Our Mathematical Senses

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Lecture - 08
Perspectivities of the Extended Euclidean Plane

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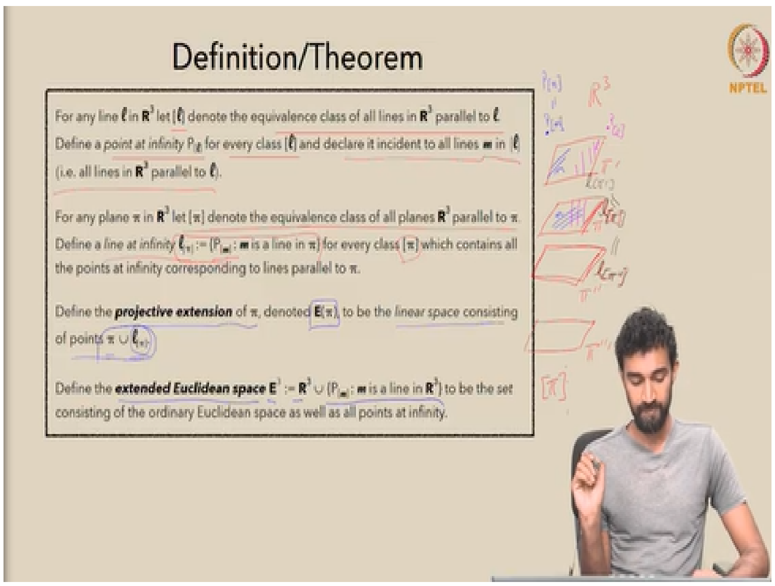

So, that is the content of the next definition/theorem. So, first of all notice that, basically what I want to do is just extend the idea of a point at infinity. So, earlier we defined points in infinity on the ground plane, specifically for a plane. So, but now we are dealing with many different planes with many different points at infinity. So, it will be convenient to have a common home for all these points at infinity.

So, I want to build a house that all these points at infinity can live in. So, given a line in R^3, let [l] denote the equivalence class of all lines in R^3 parallel to l. We will define a point at infinity $P_{[l]}$ for every class [l] and declare it to be incident, declaring it to lie on all of the lines m in [l]. So, hopefully this definition, this first part of the definition, looks familiar, because it is almost exactly the way that what we did when we defined points at infinity for the plane.

The only difference is that now I am looking at R^3 and I am defining my equivalence class to be the set of all lines in R^3 parallel to l, not just all lines in the plane. So, I am looking at all lines in space that are parallel to l. I am denoting the set of those equivalence classes by this same [l] notation. So, it is almost the same thing as before it is just that I am letting my lines range over all lines in space instead of all lines in the plane.

And my point at infinity now, I am defining one for every class [l], every equivalence class [l] as l ranges over all lines in space. But the incident relations are the same. A given point at infinity is incident, it lies on or it connects with all lines in space that are parallel to l. So, it is just a slight extension of the notion that we had earlier a point at infinity. So, now that we have extended our idea, our notion of a point at infinity.

So, we have points at infinity now for any line in space. Any family of parallel lines in space corresponds to a new point at infinity. In this way, I also want to expand our notion of lines at infinity. The reason is that I want for any plane in space, I want to be able to talk about its points at infinity and I want all its points at infinity to lie on a single line at infinity for that space.

There is a slight issue here which means the definition needs a little bit more explanation. If a line if we have a plane in space, let us call it π, it will have some collection of points at infinity which are based on the different equivalence classes, different families of parallel lines. But if we have another plane π' which is parallel to π in R^3, then all of the lines in π' will in fact be parallel to lines in π.

So, as a result the points at infinity for π will be the same as the points at infinity for the plane π'. Both planes, although they are distinct planes in space, will actually have the same collection of points at infinity. Because every equivalence class of lines in space gives us one point in infinity.

So, these blue lines in π' and these blue lines in π, since they are parallel to each other, will give us one blue point at infinity and the same with the violet ones. Let this line m and this line n.

Well, $P_{[m]}$ is the same as $P_{[n]}$; they correspond to the same point at infinity. So, I have to be a little careful in my definition here.

So, for that reason I want to do the following; before defining lines in infinity, I want to define equivalence classes or parallel families of planes in R^3. So, for any plane π in R^3, let [π] denote the equivalence class of all planes in R^3 that are parallel to π. And now let us define a line at infinity $l_{[\pi]}$ which is for that entire equivalence class, that entire family of planes. You can think of it as a stack of planes.

So, for this entire stack (equivalence class) of planes that are parallel to the plane π, which I am going to denote [π], let us define a line at infinity $l_{[\pi]}$ to be the set of all points at infinity $P_{[m]}$ for any line m in π.

Here, I do not want to say more than I need to say. I could have also said let m range over all lines in all planes in [π], in all planes parallel to π, but there is no reason to look at other planes, we do not get any new lines that way, all these planes are parallel. So, the collection of lines that lie in π is the same as the collection of lines that lie in any of these planes in the stack as far as equivalence classes of parallelism go.

We do not get any new families of parallel lines by going into planes other than π. So, it is ok to just say, let m range over lines in π here. So, we will define a line at infinity to be denoted $l_{[\pi]}$ and it is just the set of all points at infinity corresponding to lines in π. And we will define this line at infinity for every class [π]. So, in particular, that means that the line infinity for π is the same as the line at infinity for this other plane.

