and whenever we say measurable I mean Lebesgue Measure sets. Because they are also Borel measurable sets. So, measurable will mean unless otherwise specified Lebesgue Measure sets.

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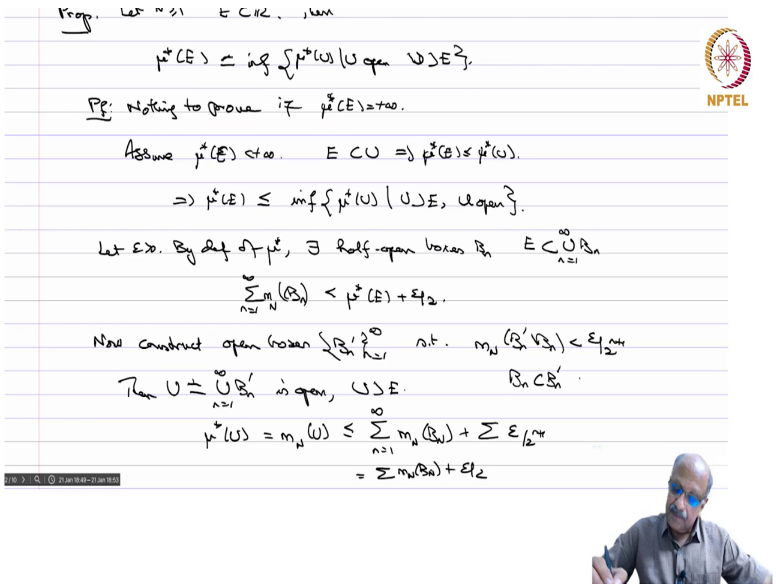

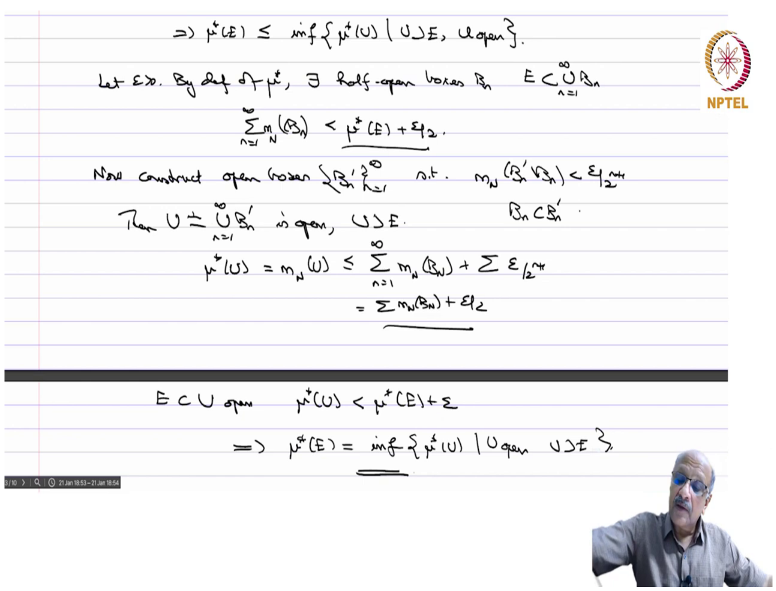

Proposition: Let $N \geq 1$, $E \subset \mathbb{R}^N$. Then

$$\mu^*(E) = \inf\{\mu^*(U) : U \text{ open}, U \supset E\}.$$

proof: nothing to prove if $\mu^*(E)$ is infinite.

So, assume that $\mu^*(E)$ is finite. Now, $E \subset U \Rightarrow \mu^*(E) \leq \mu^*(U)$. Therefore, you have

$$\mu^*(E) \leq \inf\{\mu^*(U) : U \text{ open}, U \supset E\}.$$

Let $\epsilon > 0$. By definition of outer measure there exists half open boxes B_n, $E \subset \cup_{n=1}^{\infty} B_n$ and

$$\sum_{n=1}^{\infty} m_N(B_n) < \mu^*(E) + \frac{\epsilon}{2}.$$

Now construct open boxes $\{B_n'\}_{n=1}^{\infty}$ $s.t.\ m_N(B_n' \backslash B_n) < \frac{\epsilon}{2^{n+1}}$ $and\ B_n \subset B_n'$.

