Measure and Integration
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Lecture No - 1
1.1-Preamble

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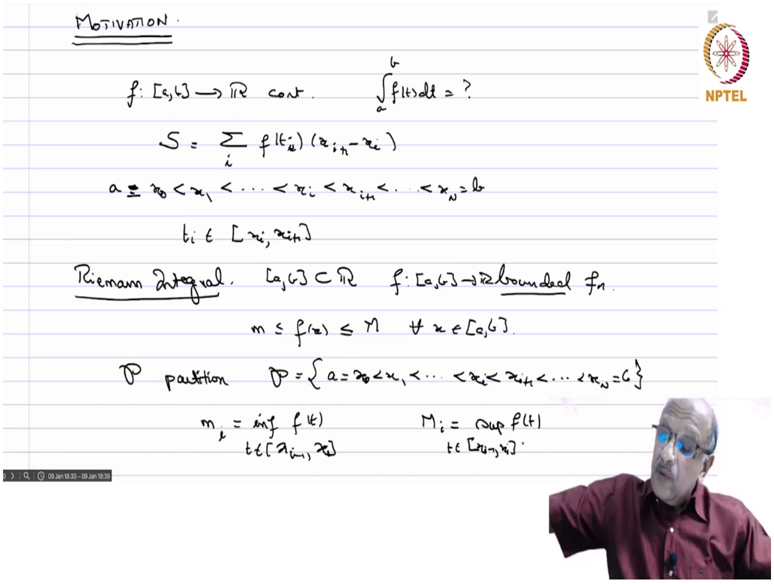

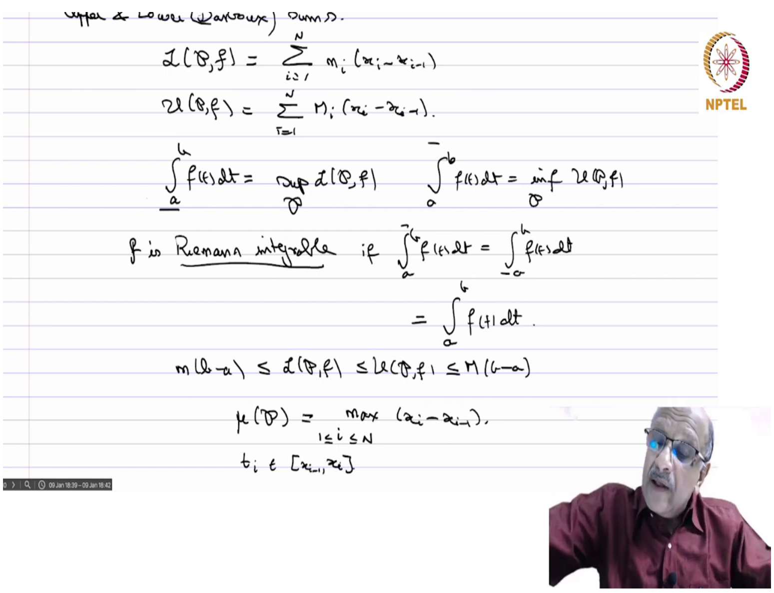

$$S(P,f) = \sum_{i=1}^{N} f(t_i)(x_i - x_{i-1}).$$

Def: $\lim_{\mu(P)\to 0} S(P,f) = A$

$\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall P$ s.t. $\mu(P) < \delta$, $\forall$ choices of $\{t_i\}_{i=1}^{N}$,

$|S(P,f) - A| < \varepsilon$

Thm. (Rudin - Principles of Math. Anal.).

f is Riemann integrable $\iff \lim_{\mu(P)\to 0} S(P,f)$ exists.

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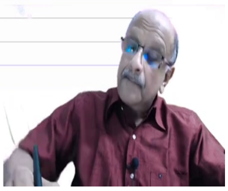
and in fact $\int_a^b f(t)\,dt = \lim_{\mu(P)\to 0} S(P,f)$.

