**Measure and Integration**
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**Lecture-12**
**2.7 The Cantor Set**

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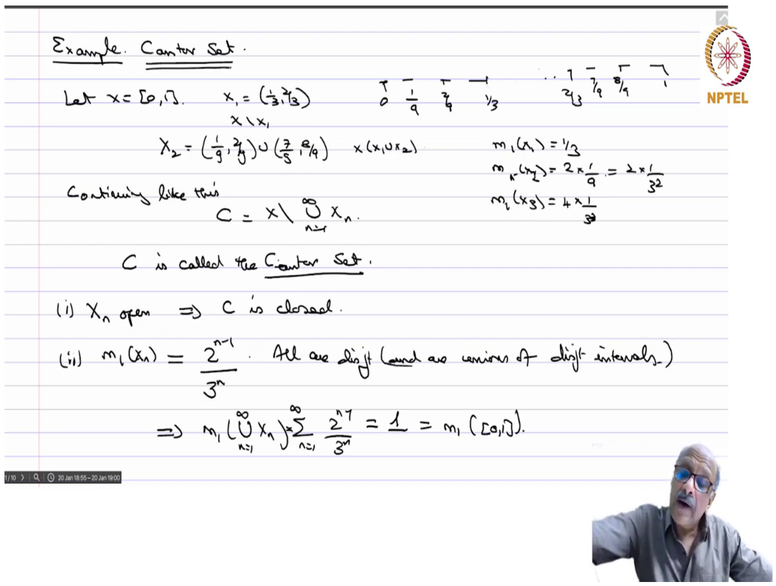


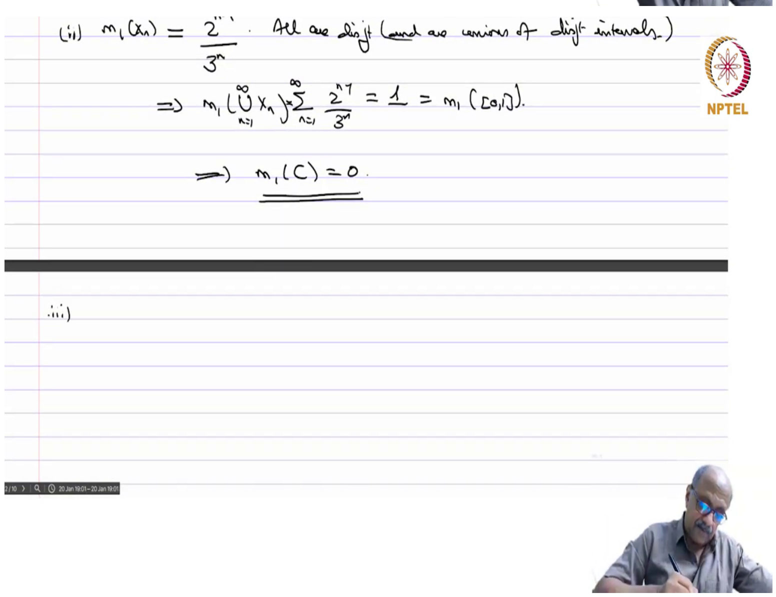


We now study for example, it is very important and very nice also, it is a good source of counter examples.

**Example: (**Cantor set)

So let $X = [0, 1]$, $X_1 = [\frac{1}{3}, \frac{2}{3}]$.

So, you take the interval $[0, 1]$. And then you divide it into three parts 1 by 3, and this is 2 by 3, and we want to remove this portion now. So, I am going to remove it, so, let me erase it, and then I write x1, so I write x1 like this.

Now, I take the middle (())(01:20) each one of them into three parts, and then this becomes 1 by 9, this is 2 by 9, and then this is 7 by 9, 8 by 9 and then I write

$$X_2 = (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}).$$

x2 is the middle of that. So, it is 1 by 9 union 2 by 9 union 7 by 9, 8 by 9 these are all disjoint open intervals, and then I removed those also. So, now, I continue like this. So, we continue like this.

So, this gives you first we had x minus x1 then you had x minus x1 union x2 and so on. And now you continue like this. We define

$$C = X \setminus \cup_{n=1}^{\infty} X_n .$$

Now $C$ is called the Cantor set.

So, next time, what would I do, I would remove a portion here. And then I would remove a portion here, I would remove a portion here, and I remove a portion here and so on. And now I will go on like this. So, I will get something very (())(03:16), everything will be turned up, there will be some collection of points, and you have this.

(i) $X_n$ - is open, because it is a joint union of open intervals. Anyway, it is a union of open intervals. So, it is open and therefore, C is closed.

(ii) Two, measure of xn so, what is measured of x1? m1 of x1 is equal to 1 by 3, m1 of x2 is 2 times 1 by 9, and.