So the line at infinity for π is the same as the line at infinity for this plane.It is also the same as the line infinity for this plane because these planes are all parallel. So, they all have the same line at infinity. Parallel planes will all have the same line at infinity. So, now I want to define something called a projective extension of a plane π.

So, it is denoted $E(\pi)$ for extension and we will define it to be the linear space consisting of all of the points in π and all of the points in the line at infinity for π. So, it is just taking π and extending it by including all these points in infinity. So, it is exactly the same construction we did earlier to define our extended Euclidean plane. So, in that same way we can do a projective extension of any plane π in space, just add in all of its points in infinity.

The collection of its points at infinity is its line at infinity. So, just add in its line at infinity. Finally this last definition is just it is, we are not going to get into this much right now. So, we are not going to look into all of the properties of it. But I want to give a home for all of these many points at infinity that we have defined. So, let us define the extended Euclidean space E^3 to just be R^3 union all the points in infinity.

And it is just a set right now, I am just defining it as a set consisting of all the ordinary Euclidean points as well as all the points at infinity. And the reason I want this there is that, as we are looking at different planes in space sometimes, they share points at infinity. So, I want the points in infinity in this plane and the points at infinity in this plane to have a common home and that home is this extended Euclidean space. That is why we need to add this here.

But we will not get into how it looks and how it is shaped and how to imagine it right now. So, given this definition, the upshot of this, the real reason we are doing this is because we want to be able to look at the projective extension of a plane π and consider its points at infinity.

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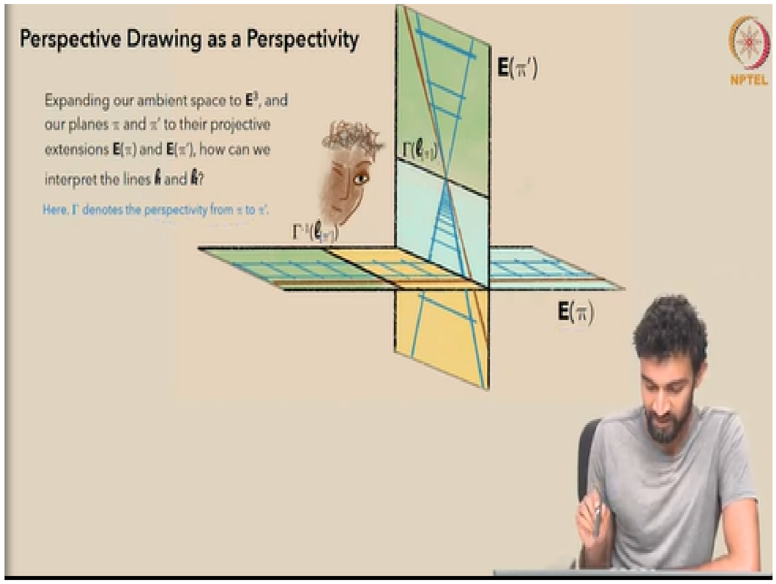

Remember, the reason we want to do that is because we want to think of what this perspectivity is doing on these lines k and h. We could always say, it is just not defined there, fine, it is defined everywhere else. But there is something going on that we want to capture. So, it is better if we can somehow give it a meaning on this line k and this line h. And we are going to do that via this notion of projective extension.

So, now expanding our ambient space to E^3, we can expand our planes π andπ' to their projective extensions E(π) and E(π'). So, we are now looking not just at planes but planes that also have a bunch of points in infinity which we cannot see in this picture but they exist, they are out there. So, now how can we interpret these lines h and k?

Well, let us call our perspectivity **Γ**. So **Γ** denotes the perspectivity from π to π'. What is h? It is the horizon line in our perspective drawing interpretation. And we can also now think of it as the literal image of the line at infinity in E(π) in the extended version of π, in the projective extension of π. So, by the way I am just giving some inspiration. We have not defined why it is equal to this yet. We will do that in a second.

And similarly, it would be nice to think of this line here as the inverse image, as the set of points that map to the points in infinity in our extended plane here.

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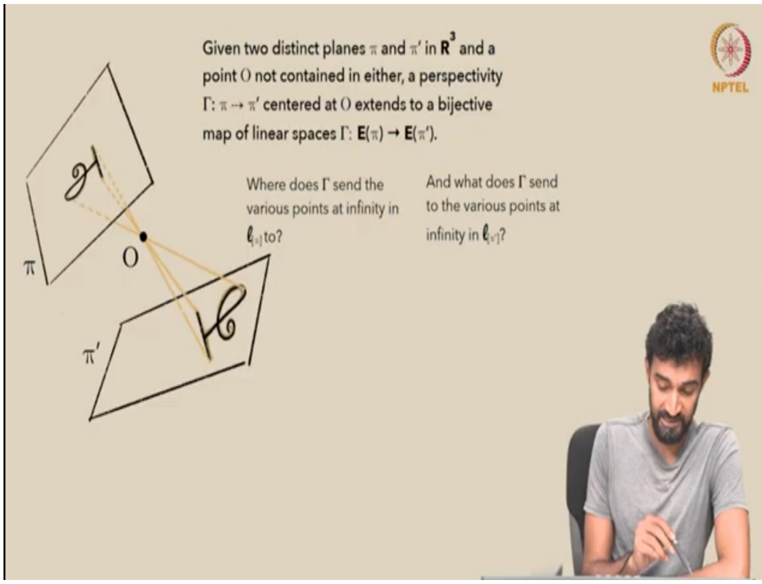

So, let us make this a little precise. So, given two distinct planes π and π' in R^3 and a point O that is not contained in either one. A perspectivity **Γ** from π to π' centred at O will extend to a bijective map of linear spaces. We will still call it **Γ**, but now it goes from E(π) to E(π'). How do we do this? Well. we have to define this extension. We kind of have some inspiration for it here, we did like these to hold, so let us define it.