Thm. If f is cont, or f has atmost a countable no. of discontinuities, then f is R-integrable.

Eg: $[0,1]$ numbering $\{r_i\}_{i=1}^{\infty}$ of the rationals in $[0,1]$.

$$f_n(x) = \begin{cases} 1 & x = r_1, r_2, \ldots, r_n \\ 0 & \text{otherwise.} \end{cases}$$



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$$f_n(x) = \begin{cases} 1 & x = r_1, r_2, \ldots, r_n \\ 0 & \text{otherwise.} \end{cases}$$

f is discont only at $r_1, \ldots, r_n$. f_n R-int. $\int_0^1 f_n(t)\,dt = 0$. (Check!).

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x \text{ rat.} \\ 0 & \text{if } x \text{ irrat} \end{cases}$$

$\forall P$ $L(P,f) = 0$ $U(P,f) = 1$ ($\because m_i = 0, M_i = 1\ \forall i$).

$\underline{\int_0^1} f(t)\,dt = 0$ $\overline{\int_0^1} f(t)\,dt = 1$.

f is not R-int.

Hello, in this first lecture, we will try to motivate the need to study Lebesgue theory of measure and integration. So, we are going into motivation. So, from the time of the Greeks, the problem of computing the area enclosed by a curve has been at the base exercising the minds of scientific thinkers. This crucial question, which is at the root of integral calculus ,was treated as early as the third century, BC by Archimedes.

He computed the area of the circle, the segment of a parabola and other such figures. What he did was to use the method of exhaustion, which means to exhaust the area by seeing the sequence of polygonal domains compute the area of the polygonal domains and somehow compute the area of the required figure by means of some kind of limiting argument.

And during the 17^{th} century, many such areas were calculated and in each, this problem was solved by some ingenious device and suited for the special case. So, one of the main achievements of calculus was to give a powerful and general method to the place, these special techniques.

So, from the time of Archimedes, until the time of Gauss, the attitude was that area is something which is intuitively known. We know what it is. We do not have to define it. We just need to compute it. But before Cauchy, there was no formal definition of the integral in the precise sense of the term. One was often limited to saying which areas 1 had to add, which areas 1 had to subtract. So, as to get the area, which is, of course we know an integer Cauchy, which is concerned for rigor, and this is characteristic of modern mathematics, define the continuous functions and their integral in much more, much the way as we do now.

So, to arrive at the integral continuous function f defined on an interval [a,b]. So,

$f\colon [a, b] \to \mathbb{R}$ is continuous. So, we want to know what do you mean by $\int_a^b f(t)dt$? So, he computed sums of the form

$$S = \sum_i f(t_i)(x_{i+1} - x_i), \quad a_0 = x_0 < x_1 <.... < x_i < x_{i+1} <... < x_N = b,\ t_i \in [x_i, x_{i+1}]$$

So, this is called a partition. And then using a suitable passage to the limit. He computed, he defined the integral. Now, for a long time some discontinuous functions were also evaluated by their integrals and evaluated by showing that Cauchy's method still applied. And it was Riemann who systematically investigated the exact scope of Cauchy's definition.

So, in what follows, we will briefly recall the Riemann integral and also examine what are its drawbacks, so that we will motivate that by the need for the new theory of integration, which will be Lebesgue theory.

Riemann integral: $[a, b] \subset \mathbb{R}$ is a fixed interval and $f\colon [a, b] \to \mathbb{R}$ is a bounded Function. So, that means $m \le f(x) \le M$, for all x we need. So, then we look at a partition. So, P is of the form, $P = \{a_0 = x_0 < x_1 <.... < x_i < x_{i+1} <... < x_N = b\}$.

