**Our Mathematical Senses**
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**Lecture – 09**
**Projectivities**

Hi, welcome back to the Geometry of Vision. This is Lecture 5 in which we will talk about Projectivities.

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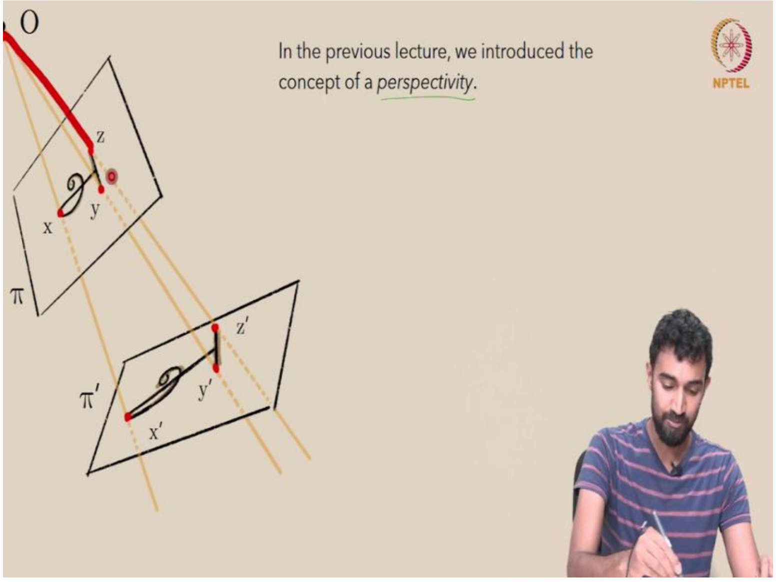


So, in the previous lecture we introduced the concept of perspectivity and we learned that as a map between ordinary planes, a perspectivity is a rational map. Because remember perspectivity is basically like central projections, it is kind of the generalization of the idea of central projections. So, from the center of perspectivity O we can imagine we are projecting out and we are relating points on these different planes whenever they are collinear with this point O.

So, in this example here this is a perspectivity from a plane π to a plane π' which is sending the points x, y and z to the points x', y' and z'.

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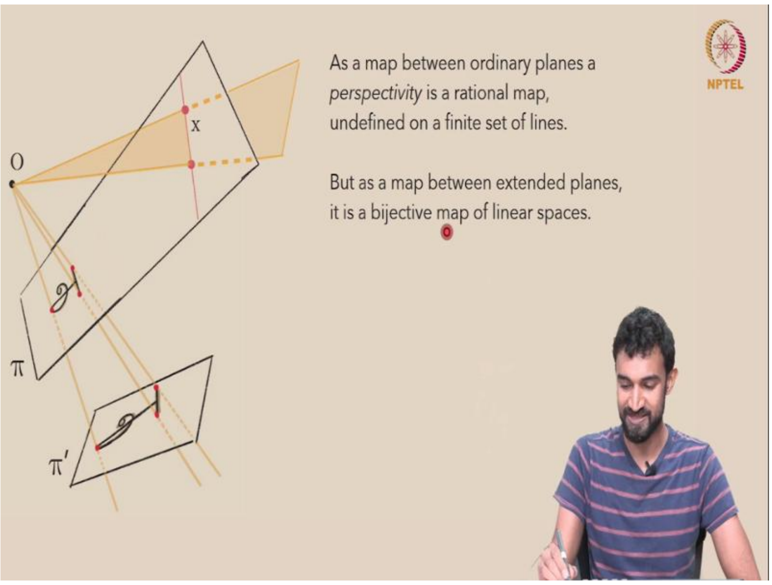


And we saw in the previous lecture that this map, if we think of it as a map between ordinary planes, then it is not defined everywhere, so it is a rational map. It is undefined on a finite set of lines, in particular on this line x, it is not defined because the lines from O to x fail to intersect the plane π'. However, as a map between extended planes a perspectivity is a bijective map of linear spaces.

And remember the linear spaces part means it takes lines to lines. That is all it means to say it is a map of linear spaces. It preserves collinearity.

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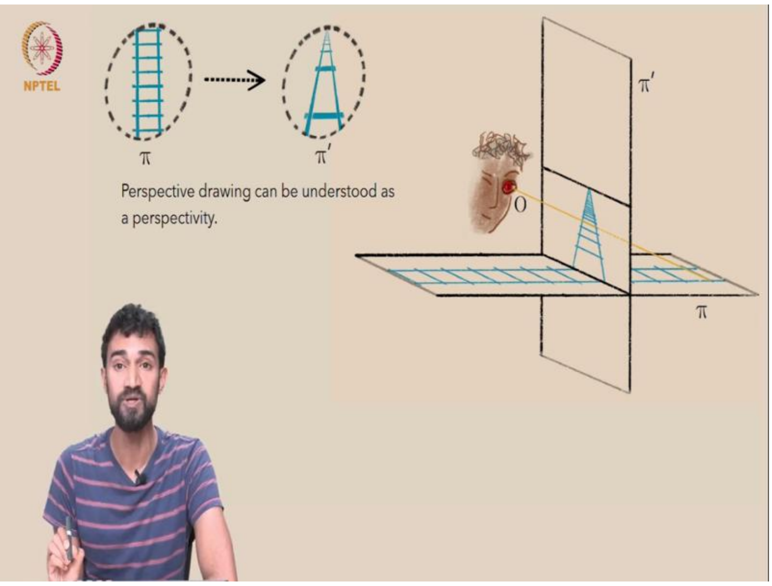


We also saw in the previous lecture that perspective drawing can be understood as a perspectivity. So, perspectivity has given us a kind of mathematical framework in which to understand perspective drawing and this simple trick that is to think of the eye of the person who is doing the drawing as being the center of the perspectivity.