So, in particular where does **Γ** send the various points in infinity in π? I have a small typo there. I should have said where does it send the line at infinity of π and what does it send to the various points at infinity in π'. In other words what does it send to the line at infinity in π'.

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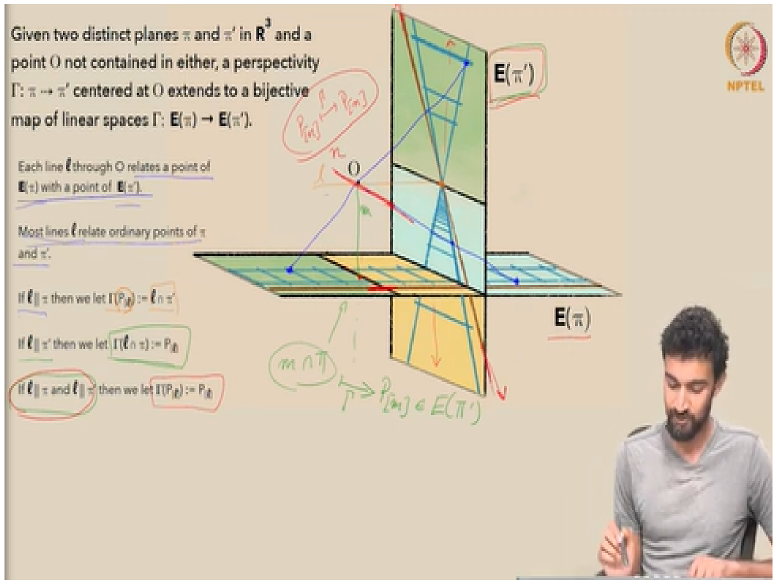

So, to answer that let us go back to this picture here and see where it sends those to. Well, how does a perspective work? Each line through O, like this line here, will relate one point of E(π) with one point of E(π'). So, in this case it is relating this point of E(π) with this point of E(π'). Similarly, this line through O relates this point of E(π) with this point of E(π').

But these are kind of simple examples because I am just relating ordinary points of π to ordinary points of π'. And most lines through O are going to do just that, they are just going to relate ordinary points of π with ordinary points of π'. Really, we are looking at E(π) and E(π') for a reason. We want to know how points at infinity get related between E(π) and E(π') .

So, how do we see that? Well, let us just look at it on a case-by-case basis. When are we not going to have a simple situation where a line intersects both planes? Well, that is going to fail whenever the line l is parallel to π or parallel to π' or parallel to both. That can also happen. So, let us look at the three cases. If a line l is parallel to π, in that case it is going to look like this. Then it is parallel to π, in that case it is not touching π anywhere but it will touch π'.

In that case, let **Γ**($P_{[l]}$) defined to be l intersect π'. In other words this l here which is parallel to π, it represents a point at infinity of π and namely $P_{[l]}$, and let **Γ** send $P_{[l]}$ to l intersect π', this point here. So, for example this l sent this $P_{[l]}$ to this point here, this ordinary point here. Now let us look at the second case.

So, in the second case, what happens to a line which is parallel to π'? Well, a line that is parallel to π', for example this line here, let us call it m, that line m does intersect π in an ordinary point. What point? That point is m intersect π, this point here. So, we need to say where **Γ** sends this ordinary point to. Well, this projection is not sending it to an ordinary point on π' because this line here, this line m, is parallel to π'.

So, in that case, let us define the image to be $P_{[m]}$. So m intersect π gets mapped by **Γ** to this point in infinity which is in the projective extension of π'. So, that is what happens if a line, in this case m, is parallel to π'. It will relate an ordinary point of π to an ideal or infinite point of π'. There is a final case to consider which is what happens if the line is parallel to both π and π'.

So, let me actually get a new colour, say red and let us consider a line n which is parallel to both π and π'. So, maybe you know it is parallel to their intersection, it looks like this and we will call that n. So it is parallel to that intersection; it does not intersect either π or π'. In other words, it represents a point at infinity in both π and π'.

It is a shared point at infinity. And in this case, we just defined $\mathbf{\Gamma}(P_{[n]})$ to be $P_{[n]}$. So, $P_{[n]}$ maps to $P_{[n]}$ via **Γ** in this case. So, **Γ** just fixes that point. So that basically takes care of all the cases and this is how we can extend our map, our perspectivity **Γ** to a well-defined map from E(π) to E(π').