And then we define

$$m_i = \inf_{t\in[x_i, x_{i+1}]} f(t) \quad ; \quad M_i = \sup_{t\in[x_i, x_{i+1}]} f(t)$$

Then we define what are called the upper and lower Darboux sums. So, we have upper and lower Darboux sums. So, the lower Darboux sum is equal to

$$L(P, f) = \sum_{i=1}^{N} m_i(x_i - x_{i-1}).$$

And the upper Darboux sum

$$U(P, f) = \sum_{i=1}^{N} M_i(x_i - x_{i-1}).$$

And then we define the lower and upper integrals. So,

$$\underline{\int_a^b} f(t)dt = \sup_P L(P, f), \quad \overline{\int_a^b} f(t)dt = \inf_P U(P, f) .$$

Now, the function is Riemann integrable if

$$\underline{\int_a^b} f(t)dt = \overline{\int_a^b} f(t)dt := \int_a^b f(t)dt$$

The common value is called the Riemann integral. So, this is how we define the Riemann integral. Now, f is bounded. So, we have that

$$m(b - a) \le L(P, f) \le U(P, f) \le M(b - a).$$

In fact, this less than is true for any two different partitions also, if I have P 1 and P2 two different partitions LP1 f will be less than equal to U P 1 f. So, therefore the Sup and inf are always existing because you have that, these sums are all bounded and therefore the lower and upper double integrals always exist, whether they are equal or not, it is a delicate question.

So, we have to investigate. So, being Riemann integrable means that these 2 are to be equal. Now, if you have a partition P. So, we define the mesh size of the partition as follows

$$\mu(P) = \max_{1 \le i \le N} (x_i - x_{i-1}).$$

And now if you take $t_i \in [x_{i-1}, x_i]$, then we define the following sum,

$$S(P, f) = \sum_{i=1}^{N} f(t_i)(x_i - x_{i-1}).$$

Definition: you say that $\lim_{\mu(P) \to 0} S(P, f) = A.$

So, what does it mean? So, this means for every $\epsilon > 0$, there exists a $\delta > 0$, such that for all partitions P such that $\mu(P) < \delta$ and for all choices of $\{t_i\}_{i=1}^{N}$, $1 \le i \le N$,

$$|S(P, f) - A| < \epsilon.$$

So, this is very, very important. So, how you have to change. So, this is a very strong condition and you have to use it.

So, now, the first theorem in Riemann integration. So, for all these results, you can look at Rudin's -Principles of Mathematical Analysis. This is a standard book, which you can find very easy to get. And therefore, you can see this in Rudin's book. And so, you must have done it in your real analysis course.

Theorem: f is Riemann integrable, if, and only if $\lim_{\mu(P) \to 0} S(P, f)$ exists and in fact

$$\int_a^b f(t)dt = \lim_{\mu(P) \to 0} S(P, f).$$

So, this is a very important theorem.

Now, the next important theorem.

Theorem: If f is continuous or f has at most countable number of discontinuities, then f is Riemann integrable.

So, let us look at an example. So, you take [0,1] and you take a numbering $\{r_k\}_{k=1}^{n}$ of the rationals. So, you define

$$f_n(x) = 1, \; if\ x = r_1, ..., r_n$$

$$= 0, \; otherwise.$$

So, this function f is discontinuous only at $r_1, ..., r_n$. So, it has only a finite number of discontinuity. So, by the previous theorem, f_n is Riemann integrable. But in fact you can apply the definition straight away, the partition definition, which we have seen just now computes the lower sum and computes the upper sum and you can easily prove that in fact,

$$\int_0^1 f_n(t)dt = 0 \qquad \text{(Check !)}$$

Now, let us now consider

$$f(x) = \lim_{n \to \infty} f_n(x) = 1, \; if\ x\ is\ rational,$$

$$= 0, \; if\ x\ is\ irrational.$$

Now, if you take any partition that is going each Sub interval will have both a rational number and an irrational number, because both sets are dense and therefore $m_i = 0, \; M_i = 1$. So, this implies for every partition P, you have

$$L(P, f) = 0, \;\; U(P, f) = 1.$$

Therefore you have that

$$\underline{\int_0^1} f(t)dt = 0 \qquad and \; \overline{\int_0^1} f(t)dt = 1.$$

and consequently f is not Riemann integral. So, we have already come to our first drawback, namely, a sequence of Riemann integrable functions need not be Riemann integrable.

