Measure and Integration
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Lecture - 10
2.5 Construction of The Lebesgue Measure

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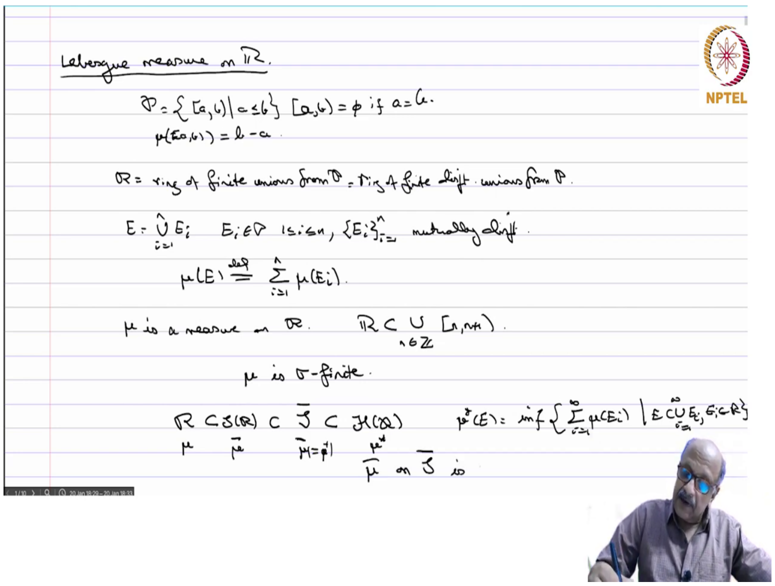

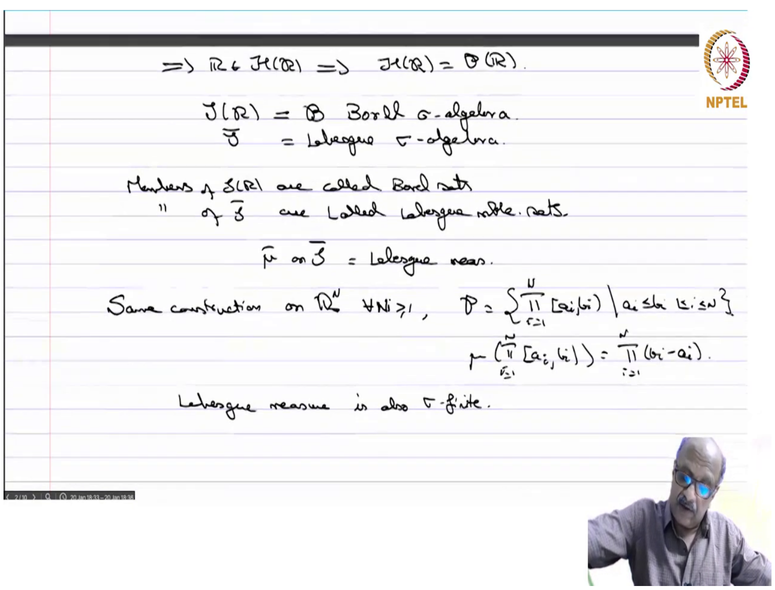

So, we now construct the Lebesgue Measure on $\mathbb{R}$. So, recall that we have a ring. So, we started with $P = \{[a, b): a \le b\}$, $[a, b) = \phi \ if\ a = b$ and we define $\mu([a, b)) = b - a$ and then we took R equals ring of finite unions from P equals ring of finite disjoint unions from P and then if $E = \{E_i\}_{i=1}^{n}$, $E_i \in P\ \forall\ 1 \le i \le n$, $\{E_i\}$ mutually disjoint. Then we define

$$\mu(E) = \sum_{i=1}^{n} \mu(E_i).$$

So, and then μ is a measure on $\mathbb{R}$, and since you can cover $\mathbb{R} = \cup_{n\in\mathbb{Z}}[n, n + 1)$.

Therefore, μ is σ-finite.

