Our Mathematical Senses

Prof. Vijay Ravikumar
Department of Mathematics
Indian Institute of Technology, Madras

Lecture - 06
Harmonic Tetrads

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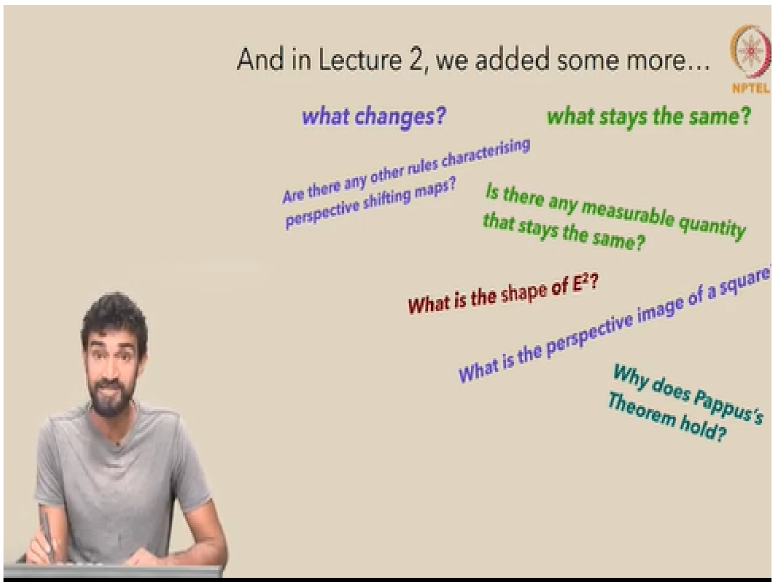

Hello, welcome back to lecture 3 of the geometry of vision. In lecture 1, we raised a lot of questions concerning the way in which we visually understand the world around us. In particular, we talked about what changes and what stays the same when we shift from one perspective to another. And some related questions that we asked are whether there are any measurable quantities, any numerical quantities that stay the same when we shift between perspectives.

For example, different perspective views of a tiled floor. And another question we asked is overall how do we characterize these maps? These perspective shifting maps from the plane to the plane. And a final question that we asked is what is the perspective image of a square? What are all the possible ways that a square can appear under a perspective shift? In lecture two, we did not actually answer any of these questions, in fact we added a couple more.

We introduced the extended plane, the extended Euclidean plane and we saw that it has a somewhat strange shape. And we also introduced a coincidence, known as Pappus's theorem. But we have not yet seen why Pappus's theorem holds. So, we have a lot of questions hanging over our heads now.

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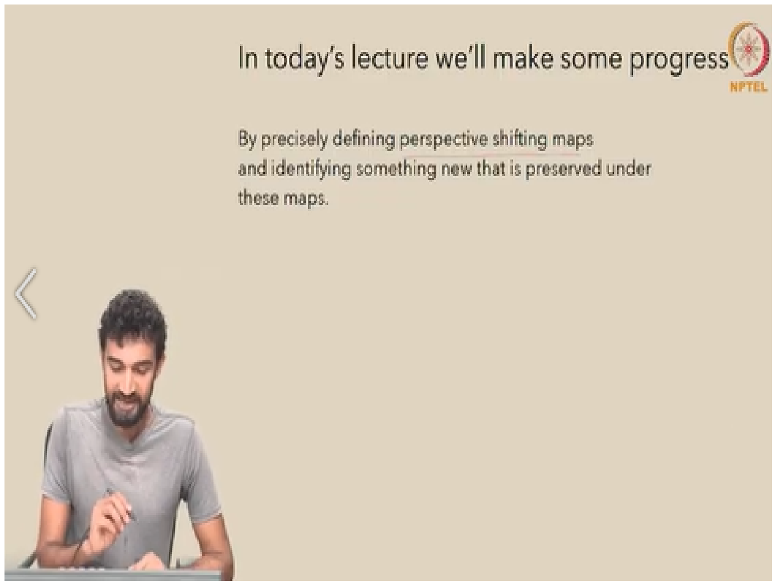

And today we will start to make some progress or rather this week we will start to make some progress in answering some of them. So, what we want to do this week is precisely define perspective shifting maps and identify something which is preserved under these maps. So, that is our mission for this week, for lectures 3 and 4.

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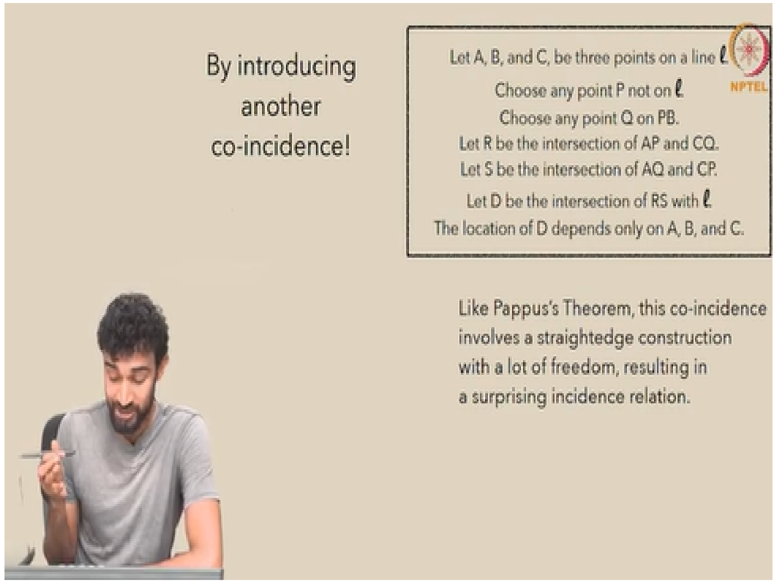

And in lecture 3, we will begin by introducing another coincidence. So, Pappus's theorem is what I am considering our first coincidence, that we have explored. Today I want to introduce a second coincidence and it works in the following way. We can take any three points on a line l, in other words any three collinear points A, B and C. For example, if we have a line here, we can imagine three collinear points on it, A, B and C and we have the following construction.