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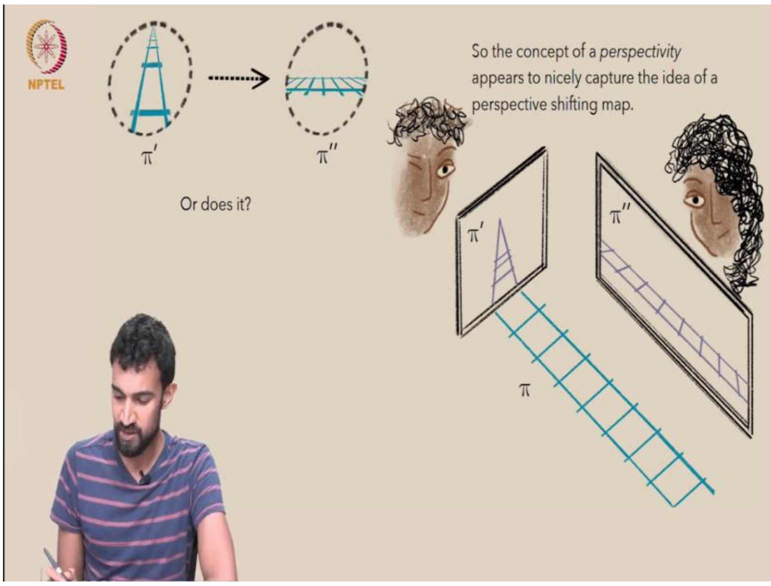


So, basically the concept of a perspectivity seems like it nicely captures the idea of a perspective shifting map which is exactly what we are trying to study in this course. Does it really capture that? So, here we can see another picture plane and another person who is viewing the same railway tracks and drawing a different perspective view of it. And to go from this picture plane here to this picture plane here will that ought to be a perspective shifting map?

So, hopefully this perspective shifting map which goes from this plane to this plane ought to be captured by a perspectivity. So, can we design or devise a perspectivity from the plane $\pi'$ to the plane $\pi''$ which takes the image of these railway tracks onto the image of these railway tracks? That is the question. So, if we were to do that, where would the center lie?

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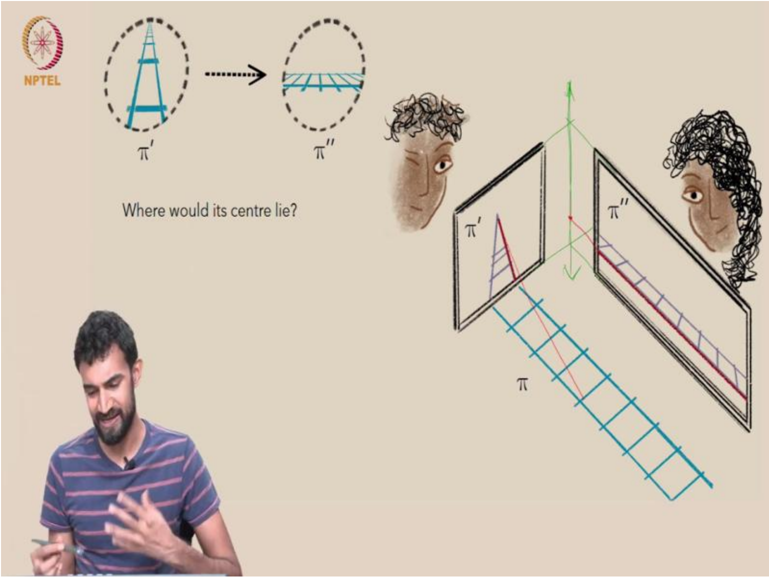


So where will the center lie if we construct a perspectivity from π’ to π”? Well clearly it is not going to lie at this person's eye or this person’s eye. Remember this person’s eye here which was the center of perspectivity to go from π to π’. It is taking this image of the railway tracks right back down to this π here and it is certainly not going to take it to this other plane π” or at least not to its image there as we see it.

So, clearly that would not work. Clearly for the same reason this center of perspectivity here is not going to work to construct a perspectivity between π’ and π”. So, we have to be a bit more clever or maybe it is a doomed experiment in the first place because maybe there is no center. So, consider this line here and this side rail here; these both correspond to this side rail here.

And if there is a perspectivity taking π’ to π” and taking this image on to this image, it also takes this line to this line. But is that actually possible? Are these lines even coplanar? If this line is going to this line under a perspectivity then they would better be coplanar. Because for example, the center of that perspectivity is here and we are literally looking at all of the lines from that center. They will map this line here to some other line somewhere which is going to have to lie in the same plane determined by this center and this line.

So, are these two red lines actually coplanar? If you look a little bit hopefully you can convince yourself that they are not coplanar. Because if they were coplanar then they would intersect somewhere, but for π’ and π”, the only place where they intersect, if we extend these picture planes out a little in this direction, is this line here.

And in other words this side rail here which intersects that line here, this side rail here is clearly going to intersect that line way down somewhere there. So, clearly they are not going to intersect.

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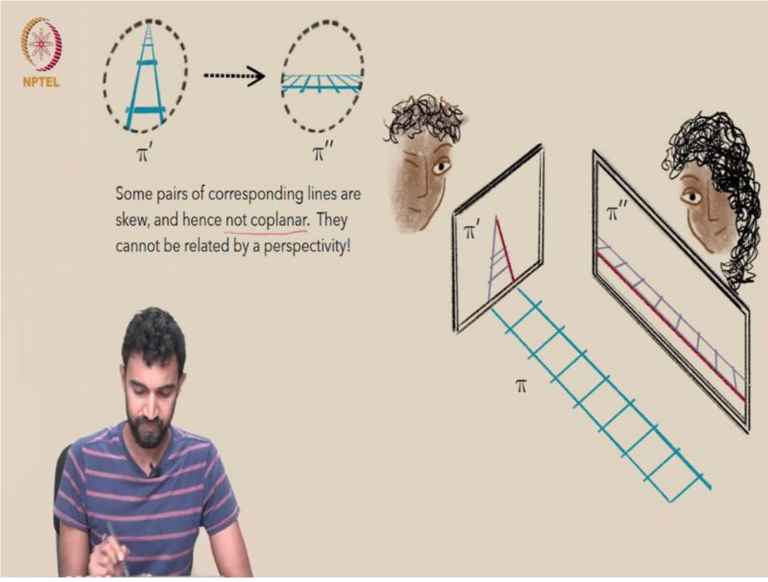


And just from your geometric intuition hopefully it does not seem too far fetched to say something is wrong, they are not coplanar and as a result these two lines cannot be related by a perspectivity, there is no perspectivity that will relate them.