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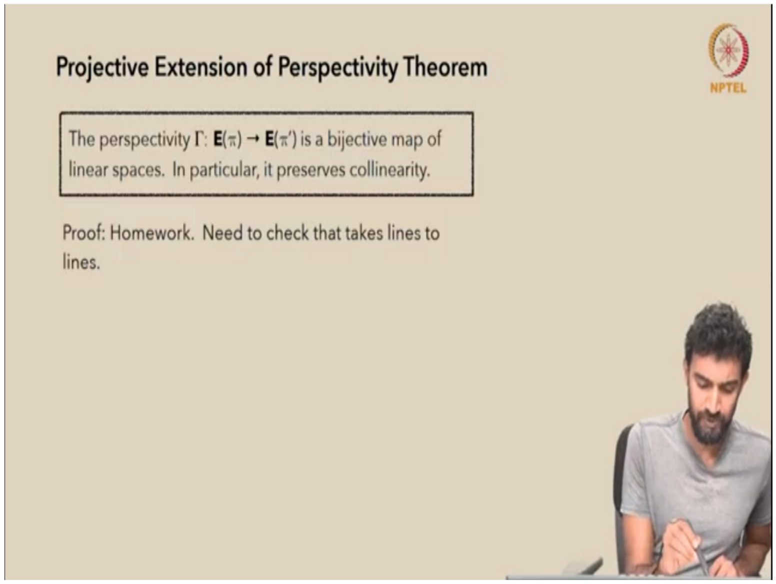

Now there is a little more to do, in particular, this is a bijective map that is not so hard to see but there is a little bit of detail to work out. It is a map of linear spaces. Both E(π) and E(π') are linear spaces. They both have a notion of lines and a map of linear spaces should preserve collinearity. So, I am leaving it as homework to prove that and one needs to check that this map takes lines to lines.

So, it is kind of clear that when we are looking at the perspectivity away from the problematic zones when it is just a simple projection from π to π'. Then it is fairly easy to see why it takes lines to the lines. I will leave that for you to think about. But the slightly more tricky part is to check that it takes the other lines that we have added to the lines. For example, it takes lines at infinity to lines.

So, that still remains to be checked but just to check that the image of a line infinity is in fact a line and that preimage of a line at infinity is also a line. So, that just needs to be checked. But if we know those are true, once we have checked that, then this takes lines to lines and therefore preserves collinearity. Therefore it is a map of linear spaces. So, it is not just a map of sets it is a map of linear spaces.

Now the other thing that you might be wondering about which we can see from this picture is that the map is a bit funny as a rational map.

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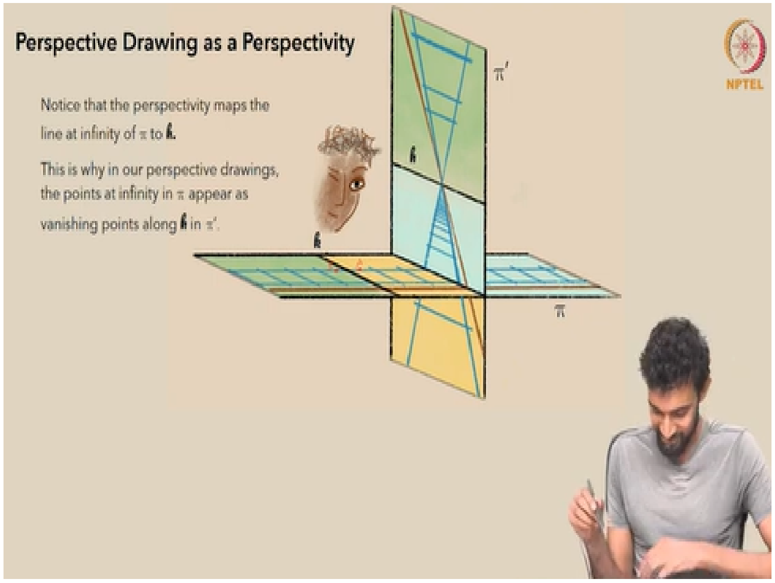

Let us go back to it being a rational map, so I have just gone back before we introduced the extended projective extensions of our planes. When we think of this perspectivity as a rational map, it is not continuous. You can see that from the three regions here, so if we just look at the blue and yellow regions it looks kind of continuous but everything breaks down at this line k. So, at this line k here the continuity breaks down.

Because on π, this point and this point are very close to each other., but in π' the images of these points are very, very far away from each other. If we call this point a and this point b where are the images of a and b in π'? Well, for a, we get the image by imagining a flashlight. So, maybe I will go back to the flashlight, that is over here.

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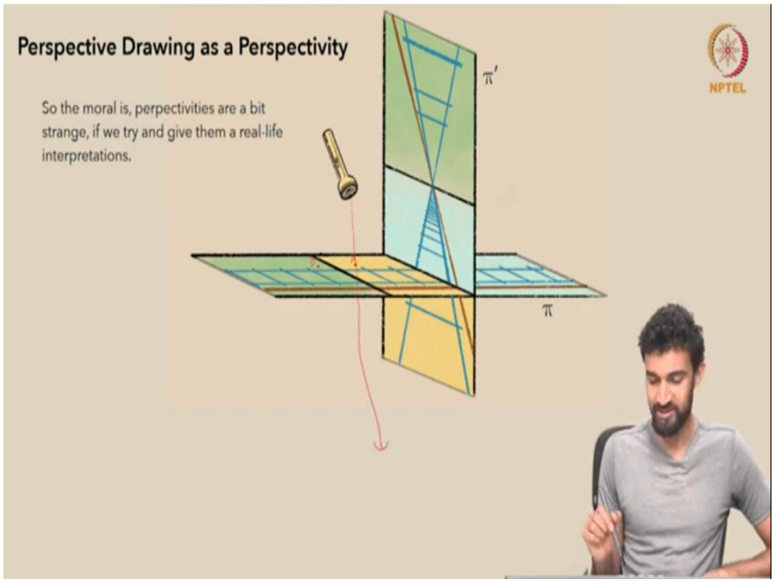

Note that a and b are over here, a is getting projected far down. So, a is getting projected way out here farther down than we can see and far off into the distance along this plane π'.