Now, it has become worse. So, f_n Riemann integrable does not imply f is Riemann integral, in general. So, taking point wise limits, you cannot assume, but assume that f is Riemann integrable, even then things cannot be very good.

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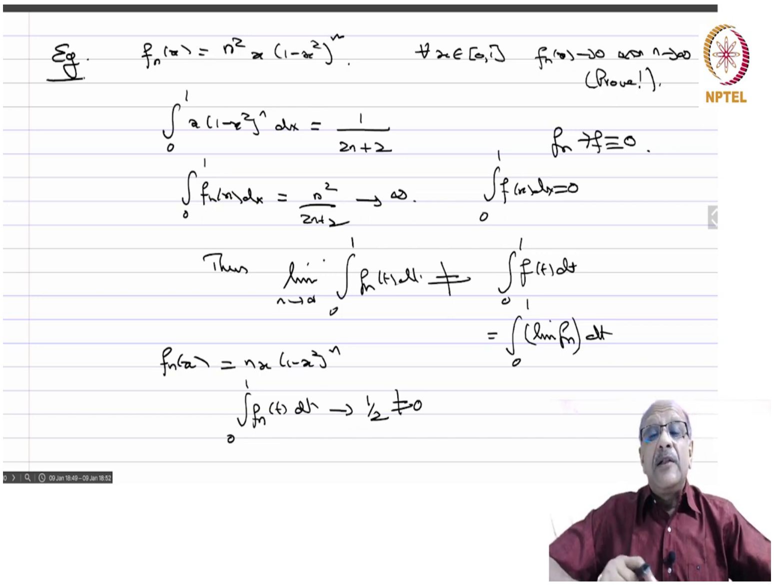