So, now we are ready to bring about the method of (())(02:44). So, we have here

$$\mathbb{R} \subset S(\mathbb{R}) \subset \overline{S} \subset H(\mathbb{R}).$$

So, $\mu * (E) = \inf\{\sum_{i=1}^{\infty} \mu(E_i): E \subset \cup_{i=1} E_i, E_i \in \mathbb{R}\} \Rightarrow \mathbb{R} \in H(R) \Rightarrow H(R) = P(R).$

Now, $S(\mathbb{R})$= B-Borel σ-algebra and $\overline{S}$-Lebesgue σ-algebra. Now, you can do the same thing same construction on $\mathbb{R}^N$, $\forall N \geq 1$ with

$$P = \{\Pi_{i=1}^{N} [a_i, b_i): a_i \leq b_i, 1 \leq i \leq N\}, \mu(\Pi_{i=1}^{N} [a_i, b_i)) = \Pi_{i=1}^{N}(b_i - a_i).$$

So, in other words this Lebesgue Measure in $\mathbb{R}$ is called the extension of the notion of length and in $\mathbb{R}^2$ it is the extension for areas of 3 volumes and so on and so forth. So, Lebesgue Measure because it comes from a σ-finite measure on a ring, so, it is also σ-finite. In fact, this itself tells you this relation itself tells you that it is sigma finite.

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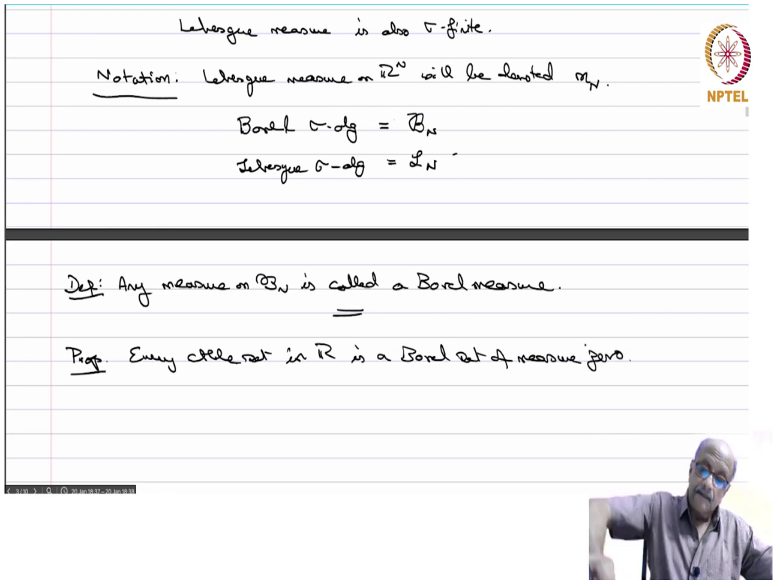

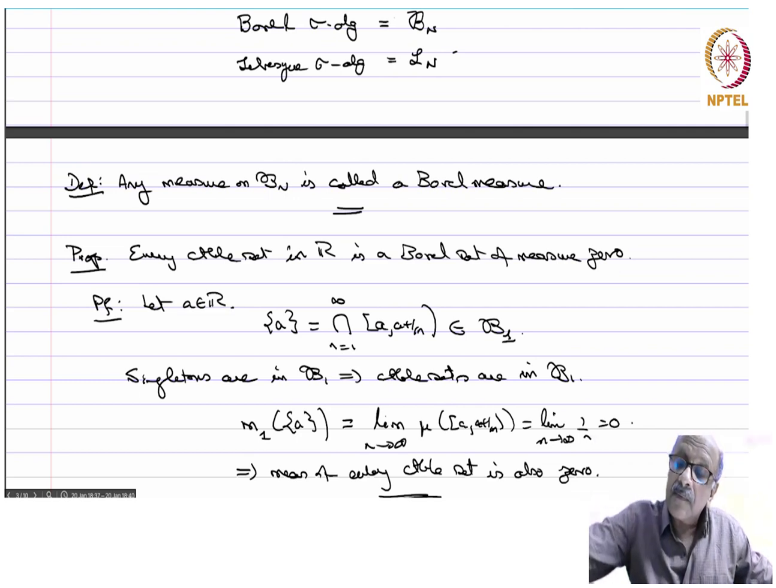

Notation: Henceforth, we will use the following convenient notation because we should be somewhat systematic, so, Lebesgue Measure on $\mathbb{R}^N$ will be denoted m_N. And then