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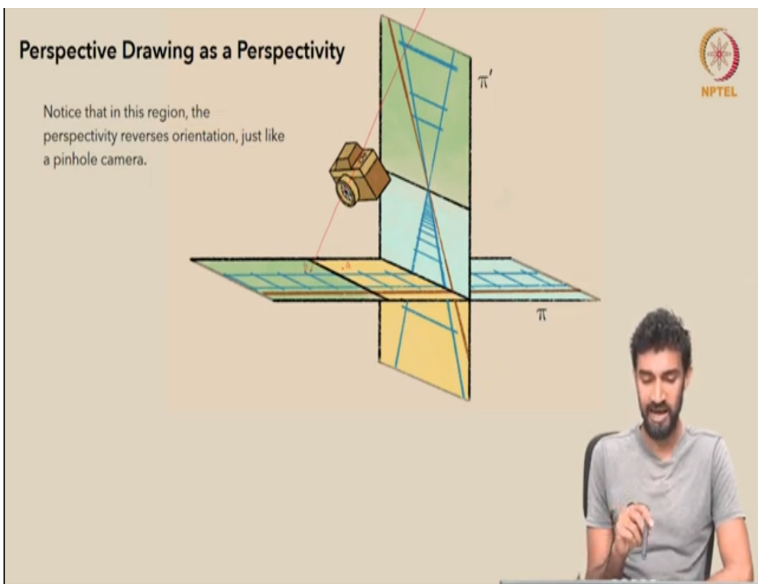

On the other hand, using this camera view, b is also getting projected far up further than we can see in this plane π'. So, neither of these are visible in our picture but they are clearly very far apart from each other, whereas in π they are very close to each other. So, continuity gets broken quite drastically along this line here. So, it is a rational map. I guess I cannot say it is discontinuous because this line k is not in its domain.

But clearly, we can see that it will not extend to any continuous map from π to π'. However, the interesting thing is that once we create a projective extension of a perspectivity which is a map from this projective extension of π to this projective extension of π'. This extended perspectivity in fact it is continuous. We will not prove that now because we have not talked or thought about the topology of these extended spaces.

But the basic reason for that is that a pointed infinity in one direction is the same as a point in the other direction. So, for example the points on this line k which map down to points in infinity for E(π'). Well, they actually connect up to, I mean they serve as the infinite limit in this direction and also in this direction here. So, that is just an interesting thing to think about ,we will not be needing to use that anytime soon.

But I just wanted to mention that somehow in this extended setting, that is another way that things become nicer, this map actually becomes continuous. So, the reason is that in the extended projective plane, in the extended Euclidean plane or these projective extensions of these planes, lines kind of loop around in a way. And you can see it in this picture, you can see that these railway tracks here somehow this line here and this line here well they are actually connecting up.

Because they are both coming from this portion of the third rail here, this portion of third rail is mapping to here and also to here. So, they are kind of linking up at that point in infinity. So, adding those points in infinity kind of brings back our continuity in an interesting way.

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So, I finally want to address some unfinished business from the previous lecture which is that I cheated in our proof that the harmonic conjugate is well defined. So, in our construction of the harmonic conjugate so far, we only looked at nice examples. So, I want to look at some not so nice examples.

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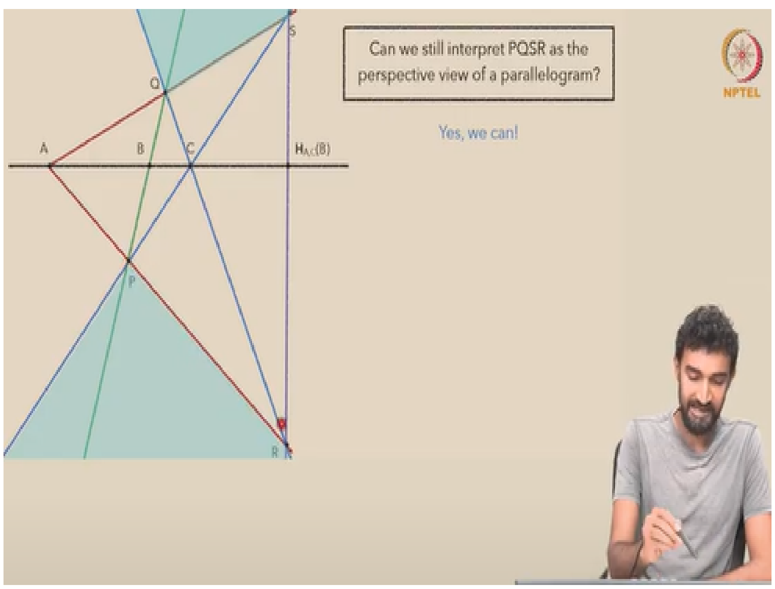

So, let us just review the construction of the harmonic conjugate it is been a little while. So, we start with three points A B and C that are collinear. Let us add point P lying off of that line, let us connect it up to B and let us choose any point Q on that line. So, far I have only chosen points Q that lie between P and B, like here or here etc. But we can also choose a point Q that lies beyond, along that line PB, on the other side of this line l.