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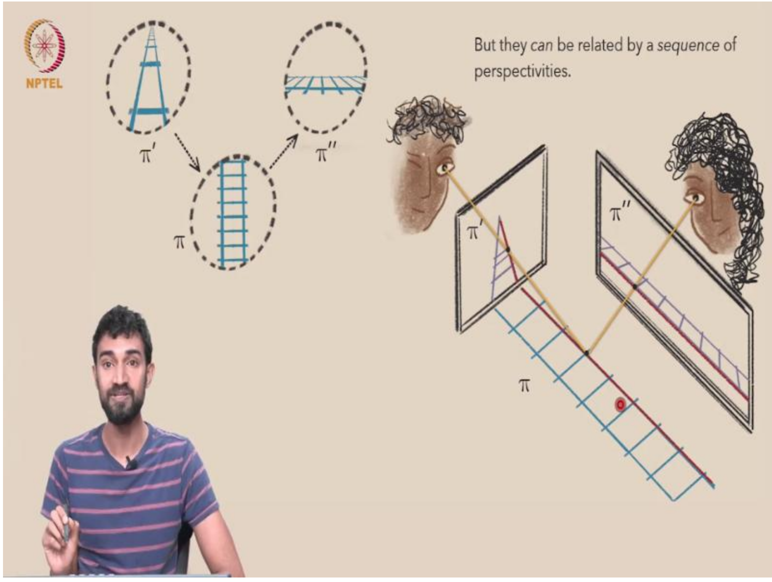


However, they can be related by a sequence of perspectivities and hopefully can see why because both of them came from this ground plane π via a perspectivity. So, if we just reverse the first one and go back from $\pi$' to $\pi$ and then go from $\pi$ to $\pi$'' we can relate $\pi$' and $\pi$'' via perspectivities which takes this image to this image here which is exactly what we do.

So, in other words we take this perspectivity here and reverse it to bring this image back down to the ground plane and then we take the second perspectivity and pull the ground plane image back to the plane $\pi$''. So, combining or composing those two perspectivities gives us exactly the perspective shifting map that we want.

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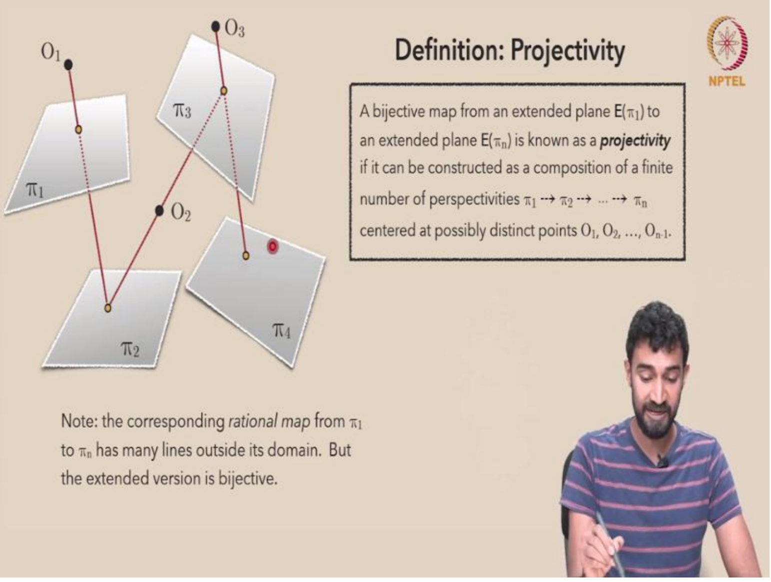


So, I am going to give you a new definition here which is a projectivity. So, which is exactly what we just saw as an example. The actual definition is that a projectivity is a bijective map from an extended plane $E(\pi_1)$ to another extended plane $E(\pi_n)$ such that it can be constructed as a sequence as a composition of a finite number of perspectivities from $\pi_1$ to $\pi_2$ , $\pi_2$ to $\pi_3$ and so on up to $\pi_n$ centered at possibly distinct points $O_1$, $O_2$ , …, $O_{n-1}$.

So, it is just a sequence of perspectivities composed with each other that is what a projectivity is. Note that the corresponding rational map from $\pi_1$ to $\pi_n$ will have many lines outside of this domain because each time we do another perspectivity between ordinary planes and space we lose another line in our domain.

So, in that case the rational map will have many things outside of this domain which does not sound that nice. However, the extended version of the map between extended planes in space is bijective, it is defined everywhere and it is a bijection so it is much nicer. So, the set of projectivities from an extended plane E($\pi_1$) back to itself is an especially interesting set. It is an especially interesting collection of maps that we want to study.

So, the previous picture, maybe I should quickly explain, is illustrating a projectivity from the plane $\pi_1$ to the plane $\pi_4$. So, it is literally just sending this point here via the perspectivity through $O_1$ this point gets mapped down to this point here then via the second perspectivity centered at $O_2$ the same point gets projected to this point here.

Finally, we have this third perspectivity centered at $O_3$. This point gets projected from the plane $\pi_3$ down to the plane $\pi_4$. So, taken together these three perspectivities they project this point from this point in $\pi_1$ to this point in $\pi_4$. So that is how general projectivity works.

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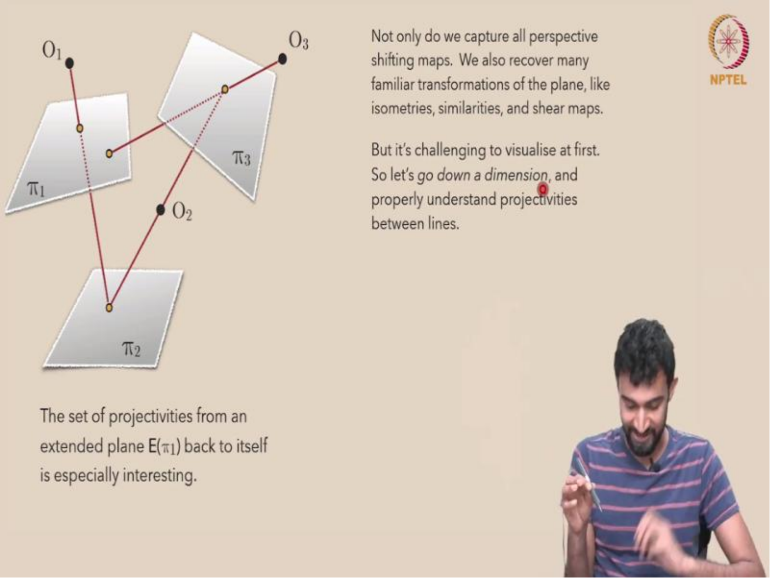


Now here is a projectivity from a plane $\pi_1$ to itself. So, again this point here is sent to here via the first perspectivity centered at $O_1$ next it is sent up to here via the second perspectivity centered at $O_2$ and finally it bounces back to $\pi_1$ via this third perspectivity centered at $O_3$. So, it is map from $\pi_1$ back to $\pi_1$ which in the case of this point is mapped to this point here.