So, m1 of x3 will be there before such intervals 1 by 3 cube for this equal to 2 into 1 by 3 squared, 4 by 1 by 3 cube. So, in general, interactively

$$m_1(X_n) = \frac{2^{n-1}}{3^n}.$$

And all are disjoint. And are unions of disjoint intervals that is how we computed the measure of each of these therefore, this implies that

$$m_1(\cup_{n=1}^{\infty} X_n) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 1 = m_1([0, 1]) \Rightarrow m_1(C) = 0.$$

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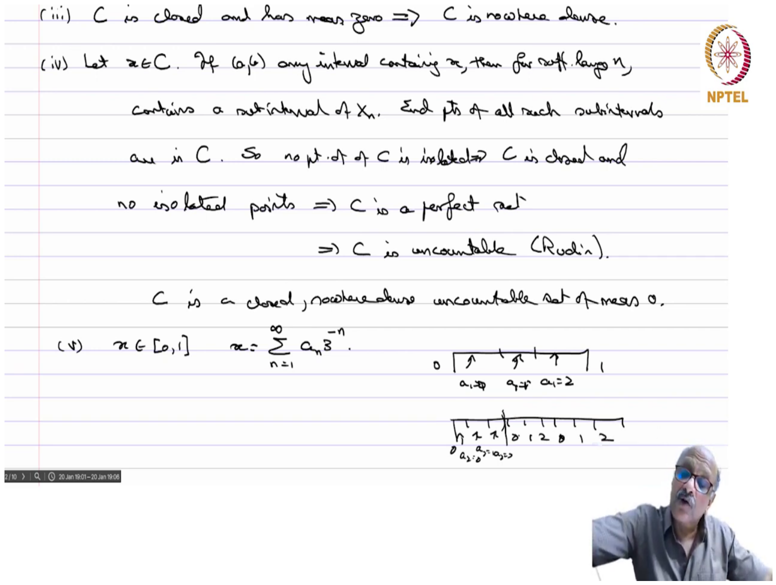


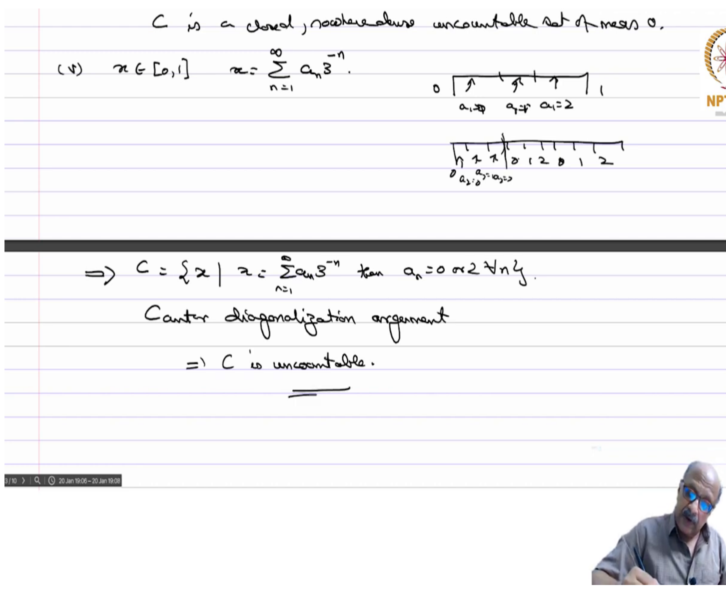


(iii) C is closed and has measure 0, and therefore, this implies C is nowhere dense.

(iv) Let $x \in C$. If $(a, b)$ any interval containing x, then for sufficiently large n contains a sub interval of $X_n$, because those are of length 1 by 3 power n end this. Now, endpoints of all such sub intervals are in C.

So, no point of C is isolated because every neighborhood you can find points in C again and therefore, say it is not so implies that it is closed so, close C is closed and no isolated points implies C is a perfect set to the definition of a perfect set and this implies C is uncountable. So, this is a theorem from Rudin.

You can check in Rudin books for instance principles of mathematical analysis that the perfect set in are should always be uncountable.

So, C is a closed nowhere dense uncountable set of measure 0. So, we know that countable sets are measures 0. So, the question is there uncountable sets of measures is 0, so the Cantor set gives you an example.

So, now, we can also show that without using this perfectness, we can also show that C is uncountable in the following way. So, you take any x in 0 1. And it is ternary expansion that means $x = \sum_{n=1}^{\infty} a_n 3^{-n}$.

How do you compute this a n? You take 0 1 divided into 3 parts. So, if x comes here, then a 1 equals 0.