And what happens if we do that? Well, let us continue the construction. The next step in the construction is to connect A and C up to Q and P. So, we can connect C to Q and P and connect A to Q and P. And doing that we can now find our points R and S; R is the intersection of AP with CQ and S is the intersection of AQ with CP. So, this is R and this is S. So, far so good. Then we connect R and S to get our fourth point which is the harmonic conjugate of B over here.

So, the construction still works it still gave us the same point but clearly our proof is not going to be as simple as compared to before. Because remember we proved this by interpreting this quadrilateral PQRS as a perspective view of a parallelogram tile. That is how we proved this and that is how we prove the theorem that this harmonic conjugate is well defined. And I am not so sure now how we can interpret this as a perspective view of a tile.

So, that is the question. Can we still interpret it as the perspective view of a parallelogram? The answer is that yes, we can. It is just a little surprising because let us extend our lines further out and the key is to imagine our parallelogram lying out here and here going out. So, in this view it is a perspective view of a parallelogram but in this perspective view it is the parallelogram appears infinitely large.

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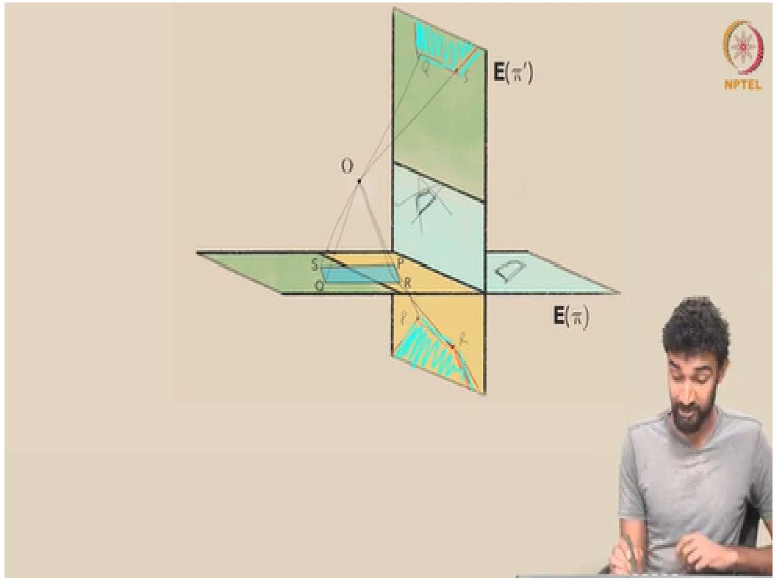

And the reason is that the parallelogram is sitting over this problematic line k. And as a result, it is mapping to a very large, seemingly disconnected region. So, our point Q ends up being somewhere here, our point S ends up being somewhere here, our point P ends up being somewhere here and R ends up being somewhere here. Let me just redraw that point P goes here, let us say and R goes here and we end up with this picture.

I will not do all the colours right now but we basically end up with a quadrilateral, something like that. So, that is how we arrived at this picture here. So, if our parallelogram is lying along across this line k that is where we get run into problems and get a perspective image that looks quite strange like this. But in fact, this is a perspective image of a parallelogram; it is just a parallelogram that is situated here.

Up till now we have only been looking at perspective views of parallelograms that are sitting over in this nice region of π which will just map to parallelograms over here that look like this. So, here we are seeing an example of a parallelogram somewhere else and the image of it under our perspectivity. So, it is the perspective view of a parallelogram, but it does not naturally arise from perspective drawing rather it arises from this more abstract notion of a perspectivity.

Which kind of generalizes and extends the idea of perspective drawing to give us a map from the plane to the plane. And it gives us a rational map from the plane to the plane but it gives us an honest to goodness map from the projective extension of the plane to the projective extension of the plane. So, that is what is really nice about a perspectivity.

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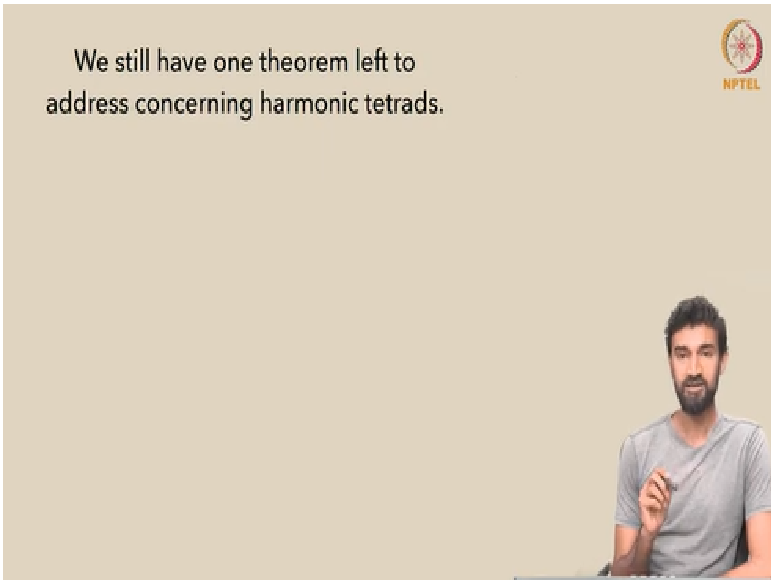

And we still have one last theorem to address which goes back to the harmonic tetrads from earlier in this week