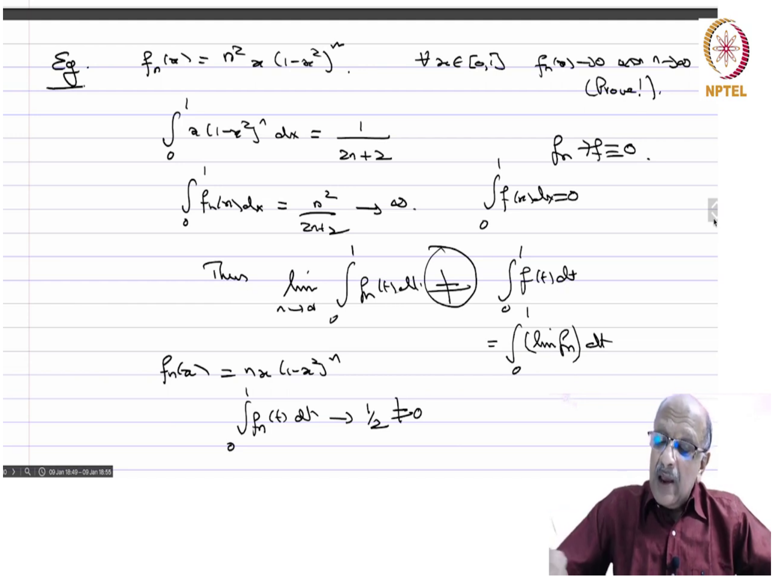

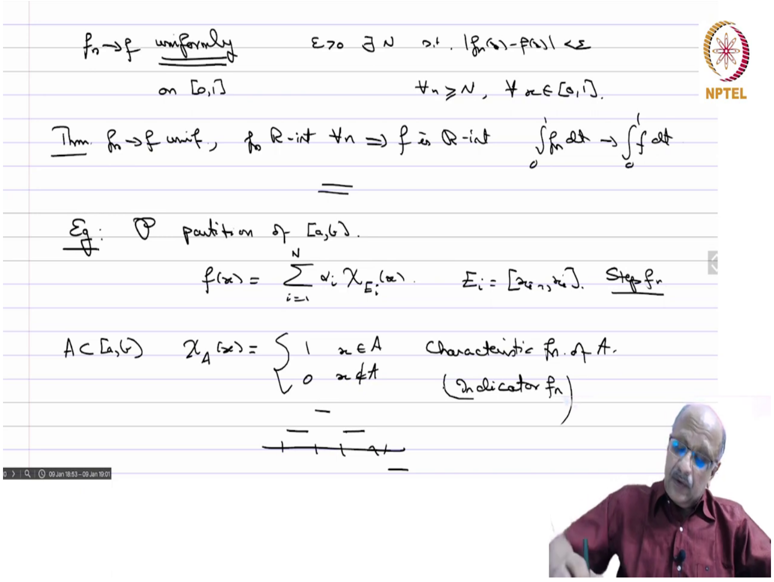

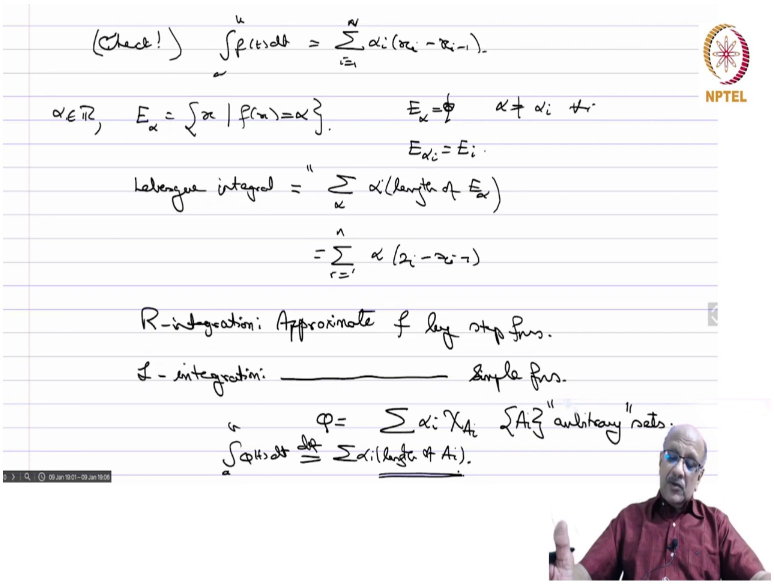

So, let us take another example.

Example: So, let us take

$$f_n(x) = n^2 x(1 - x^2)^n, \;\; \forall x \in [0, 1].$$

Then $\forall x \in [0, 1]$, we have $f_n(x) \to 0$ as $n \to \infty$ (Exercise !).

Now take $\int_0^1 x(1 - x^2)^n dx = \frac{1}{2n+2}$. So, this is a straightforward calculation. So,

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt = \infty \neq 0 = \int_0^1 f\ (t)dt = \int_0^1 \lim_{n\to\infty} f_n(t)dt.$$

So, you cannot take the limit inside the integration side. So, this is a major thing. Now, even if, so, here of course, it went to infinity, but let us take, for instance, that if you take

$$f_n(x) = n\ x(1 - x^2)^n,$$

then $\int_0^1 f_n(t)dt \to \frac{1}{2} \neq 0.$

And that is again still not equal to 0 here it went to infinity here it goes to a finite number. And still, of course, we do not have this condition here, that the limit of the integrals is not equal to the integral of the limit.

So, we have that, the limiting 2 limiting process integration is itself a limiting process. And then we are taking a point waste limit of functions. And we have just seen that these 2 do not commute. So, we have two problems. One is that the limit of a pointwise limit may not be the Riemann integrable, even though every member of the sequence is Riemann integrable. And secondly, even if it is Riemann integrable, we cannot assert that the limit of the integral is equal to the integral of the limit.