But it will be defined at any point so in that way it is actually going to be a complete bijective map from $\pi_1$ to itself it is going to rearrange the points in the plane. So, these kind of maps are going to be especially interesting because they capture all perspective shifting maps that we have been studying. We can think of a perspective view of something as sitting in our frame of vision.

And if you look at another perspective view in our frame of vision we can think of that as a map from that picture plane to itself. So, this is actually going to turn out to capture all perspective shifting map, but it is going to do more than that, it is going to recover a whole bunch of maps that we are already familiar with, transformation of the plane that we know from previous mathematical studies like isometries, rigid motions of the plane like rotations and reflections and translations.

It is also going to capture similarities maps that uniformly scale and it is up and contracts an image. And finally it is also going to capture shear maps, the maps that slants or that alter angles that you study when you study linear algebra. So, all of these maps that you may have studied of planes, transformation of planes are going to be examples of projectivities.

So they are very nice, a general type of transformation of a plane. Unfortunately, it is a little challenging to visualize them at first. So, to begin with let us go down a dimension and let us properly understand projectivities between lines.

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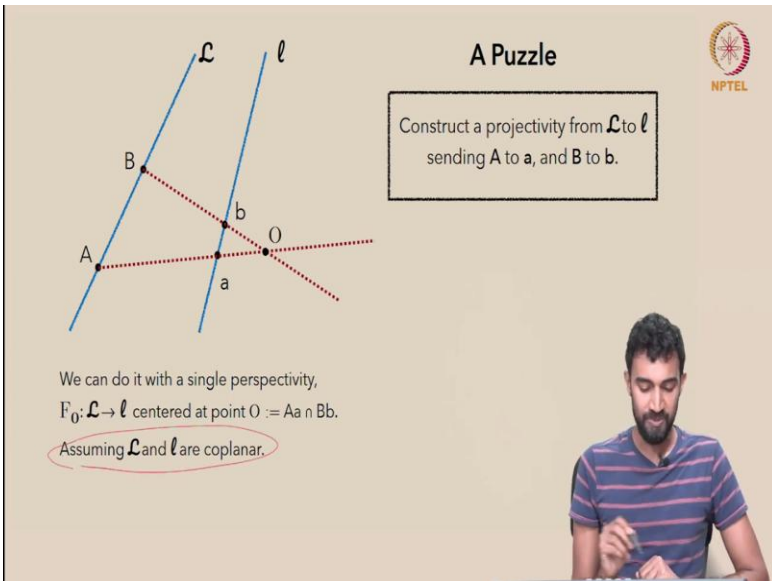


So, let us start with a small puzzle. Let us construct a projectivity from the line L to the line l which sends A to a and B to b. So, we want a projectivity sending A to a and B to b and overall mapping this line L to this line l. So, how can we construct such a projectivity? Well, we can think about a central projection, perspectivity, now.

We want a single point that will relate A and a and relate B and b. So, to do that maybe in this case you might want to just try drawing a line between A and a, and a line between B and b and seeing where they intersect. That seems like a good candidate for our center of perspectivity. And indeed if we have a perspectivity centered at O, then it will relate A to a and B to b. As a perspectivity from L to l, it is going to carry A to a and B to b as desired.

So, in fact in this case we can do it, we can solve the puzzle with a single perspectivity and we will call that perspectivity $F_O$ from L to l and it is centered at the point O, where O we have defined to be the intersection of the line Aa with a line Bb, but all these works assuming L and l are coplanar.

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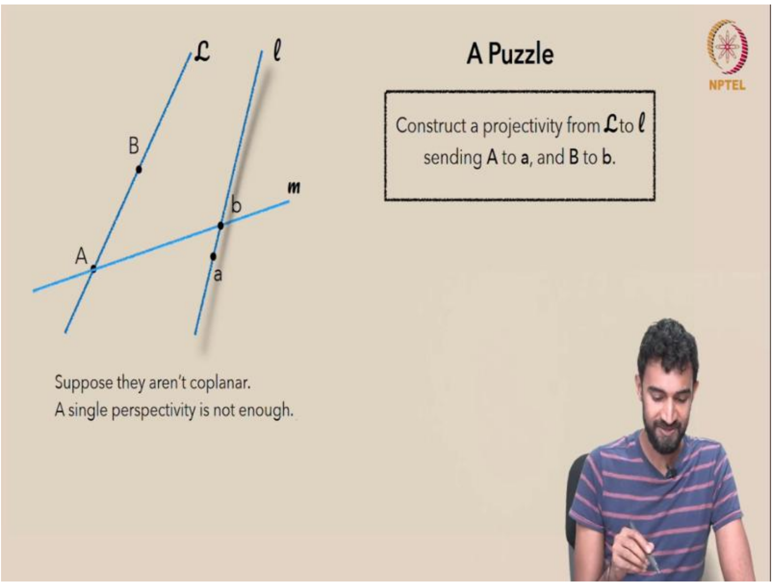


What if we have two lines in space that are not coplanar. Let us imagine tilting l up so it is no longer coplanar with L. The way I have drawn it here it looks like I have just made it rise, the shadow looks like a uniform so just imagine that it is not just that I have lifted off a table, I have actually tilted it. So, it is no longer coplanar with L rather it now L and l are skew lines in space.

So, can we solve the puzzle? In this case, can we construct a projectivity from L to l that sends A to a and B to b? So, a single perspectivity is clearly not going to be enough anymore because there are skew lines. We certainly cannot relate them with a single perspectivity.

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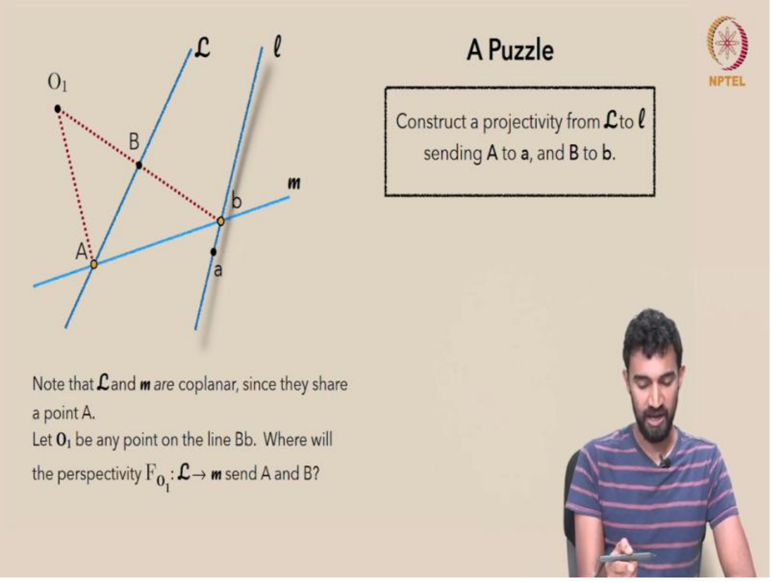


But what if we introduce an intermediate line m. So, note that L and m are coplanar because L and m share the point A. I have deliberately chosen m to go through A and b because now m is coplanar with L and also coplanar with l because it shares the point b. So, we have introduced an intermediate line that links them up in this way and now maybe you can see what to do.