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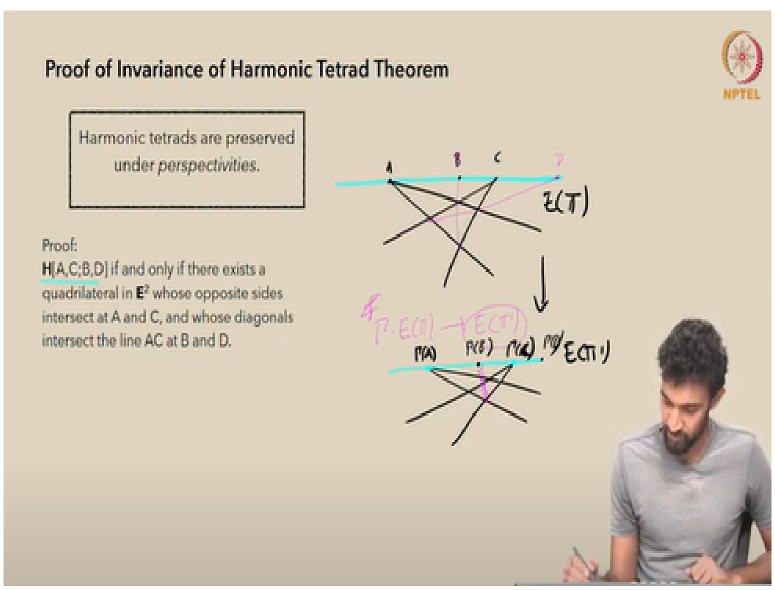

And this is the proof of the invariance of the harmonic tetrad. So, I mentioned that harmonic tetrads are preserved under shifts in perspective. And now we know what a shift in perspective is. A shift in perspective is just a perspectivity. So, the theorem that I want to discuss now is that harmonic tetrads are preserved under perspectivities. So, how do we prove that? Well, remember a harmonic tetrad H[A,C;B,D] if and only if there exists a quadrilateral in E^2 whose pairs of opposite sides converge to A and C or intersect at A and C, and whose diagonals intersect that line AC at B and D.

So, it is the same picture as before. So, in this case A, B, C and D is a harmonic tetrad I have just drawn, I have simplified the colours a little the diagonals are in violet. If we have a quadrilateral whose pairs of opposite sides intersect at A and C respectively. And whose diagonals hit the line AC at B and D then A, B, C and D is a harmonic tetrad.

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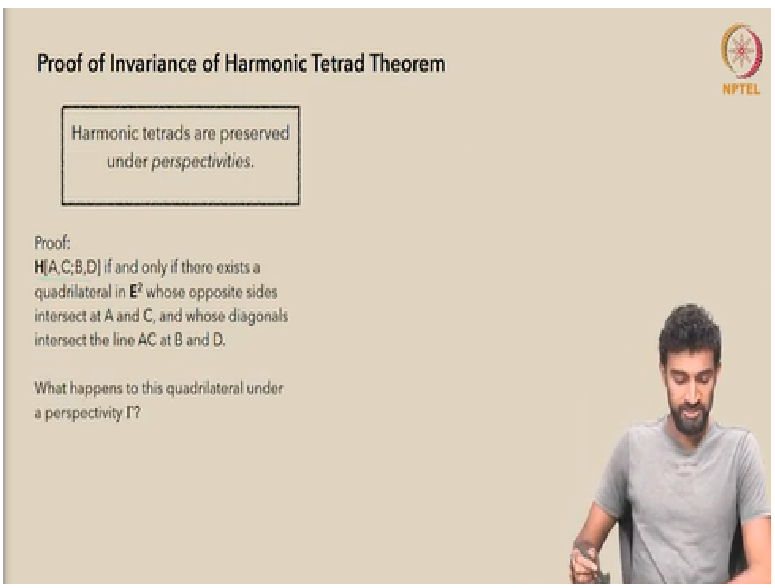

So, what happens to this quadrilateral under a perspectivity **Γ**? So, for example, if we have a harmonic tetrad in our plane π and we perform a perspectivity **Γ** to send π to another plane π' what happens to this quadrilateral which is associated to our harmonic tetrad?

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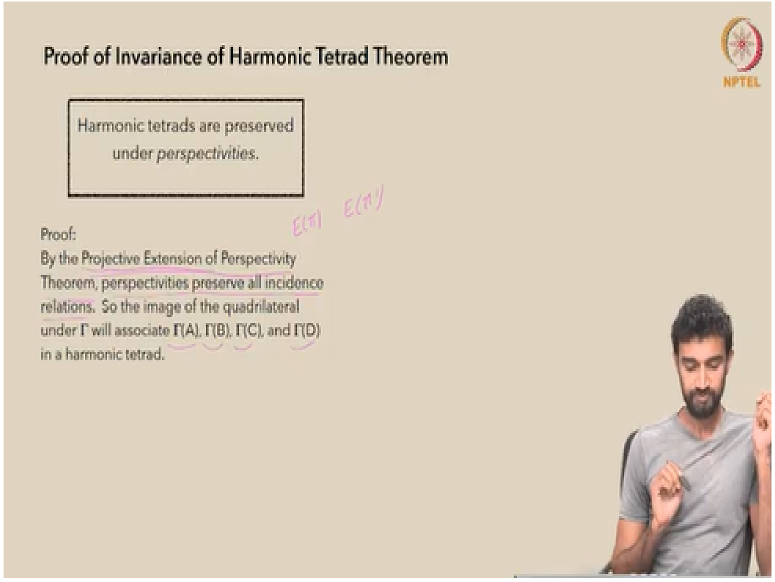

Well, I claim that it goes to another quadrilateral and that new quadrilateral will be associated with the points **Γ**(A), **Γ**(B), **Γ**(C) and **Γ**(D). Meaning that it will show that those also form a harmonic tetrad. So, how do we see this? Well, by the projective extension of perspectivity theorem, by the theorem that showed that the projective extension of a perspectivity is a well-defined bijection between E(π) and E(π').