So, now, is there any hope, I mean, is there any condition for which you have this? So, $f_n \to f$ uniformly, that means that given any $\epsilon > 0,$ there exists a capital N such that

$$|f_n(x) - f(x)| < \epsilon,\ \forall n \geq N,\ \forall x \in [0, 1].$$

So, then you suppose you have this, this, again, a very strong condition then of course, you have a theorem.

Theorem: If $f_n \to f$ uniformly, f_n is Riemann integrable for all n ⇒ f is Riemann integrable and $\int_0^1 f_n(t)dt \to \int_0^1 f\ (t)dt.$

So, this is of course a very costly condition and therefore, it is so we find that for the theory of integration, we, therefore we need a theory of integration, which is more friendly towards taking limits. After all, in analysis limits, taking limits is our bread and butter.

And we have several limiting processes. We would like limiting processes to compute, so that computation will be easier. And for many other reasons also, and therefore we would like to know if there is some better integration theory than this one, such that more functions become integrable, like. So, for instance, this, in the example, which we saw the rational, f where f is 1 is if it is rational, 0 for xi rational.

Now, that sequence is obviously not uniformly convergent, because if it was uniformly convergent, then the integrals f would be the Riemann integral, which it is not. And therefore, in fact, the validity of this theorem if this becomes equal or not, that is itself a test, whether 1 of the tests for uniform convergence of a sequence of functions, we say, fn goes to f and if you want to say, does it convert uniformly? And then you check if this condition holds, if it does not hold, that means the convergence has not been uniform because of this theorem which we just stated.

So, we want to know if you have a theory of integration where more functions become integrable, not just for the foundation of generalization, because we would like under reasonable hypotheses that see limits of integrable functions are integrable. And also the integral of the limit is equal to the limit of the integrals. So, these are the properties which I would like to know.

Now, the Riemann integral is defined with a certain amount of continuities forced into the theorem because we see we have this following theorem here f is Riemann integrable, if and only if limit S Pf exists and this equal to the same. So, we want that for the function to be the Riemann integrable S Pf exists.

Now, what is S Pf? S Pf is nothing but for all choices of points. So, if you take mu P mu t going to 0. So, for small intervals, if you change a bit inside the sub interval, S Pf should not change too much because otherwise then the limit will not exist. This is what is happening in the case of the function which is 1 if rational 0, if is rational.

If you change a point from rational to irrational, S Pf will collapse. So, that is why that function was failing to be Riemann integral. So, we have that small close by values should not, should still be close by this is it, but that is what continuity is? So, some amount of continuity is enforced. Of course, we can excuse a small number of discontinuities as was mentioned in that theorem, and that is the theorem which says how much we can excuse.

So, the function may be discontinuous but right now in Riemann theory, if it has countable number of discontinuity then of course, we can excuse it and still, do something, for the function will be Riemann integrable. So, the function is more or less continuous as far as the Riemann, to be Riemann integrable. And the number of discontinuities is merely limited.

Now, the idea of Riemann integration is the definition of the integral. What we do, we take a partition and then we take the points in the partition. So, we are following the function along the x axis, along the abscissa. So, we are taking the function as we proceed along the x axis and therefore, we are forced to compare nearby values.

Now, differentiation is a local phenomenon. If you want to differentiate a function, then of course, we have to look at the values nearby to find the rate of change of the function. But integration is something global. You do not have to, we are not worried about what happens nearby values. What really we are worried about is the global behavior of the function. Let us take the example of floods. Suppose you have floods somewhere as it happened in Chennai recently.

Now, if you are the corporation Counsellor, you would be walking along in your ward and seeing how the situation is because that is all you are interested in, but the Chief Minister or the Prime Minister will not do that. He, the Chief Minister or Prime Minister needs to have a global picture of the disaster, so, he will fly. So, he will have an aerial survey of the entire damage.