We can proceed in steps. We can first construct a perspectivity $F_{O1}$ from L to m which sends B to b. By the way I am constructing it by just letting $O_1$ be any point on the line Bb. Now the perspectivity $F_{O1}$ from L to m is going to send B to b. Where is it sending A? Well it is fixing A.

So, it is sending A and B to A and b, and similarly we now want to complete our operation so we need second perspectivity.

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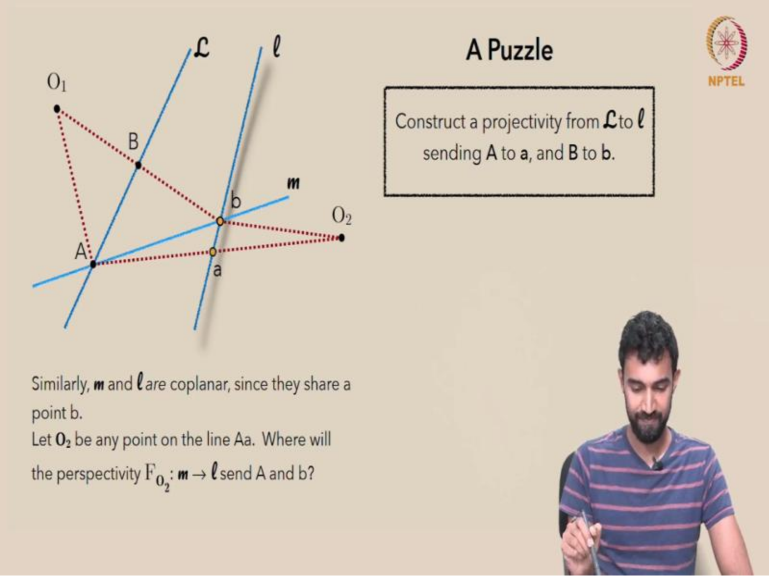


So, similarly notice that m and l are coplanar since they share the point b and we can construct a second perspectivity $F_{O2}$ centered at the point $O_2$, where $O_2$ is any point on the line Aa. I am just doing the exact same thing. So, drawing this line Aa, $O_2$ is any point on that line and I am looking at the perspectivity $F_{O2}$ from m to l.

And hopefully you can see where it is sending A and b. It is fixing b and it is sending A to a, so relating those two points.

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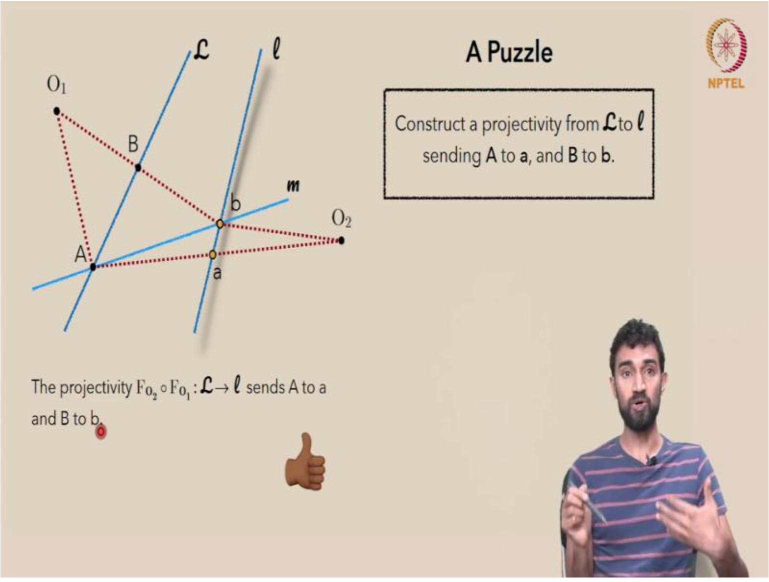


So, the projectivity $F_{O1}$ followed by $F_{O2}$ takes L to l and it sends A to a and B to b as desired. So it solves the puzzle. So, it is a slightly more general solution which works if L and l are not coplanar, skew lines in space.

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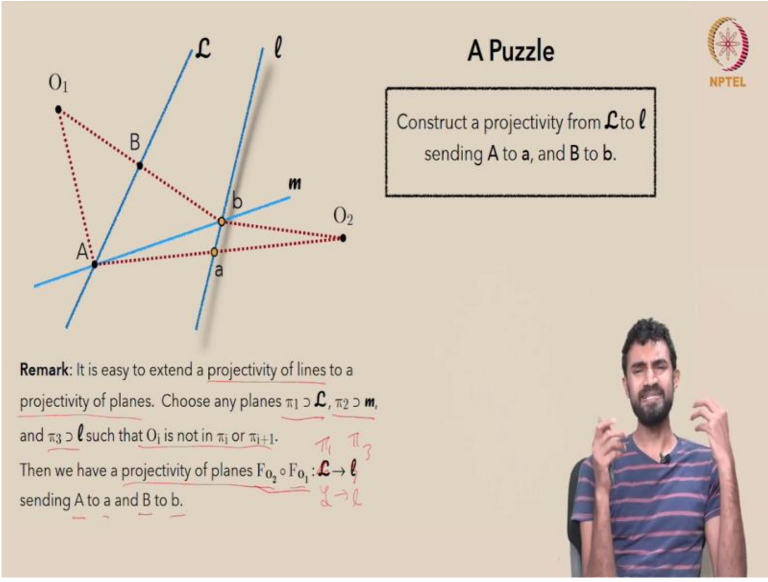


And just important remark here, it is easy to extend a projectivity of lines like the one we just created to a projectivity of planes when you have two lines in space we can extend it to a projectivity of planes and to do that we just choose any planes containing all of the lines that were involved in our projectivity. So, choose a plane $\pi_1$ that contains L, choose a plane $\pi_2$ that contains m and choose a plane $\pi_3$ which contains l, in such a way that all of our centers of projectivity do not get contained in any planes that they are supposed to be relating through prospectivities.