If x comes here, a1 equals 1 and here it is a1 equals 2, because here is 2 by 3 plus something here it is 1 by 3 plus something here something less than 1 by 3. So, a1 will be 0. Now, if you want to do a2, you have to do the same thing and now we will divide it into 1 2 3 4 5 6 7 8 9 parts. So, then if it is you take this here, and then if a1 is if it comes here, then you have a2 equal to 0, if it comes here you have a2 equal to 1, if it comes here equals a2 equals 2 and so on it repeats here, 0 1 2 0 1 2 and so on.

And like this, you can continue to write down the start and end of the expansion of the sets. So, this implies, so, what does it mean we have removed all the middle thirds to get C. So, C equals set of all x such that if you write x equals sigma an 3 power minus n, n equals 1 to infinity, then a n equals 0 or 2 for all n. So, you do not have a1 equal to 1.

And now, you can, so, if you know, it becomes a simple Cantor's diagonalization argument, so, you have only two possibilities and should be 0 or 2. So, if you make a listing of all the x in C if possible, then you look at the first point, if it is 0, you put 2 if it is 2, you put 0 for each number, then you go to the second point and do like this.

So, that new number x which you get will be different from every one of the numbers. So, this is a cantor diagonalization argument. How did you prove that [0,1] is uncountable, so, you do the same thing you take the binary expansion there, and then you have only 0 or 1 other 2 possibilities. So, if it is 0, you put 1 in that place, if it is 1 you put 0 in that place, you create a new number, which is different from all the previous numbers. And therefore, you cannot exhaust by numbering them in a countable way. That is the same argument here. Instead of 0 and 1 you have 0 and 2 and therefore, the argument implies C is uncountable. So, this is about that example.

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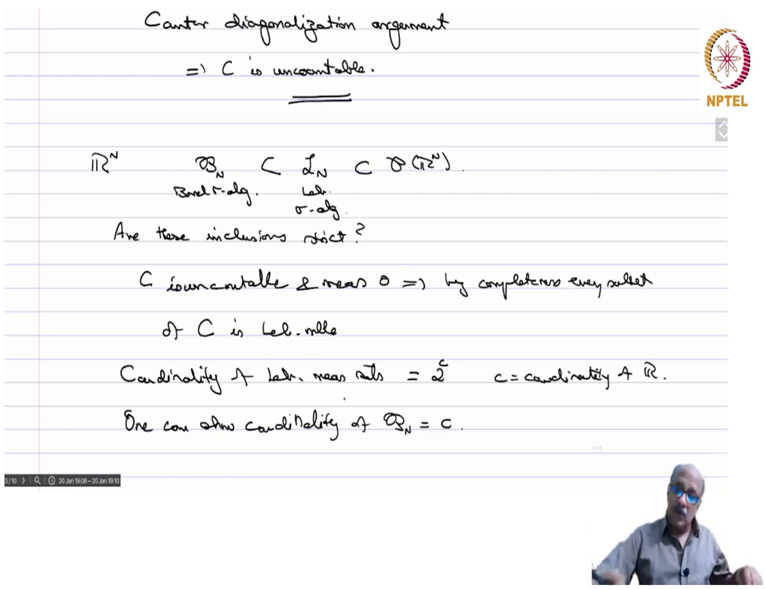


So, now let us see whether we have to go. So, if you are looking at $\mathbb{R}^N$, then you have three distinguished sets: you have $B_N$-the Borel sigma algebra $\subset$ $L_N$- the Lebesgue sigma algebra, $\subset$ $P(\mathbb{R}^N)$ power set of $\mathbb{R}^N$. So, the question is, are these inclusions strict?

So, can you have a Lebesgue measurable set which is not Borel measurable, can you have a subset of $\mathbb{R}^N$ which is not Lebesgue measurable. So, in other words (())(13:34) ln equal to the power set or not or $B_N = L_N$. Now, we will see these again later on, but for the moment I will give you an argument which is not complete.

Because I will not be able to prove everything I say, but it certainly tells you one of the uses of the cantor set now C is uncountable and measures 0, which implies by completeness every subset of C is Lebesgue measurable. Therefore, cardinality of Lebesgue measurable sets is exactly $2^{c}$, $c$ is the cardinality of the content of ℝ.

C is the uncountable first uncountable number and then so, this means there is one to one correspondence between subsets of ℝ and Lebesgue measurable sets. So, there you have that the cardinality is $2^{c}$, one can show the cardinality of $B_{N}$ is nothing but c. So, obviously this is much less than that. So, there do exist Lebesgue measurable sets, which are not Borel measurable. So, that comes from the argument of the cantor set, but we will also later use what is called the cantor functions and produce an explicit example of a Borel Lebesgue measurable set which is not Borel measurements. So, we will now continue with the properties of the Lebesgue measure next time.