$$\text{Borel } \sigma\text{-algebra} = B_N\,,\ \ \text{Lebesgue } \sigma\text{-algebra} = L_N\,.$$

So, this is a notation which we will henceforth follow in this course.

Definition: Any measure on B_N is called Borel measure.

Properties: Every countable set in $\mathbb{R}$ is a Borel set of measures zero.

proof: Let $a \in \mathbb{R}$ and then you have $\{a\} = \cap_{n=1}^{\infty} [a, a + \frac{1}{n}) \in B_1$.

So, Singleton's are in B_1 implies countable sets because it is a countable union of singleton are in B_1. Now $m_1(\{a\}) = \lim_{n \to \infty} \mu([a, a + \frac{1}{n})) = \lim_{n \to \infty} \frac{1}{n} = 0 \Rightarrow$ The measure of arbitrary countable sets is also zero.

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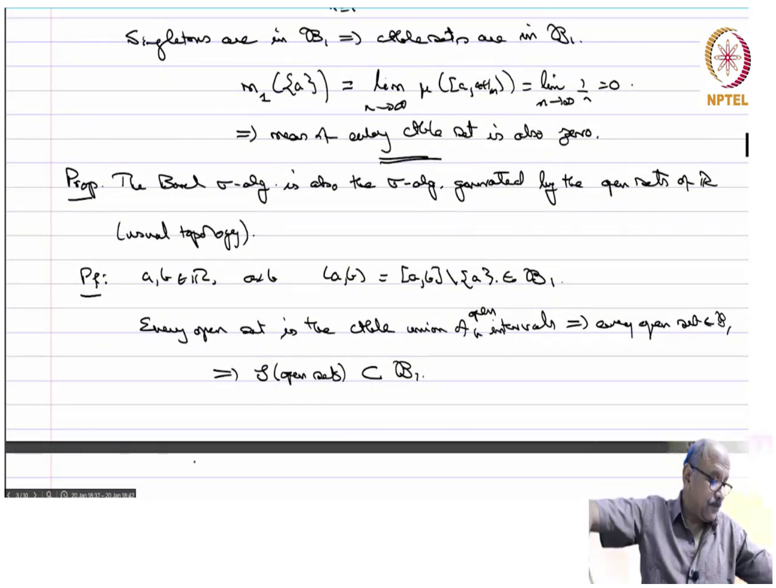