So, $O_1$ should not be contained in $\pi_1$ or $\pi_2$ and $O_2$ should not be contained in $\pi_2$ or $\pi_3$ that is what this is saying and then we immediately have a projectivity of planes $F_{O1}$ composed with $F_{O2}$ taking the plane $\pi_1$ to the plane $\pi_3$.

It also takes L to l within those planes and it is also sending A to a and B to b. So, it is easy to kind of extend a projectivity of lines to projectivity of planes and that will come in handy later that is why I wanted to mention that now.

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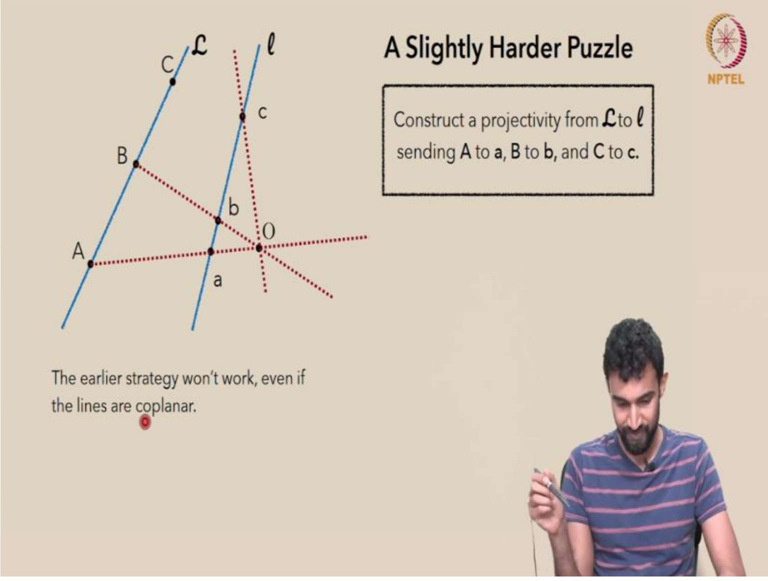


So, let us now move on to a slightly harder puzzle. Let us try and construct a projectivity from L to l sending A to a, B to b and C to c. So the same thing we did earlier, but now we have three points on the first line and three points on the second line that we are trying to relate. So, how do we start this? First step to do exactly what we did when we have two points, just take the intersection of Bb with Aa that gives us the center O.

And that O will not relate C and c except we were really lucky. So, even if the lines are coplanar, in general this strategy is not going to solve our problem.

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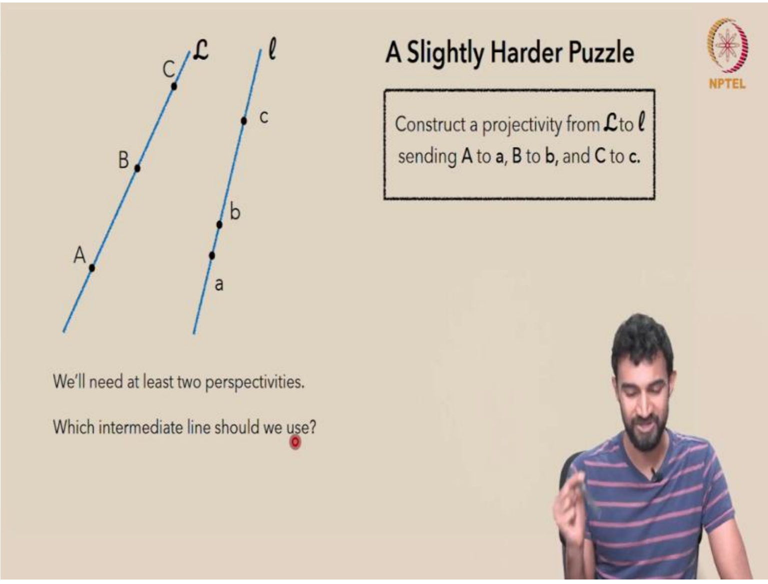


So, we will need at least two perspectivities to get the job done. So, what will the intermediate line be? What intermediate lines should be used? Well, I want to pause the video for a second and play around with it a little because this is a fun exercise to see if you can get it to work. So, maybe just pause it if you are interested and try and work it out yourself it will actually be very useful.

And once you have done that let us take a look at which intermediate line we should use. By the way, there are many solutions to this problem. This is not the only one, but here is one solution which I like.

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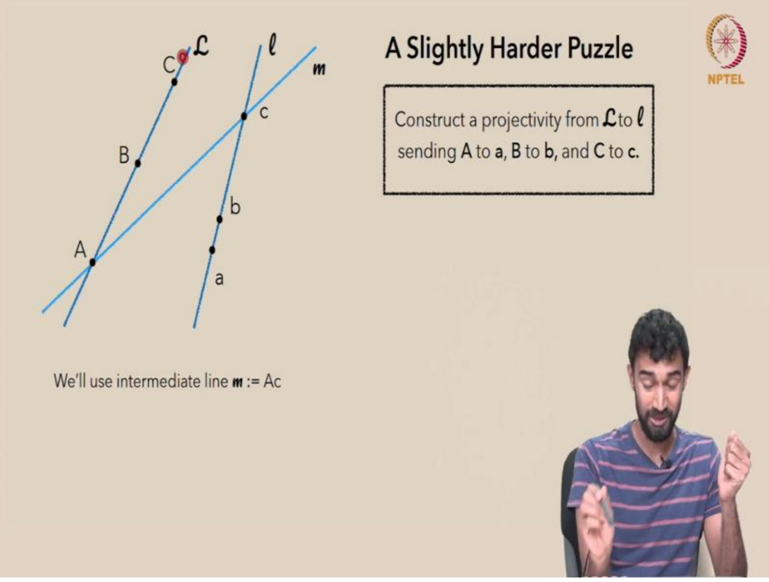


Which is to use the intermediate line m connecting A and c which is kind of like what we did in the earlier case of two points when it was not coplanar because this intermediate line is coplanar with L and is coplanar with l, so it is going to solve any issues that we might face on that front, but also another nice thing about this strategy is that now each of our individual perspectivities, remember we are going to want to now map L to m via perspectivity.

And then we are going to map m to l via another second perspectivity. And in each stage we already have part of our work done for us. When we are going from L to m, A is already mapped to A. So, we only have two points to worry about. So, suddenly our work becomes quite a bit easier. So maybe an idea of how we can proceed is going to be a lot like what we did earlier. We can connect C and c via line.