So $U = \cup_{n=1}^{\infty} B_n'$ is open, $E \subset U$, and

$$\mu^*(U) = m_N(U) \leq \sum_{n=1}^{\infty} m_N(B\ ) + \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} = \sum_{n=1}^{\infty} m_N(B\ ) + \frac{\epsilon}{2}.$$

So, we have $E \subset U$ open and $\mu^*(U) < \mu^*(E) + \epsilon$. So, this implies that

$$\mu^*(E) = \inf\{\mu^*(U): U\ open,\ U \supset E\}$$

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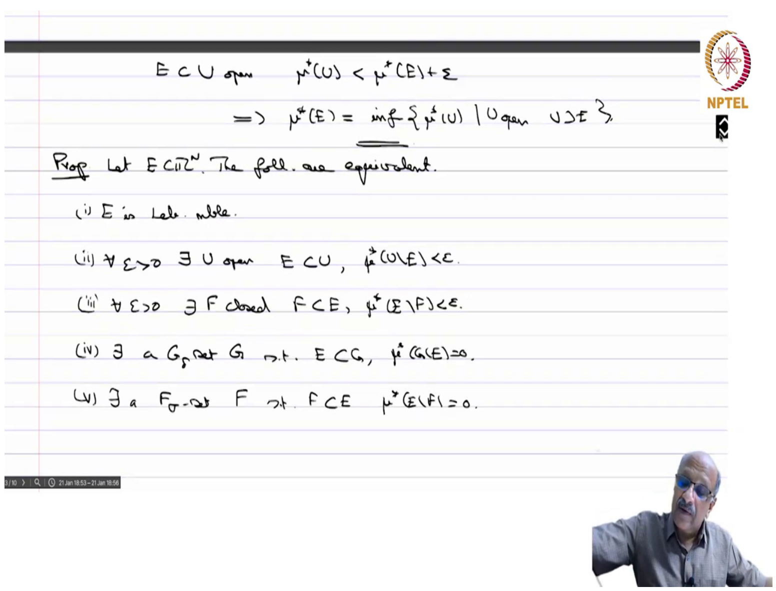

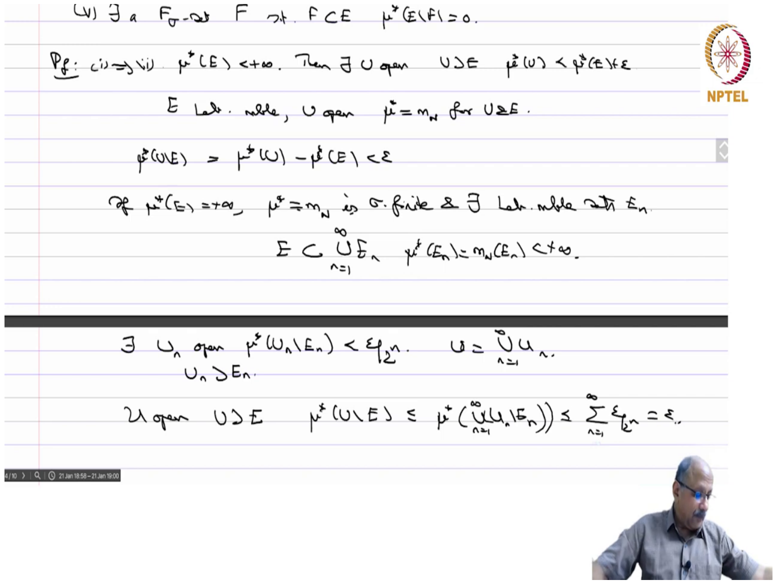

So, now we have a proposition which characteristic Lebesgue Measurable sets.

Proposition: So, let $E \subset \mathbb{R}^N$. The following are equivalent.

(i) E is Lebesgue measurable.

(ii) for every $\epsilon > 0,$ there exists U open, $E \subset U,$ and $\mu * (U\backslash E) < \epsilon$.

(iii) for every $\epsilon > 0,$ there exists F closed, $F \subset E,$ and $\mu * (E\backslash F) < \epsilon$.

(iv) there exists a G_δ set G s.t. $E \subset G$ and $\mu * (G\backslash E) = 0$.

(v) there exists a F_σ set F s.t. $F \subset E$ and $\mu * (E\backslash F) = 0$.

proof: (i)⇒(ii): Assume $\mu * (E) < \infty$. Then there exists U open s.t. $E \subset U$ *and*

$\mu * (U) < \mu * (E) + \epsilon$. But then everything is finite and therefore, you have E is Lebesgue measurable, U is open, therefore, Borel and hence Lebesgue measurable so, $\mu * = m_N$ for U, E. So, you have $\mu * (U\backslash E) = \mu * (U) - \mu * (E) < \epsilon$.