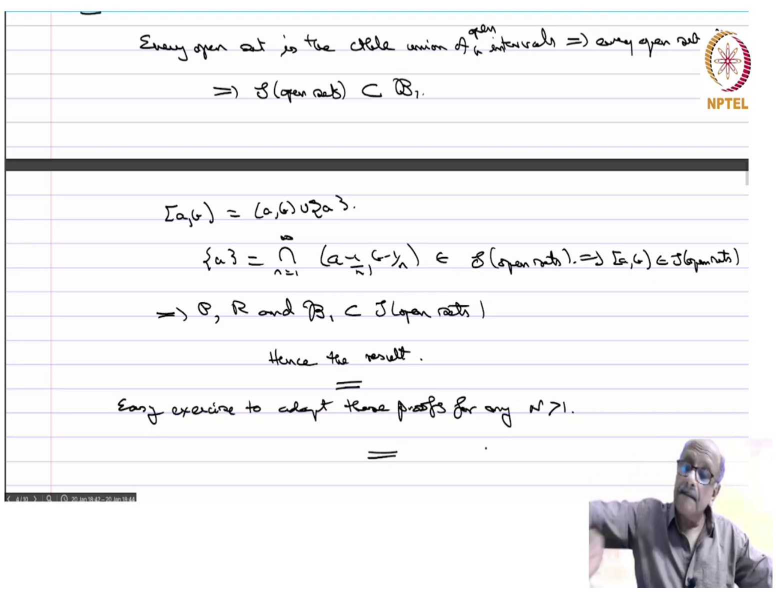

Proposition: The Borel σ-algebra is also the σ-algebra generated by the open sets of $\mathbb{R}$ (usual topology)

proof: Let us take $a, b \in \mathbb{R}$, $a < b$. Then you have $(a, b) = [a, b)\setminus\{a\} \in B_1$.

But every open set is the countable union of open intervals ⇒ every open set belongs to B_1

$\Rightarrow S(B_1) \subset B_1$.

Conversely, $[a, b) = (a, b) \cup \{a\}$. And $\{a\} = \cap_{n=1}^{\infty} (a - \frac{1}{n}, b - \frac{1}{n}) \in S(open\ sets)$

$\Rightarrow [a, b) \in S(open\ sets)$.

$\Rightarrow P, \mathbb{R}, B_1 \subset S(open\ sets)$.

Hence the result.

So, hence we have both the inclusion and therefore, now, we know the Borel sigma algebra is nothing but the thing. So, in general if you have a topological space and you define the measure on it and gender I mean topological space then this smaller sigma algebra containing open sets is called the borel sigma algebra even in the abstract if you have an abstract topological space. So, now, easy exercise to adapt these proofs for any N so, all these results are true both in R as well as R.

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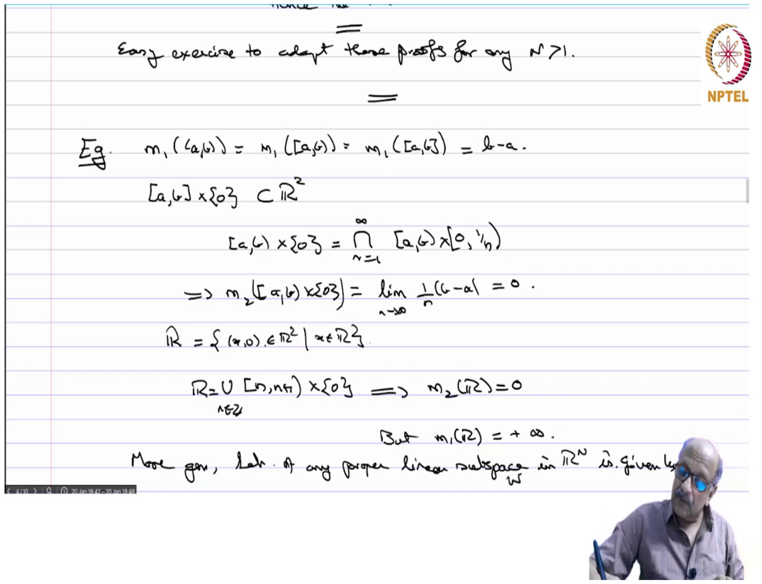