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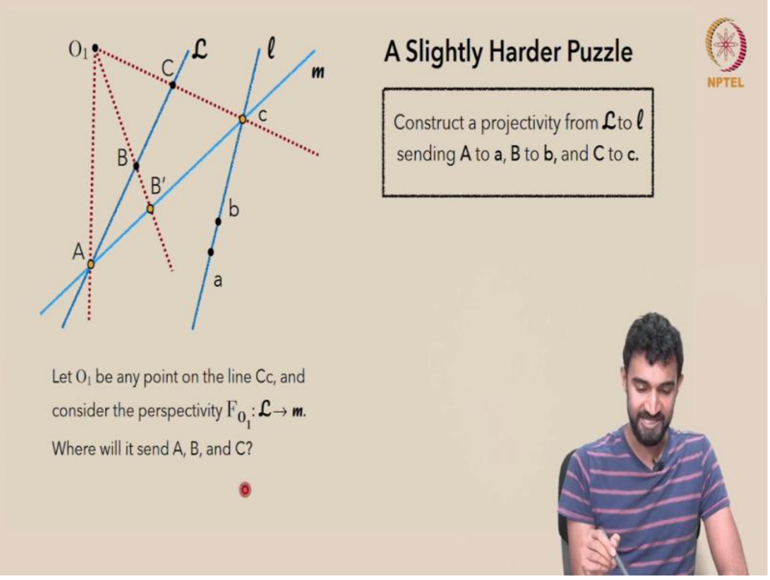


And let us just let $O_1$ be any point on the line Cc and let us consider the perspectivity $F_{O1}$ from L to m. Where is this perspectivity going to send A, B and C? Well it is clearly going to send A to itself because it is on the intersection of L and m. What about B and C? Well we have carefully constructed this perspectivity to take C to c.

But what about B? We do not have a name for that point yet it is going to take B to whatever this point is here. So, let us give that point a name, let us call it B'. Now we see that A, B and C are going to A, B' and c. So, that is our first perspectivity. So now where do we go from here? We need to construct a second perspectivity taking these three points to these three points and part of our work is done for us already.

And these two lines are coplanar. So, again we are in a situation we have seen before we just have to take two points, this point to this point and this point to this point.

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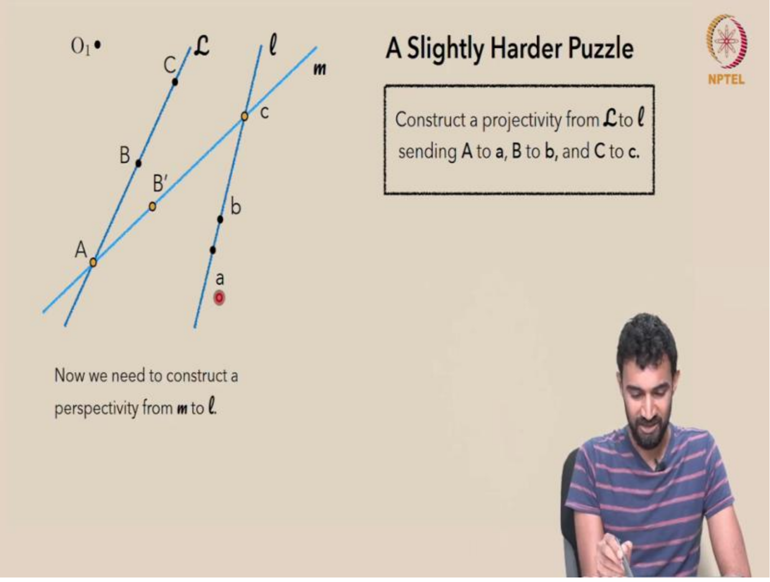


So, let us just draw the intersections of Aa and Bb.

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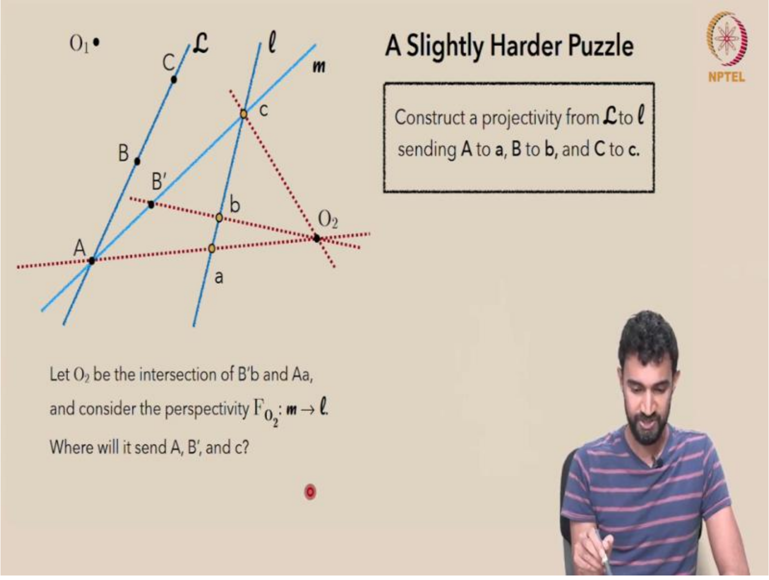


That gives us point $O_2$ and we can consider the perspectivity $F_{O2}$ from m to l. Where is it going to take A, B', c? Well it is going to take A to a, it is going to take B' to b and c to itself, because it is in the intersection. So, you can see that it is pulling our points to exactly where we want them to be.

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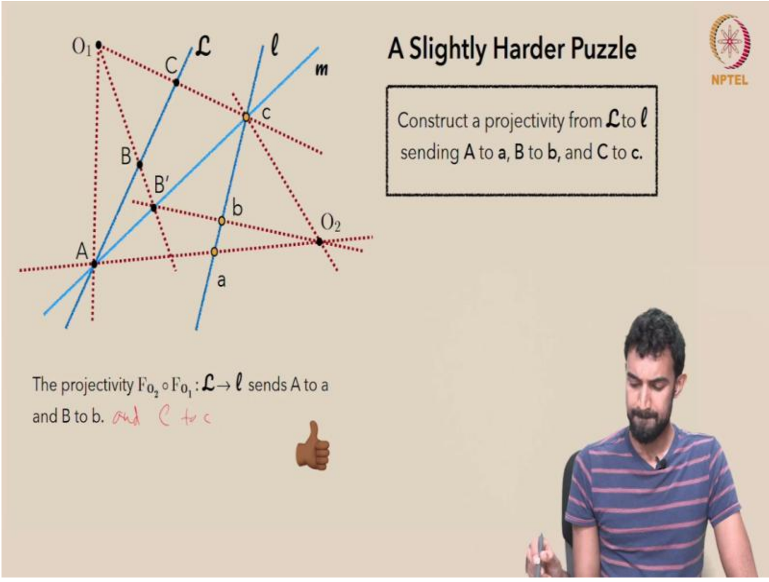


So, in other words the projectivity $F_{O1}$ composed with $F_{O2}$ from L to l sends A to a,B to b and C to c. We are done.