So, if $\mu^*(E) = +\infty$, then $\mu^* = m_N$ is sigma finite and there exists Lebesgue measurable sets E_n such that $E \subset \cup_{n=1}^{\infty} E_n$, $\mu^*(E_n) = m_N(E_n) < \infty$. So, there exists $U_n \supset E_n$ open and you have that $\mu^*(U_n \backslash E_n) < \frac{\epsilon}{2^n}$. Then you take $U = \cup_{n=1}^{\infty} U_n$ and therefore, U is open, $U \supset E$,

$$\mu^*(U \backslash E) \le \mu^*(\cup_{n=1}^{\infty}(U_n \backslash E_n)) \le \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

So, this completes the proof of one implies two.

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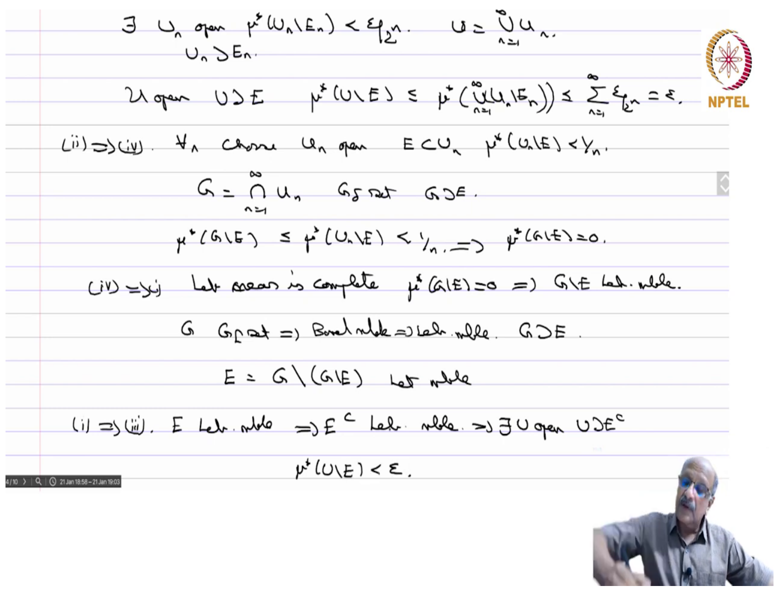
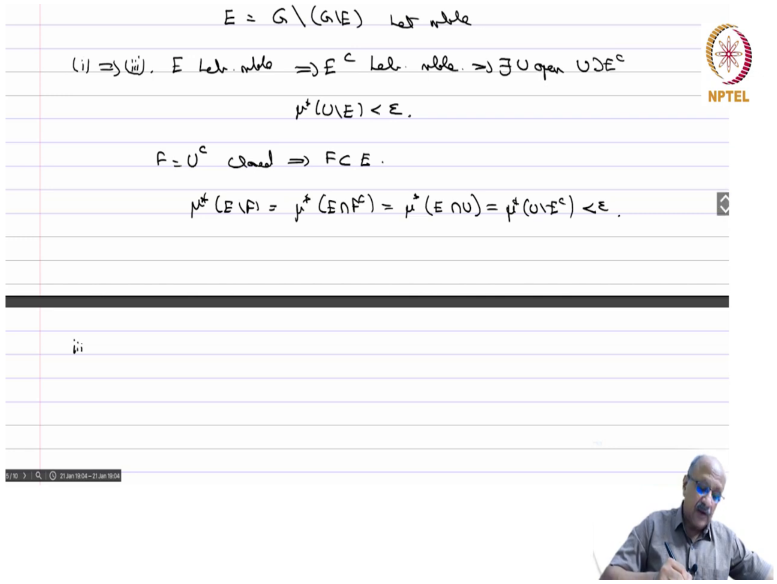




(ii)⇒(iv): For each n, choose U_n open s.t. $E \subset U_n$ *and* $\mu*(U_n \setminus E) < \frac{1}{n}$.

So, then you have that $G = \cap_{n=1}^{\infty} U_n$ and then G *is* G_δ and $E \subset G$ and