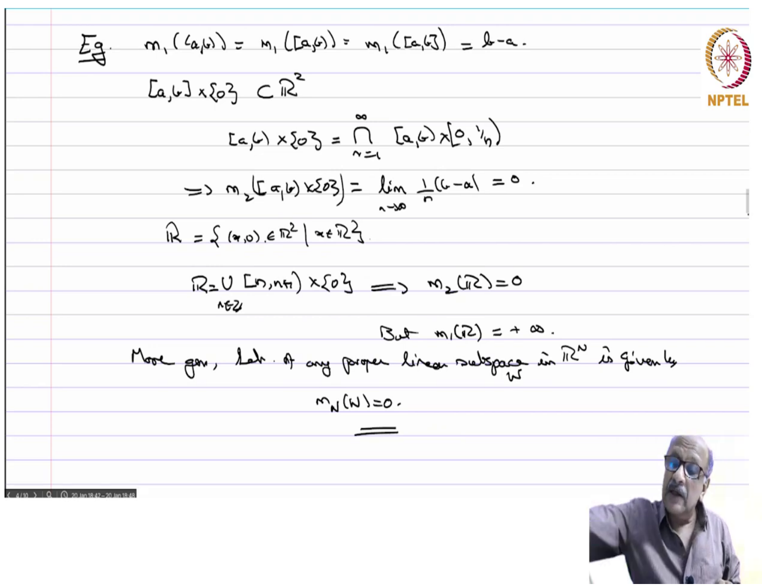

Example: $m_1((a,b)) = m_1([a,b)) = m_1([a,b]) = b - a$.Now, let us take $[a,b] \times \{0\} \subset \mathbb{R}^2$

So, $[a,b] \times \{0\} = \cap_{N=1}^{\infty}[a,b) \times [0,\frac{1}{n}) \Rightarrow m_2([a,b] \times \{0\}) = \lim_{n\to\infty} \frac{1}{n}(b-a) = 0.$

So, now you consider $\mathbb{R} = \cup_{n\in\mathbb{Z}}[n, n+1) \times \{0\} \Rightarrow m_2(\mathbb{R}) = 0 \; but \; m_1(\mathbb{R}) = +\infty.$

More generally Lebesgue Measure of any proper linear subspace in $\mathbb{R}^N$ is given by let us say W, $m_N(W) = 0$. So, if you have any proper subspace it will always be of measure 0.

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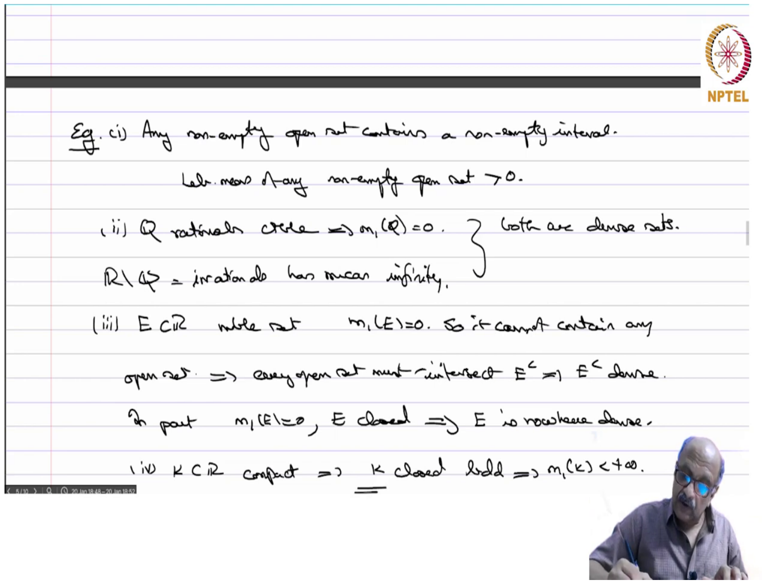

Example: (i) where several little things which we will observe so, any non-empty open set contains non-empty interval therefore, Lebesgue Measure of any non-empty open set is strictly positive.

(ii) $\mathbb{Q}$= rationals, countable ⇒ $m_1(\mathbb{Q}) = 0$, $\mathbb{R}\backslash\mathbb{Q}$ = irrationals has measure infinity.

and both are dense sets.

(iii) $E \subset \mathbb{R}$ is a measurable set, such that $m_1(E) = 0$. So, it cannot contain any open set so, this implies every open set must intersect E^c ⇒ E^c is dense. In particular, $m_1(E) = 0$, E closed.

⇒ E is nowhere dense.

(iv) you have $K \subset \mathbb{R}$ compact ⇒ K closed bounded ⇒ $m_1(K) <+ \infty$.

So, we will continue the examples next.