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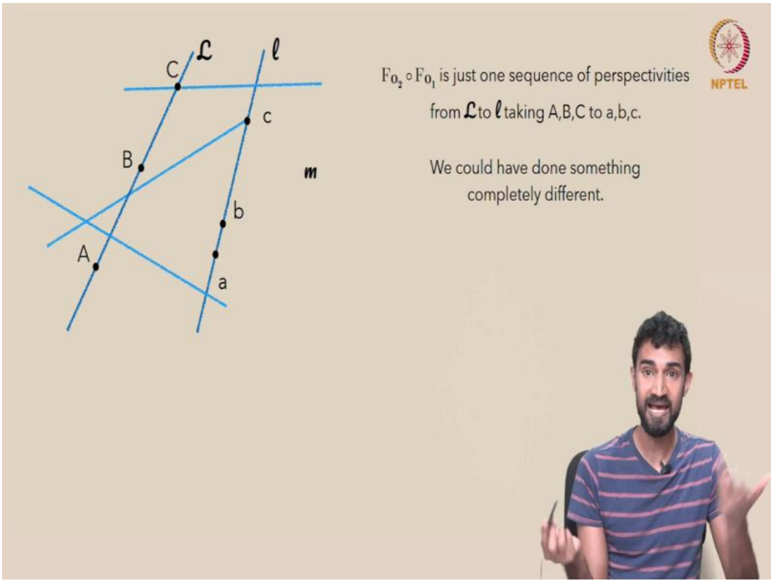


Now another remark here is that projectivity $F_{O1}$ composed with $F_{O2}$ is just one sequence of perspectivities from L to l taking A, B, C to a, b, c. There are actually many more and we could have done something completely different. We could have used 100 different intermediate lines, we could have done many different things and found a way to get back to a, b, c. So, this is not a unique sequence of perspectivities by any means.

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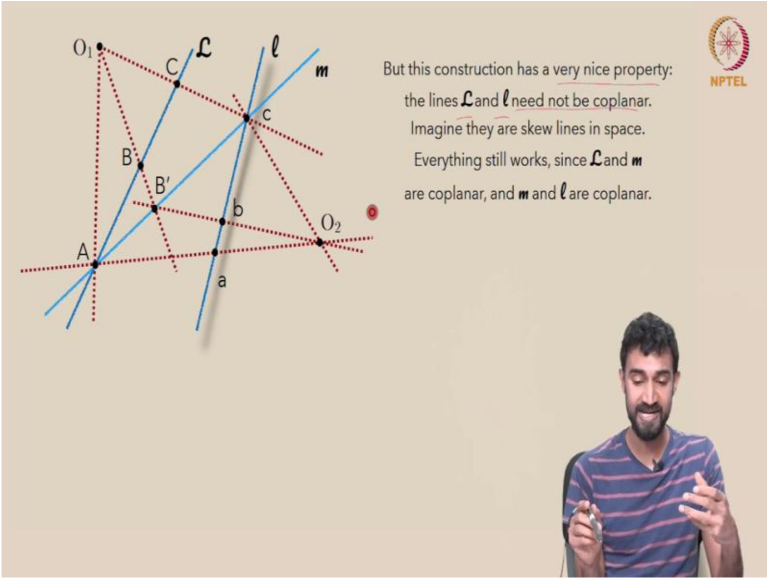


But this construction has a very nice property that the lines L and l do not have to be coplanar. I already mentioned that once, but just want to mention it again. The reason is because the intermediate line m is coplanar with both L and l. It shares the point A with L and the point c with l. So, this construction works for any two skew lines in space so I like this construction for that reason and you can verify for yourself that there is nothing here that we did which breaks down if these lines are skew lines in space.

And in the same way this will extend to a projectivity of planes as long as we choose our planes carefully.

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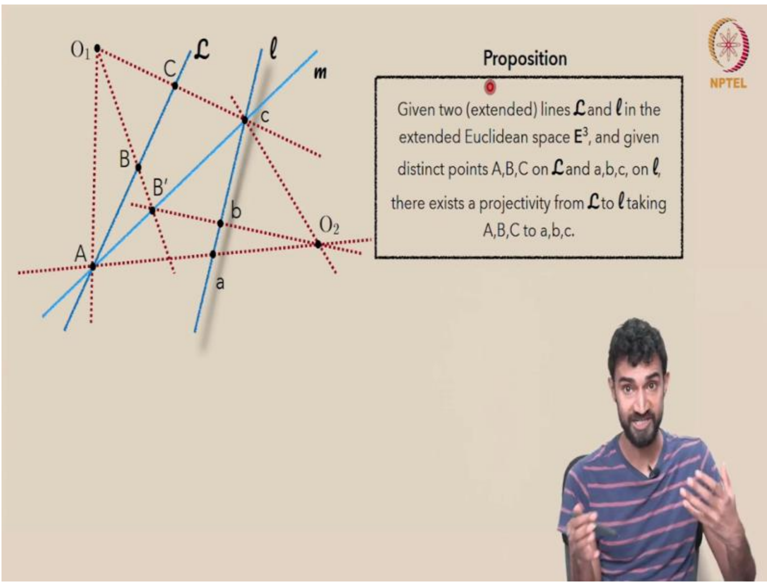


So, I mention this as a proposition because this is an important fact. Given two extended lines L and l in the extended Euclidean space $E_3$ and given distinct points A, B and C on L and a, b, c on l, there exists a projectivity from L to l taking A, B and C to a, b and c. So, we have shown the existence through this construction of such a projectivity.

So, this is an important proposition and we are going to come back a little later and just talk about how many projectivities there are, if there are indeed other projectivities. There certainly are the sequences of perspectivities, but maybe they agree with each other or maybe there are restrictions on how varied they can be as maps from L to l. So, we will come back to that in a little bit.