In that theorem, we also showed that this is a map of linear spaces and that takes lines to lines in particular. So, by that theorem, perspectivity preserves all incidence relations, all these lines will map to lines under **Γ**. So, **Γ** is a map from E(π) to E(π') and it takes lines to lines.

So, it will take this quadrilateral to another quadrilateral. In fact it will not just take the quadrilateral to another quadrilateral, it will take this configuration of seven lines also. If you notice, there are seven lines in this configuration. There are four black lines, two violet lines and this blue line. So, there are seven lines in this configuration and **Γ** is going to take those seven lines to another seven lines in E(π').

So, when it does that it will preserve all the incidence relations, so it will take its seven lines which have the same incidence relations between them. Meaning that these black lines in E(π) will map to some other black lines in E(π'). The blue line will still go through the points of intersection of the sides of the black lines. So, the blue line will still go through these two points because incidence relations are preserved.

This point A is on the blue line and it is also on these two black lines so the image of it will continue to be on the images of those lines and similarly the image of C will continue to be on the images of these lines. So, **Γ**(A) and **Γ**(C) will continue to be incident to the images of all the same lines. And similarly, **Γ**(B) and **Γ**(D) will be incident to all of the same lines as well.

So, as a result we will get another image of the quadrilateral that will be associated to **Γ**(A), **Γ**(B), **Γ**(C) and **Γ**(D) and we will associate them in a harmonic tetrad. So, because perspectives preserve incidence relations harmonic tetrads are guaranteed to be preserved under perspectivities. And this might seem a bit like a strange quantity to be preserved under perspective shifts.

But this harmonic tetrad actually comes up in a surprising number of places and it will be key to an even more and more relatable and numerical quantity which is also preserved under perspective shifts which we will see next week. But this at least gives a partial answer to the question of what is preserved under a change in perspective which is that harmonic tetrads are preserved under perspectivities.

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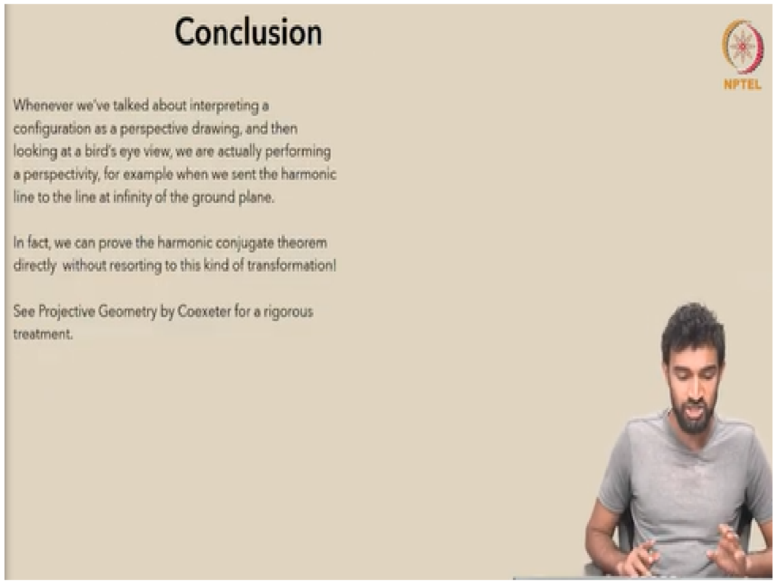

So, that concludes this lecture and I will just mention that these perspectivities we have actually been using them, even before we defined them. Because even when we talked about interpreting a configuration as a perspective drawing and looking at it from a birds-eye view that is an example of a perspectivity. We are actually performing a perspectivity there in which we are sending that harmonic line, that line containing A, B, C and D, we are sending that to the line at infinity on the ground plane.

So, our proof of the harmonic conjugate theorem was really using these projective transformations. These perspectives change our situation to one which is more favourable for proving or seeing why the theorem was true. But there is another approach to projective geometry and to a lot of this material which does not rely on these kinds of transformations and instead proves things in whatever setting we happen to be in.

And an example of that approach which is a more axiomatic approach and as a result it has advantages and disadvantages. An advantage is that because everything is very axiomatic it can be done in a very airtight kind of way. But the disadvantage is that it requires an enormous amount of time to build up fairly intuitive ideas from those axioms and we lose some of the visual interpretations.

For example, there is something very visual and approachable about the idea of looking at something from a bird's eye view or interpreting a drawing as a perspective view. It relates to our intuitive sense of vision very nicely. Whereas the more axiomatic approach in my opinion, it steps away from that connection to human vision. So, that is a disadvantage.

But if you are interested in checking it out, the book by Coexeter called Projective geometry is a great place to start and see this more rigorous treatment from the ground up. So, for people who are interested I did recommend that source but, in this course, we are not going to take that axiomatic route. So, stay tuned and see you next week.