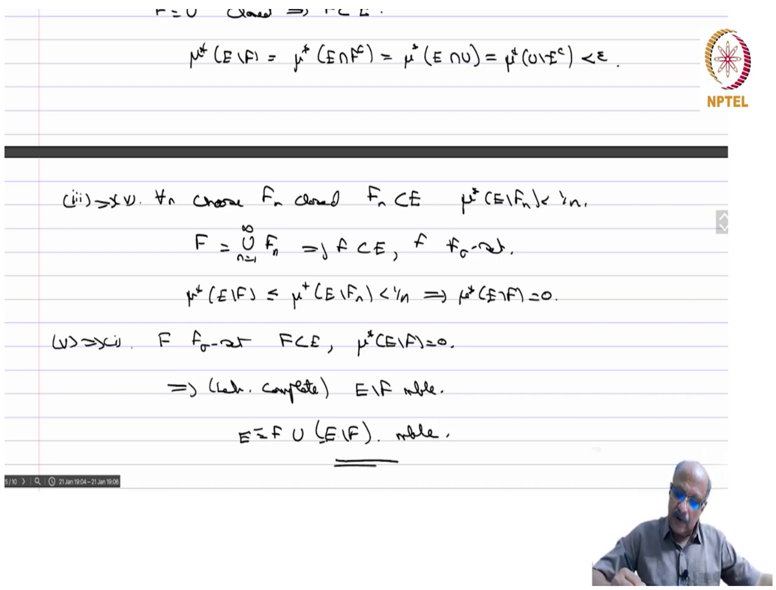
$$\mu*(G\setminus E) \le \mu*(U_n \setminus E) < \frac{1}{n} \Rightarrow \mu*(G\setminus E) = 0.$$

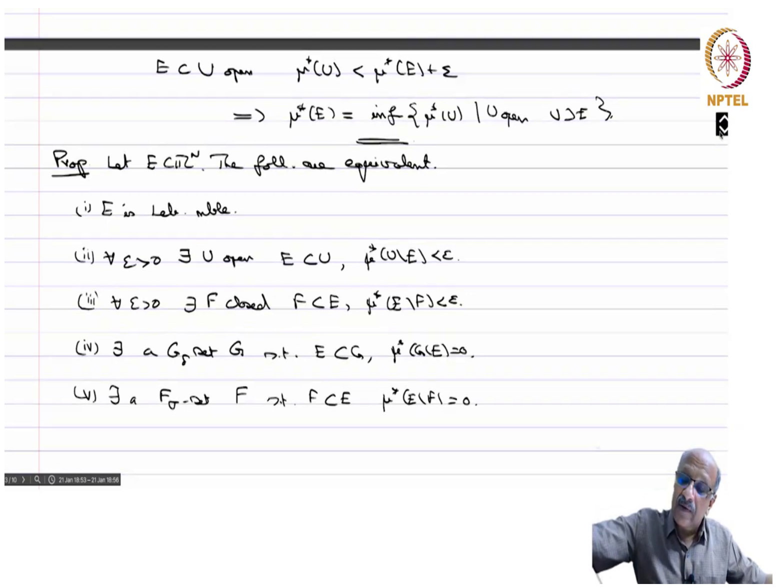
(iv)⇒(i): So, Lebesgue measure is complete so, $\mu*(G\setminus E) = 0 \Rightarrow G\setminus E$ is Lebesgue measurable and therefore, now G is a G-delta set implies Borel measurable implies Lebesgue measurable also. And then $E = G\setminus(G\setminus E)$ is Lebesgue measurable.

(i)⇒(iii): E Lebesgue measurable ⇒ E^c Lebesgue measurable because it is a sigma algebra therefore, there exists a U open, $U \supset E^c$ and $\mu*(U\setminus E) < \epsilon$. So, $F = U^c$- closed and therefore,

$F \subset E$ and then $\mu*(E\setminus F) = \mu*(E \cap F^c) = \mu*(E \cap U) = \mu*(E\setminus U^c) < \epsilon$.

So, this proves. (Refer Slide Time: 19:19)





(iii)⇒(v): for every n, choose F_n closed, $F_n \subset E,\ \mu^*(E \backslash F_n) < \frac{1}{n}$. And you take

$$F = \cup_{n=1}^{\infty} F_n \Rightarrow F \subset E,\ F\ is\ F_{\sigma}\ set \text{ and}$$

$$\mu^*(E \backslash F) \leq \mu^*(E \backslash F_n) < \frac{1}{n} \Rightarrow \mu^*(E \backslash F) = 0.$$

(v)⇒(i): So, $F\ is\ F_{\sigma}\ set,\ F \subset E,\ \mu^*(E \backslash F) = 0 \Rightarrow E \backslash F$ is Lebesgue measurable as Lebesgue measure is complete and therefore, $E = F \cup (E \backslash F)$, and therefore, this is measurable.

So, this completes this thing. So, we have various equivalent forms of so, you can approximate to the such E is Lebesgue measurable then you can approximate it by means of an open set from above that means, the difference between the two is a very small measure.

Next, we will investigate the approximation by compact sets so that we will do it next time.