Now, that is the idea of Lebesgue. Now Riemann said, I will walk along the X axis, therefore, and look at the function, take it as it comes. But Lebesgue said, no, let us fly on the Y axis and take a global view of the function. So, we take a particular value on the Y axis. So, we do not work on the domain. We work on the range, we take a value on the Y axis, and then look at the entire set, which takes this particular value and study it and use it to define the integral. So, let us give an example,

So, let us take P to be a partition of [a,b] and you take,

$$f(x) = \sum_{i=1}^{N} \alpha_i \chi_{E_i} \ , \ where\ E_i = [x_{i-1}, x_i].$$

So, this is called the characteristic function. So, also called the indicator fn. Another name for it is the indicator function of the set A.

So, we take this and so, such a function is called a step function because you take various intervals.

So, now this function, of course, only has a finite number of discontinuities. So, it is Riemann integrable and you can check. So, you can check that

$$\int_a^b f(t)dt = \sum_{i=1}^{N} \alpha_i (x_i - x_{i-1}).$$

This is standard, just apply the definition and we will get it. So, it is a good exercise to do these things, because now by Lebesgue method, what we are going to do is we will look at for every $\alpha \in \mathbb{R}$, we take $E_\alpha = \{x |\, f(x) = \alpha\}$. So, we have that $E_\alpha = \phi\ \forall \alpha \neq \alpha_i,\ \forall\ i\ and$

$E_{\alpha_i} = E_i$. And then you take the length of the interval over which, you have this and you have the alpha and you the integral is so the Lebesgue integral equals

$$\text{Lebesgue integral}=\sum_{\alpha} \alpha(length\ of\ E_\alpha) = \sum_{i=1}^{N} \alpha_i (x_i - x_{i-1}).$$

But it is just a method. So, let us assume you have a merchant who has been doing sales all day and at the end of the day when he closes his shop, he wants to know how much money he has earned today.

So, there are 2 ways he wants to compute the value, first, he can put his hand into the till, take the number coins or notes 1 by 1 and add the values. That is one way other is to empty the till, sort out all the coins and notes according to that denomination, see how much, how many coins are there in each denomination, find those values and add those values.

So, that is the first, just a method of Riemann where it takes one by one and adds some, the other is taking the way these denominations see how many notes are coins are there in each denomination, and then adding those values. So, that is the method of limit. So, if you both will, if you are careful, what should give you the correct answer? But the second one is definitely more efficient, especially if it involves a lot of money.

If you are going to some huge temples like Tirumala for instance, you would have seen how they see walls in the thing and separate the contributions regarding denominations and count them. So, this is a practical example of this method. So, the Riemann integral, what does it do? It approx., so, Riemann Integration, you approximate f by step functions. That is exactly how we have defined the integral.

Now, what is Lebesgue integration? Approximate ϕ by simple functions, that means

$$\phi = \sum_{\alpha} \alpha_i \chi_{A_i}, \quad A_i - \text{"}arbitrary\ sets.\text{"}$$

Again, not it cannot always be arbitrary. I will put it within an inverted comma. We will see these things. This is the second giant clip. So, the two leaps of imagination, which we have here first is to walk along the Y axis so that you get a global view of the function. And the second is to replace the step functions, namely, which depend on partitions and intervals by writing this now you compute.

So, the integral

$$\int_a^b \phi(t)dt = \sum_{\alpha} \alpha_i (length\ of\ A_i)$$

Now, here is the catch. What do we mean by length of Ai when Ai is not an interval? So, we said, I see this, I got a very big generalization by looking at simple functions, but I do not know what is meant by length of Ai.

So, that is where we come to the theory of measure. So, we have to generalize the notion of the length of interval, area of a square or rectangle volume of a rectangular parallel (())(33:18) in two and three dimensions, respectively, to length, area, volume of arbitrary sets. Well, we cannot do it for all sets. We will see it, but then this is the notion of a measure.

So, this notion is a generalized notion of length or area or volume, or in general, you can do it on any arbitrary set. And that is what we are going to do. And that is the abstract theory of measure from which we will develop the theory of integration. So, next time we will look at that. Thank You.