And by the way, like Pappus's theorem, this coincidence also involves a straight edge construction and there is a lot of freedom in that construction. So, the surprising thing about the theorem is that despite that freedom something unexpected always happens and it is always guaranteed to happen. So, let us see exactly what that is by following this construction. So, the construction works as follows. Maybe rather than draw it live, let us use the slides that I have already prepared.

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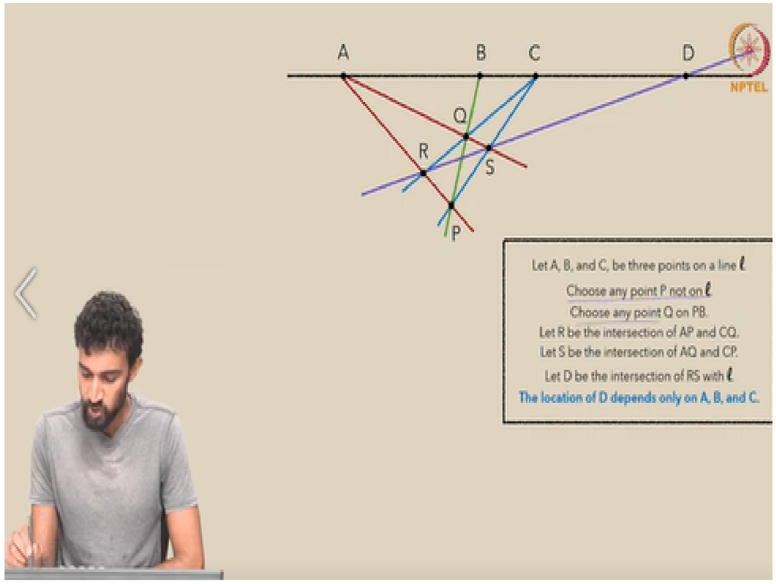

So, we start with A, B and C as three collinear points. Now we choose any point P which is not on the line l, this is the line l. So, choose any point P not on that line l. Now draw the line from P to B and choose any point Q on PB. So, you can see, I have chosen my point Q here and it is lying along this line PB. So, now we have made two choices. So far we have chosen P and we have chosen Q along PB.

The next step is to draw some lines and define two more points R and S. So, what we want to do is draw the lines connecting A and C to P and Q. So, in particular I want to draw all four of those lines connecting A to P and Q, and connecting C to P and Q. So, I am connecting C to P and Q and connecting A to B and Q. We have done that. Let R be the intersection of AP and CQ.

Similarly, let S be the intersection of AQ and CP. So, AQ and CP intersect here, so that is our point S. So, we have now defined R and S, the next step in our construction is to let D be the intersection of RS with l. In other words, draw this line RS, extend it till it hits l and let us define that point here to be D. So, this is our construction. Now nothing surprising so far, the surprising thing is that the location of D depends only on A, B and C.

It is independent of the two choices that we made. So, we could have picked P and Q to be anywhere else and we still would have gotten the same point D. Now it is easier to see it than maybe hear it. So, let us see it again. So, let us do the same construction again. Let us keep P the same this time. There are two choices. So, let us keep P fixed and let us choose a different Q lying along the line PB.

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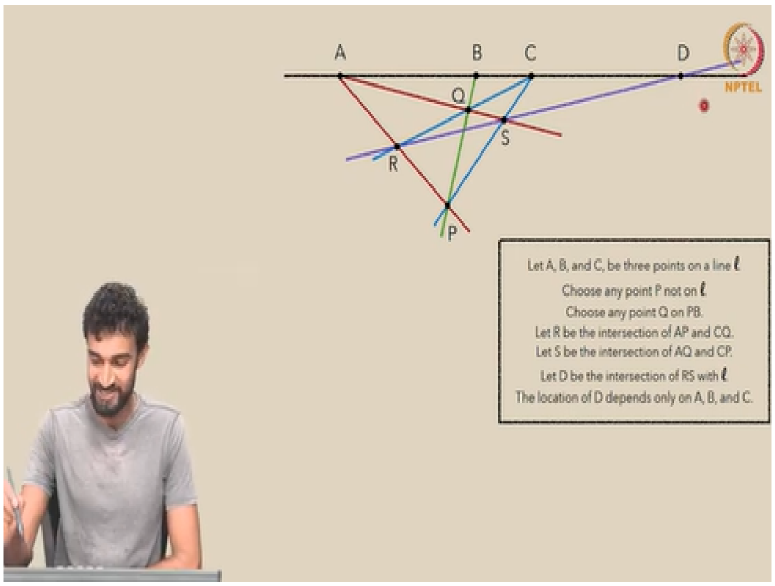

So, I am keeping the same P and I am repeating the construction but I am choosing Q all the way up here this time. And as before I am connecting C to Q and P and connecting A to Q and P. So, I am drawing all these lines. Now let R be the intersection of AP and CQ and S be the intersection of AQ and CP. So, R is the intersection of AP and CQ.

And finally let us draw the line RS and see where it hits l. It hit exactly the same place. I had left that point from before just for comparison's sake. Yes, it is going through the same point D, it is in the same location. So, that is two experiments we have done and both times D has come out in the same place. But the claim is, of course, that it comes out in the same place no matter. Let us see a third demonstration and this time let us vary the location of the point P.

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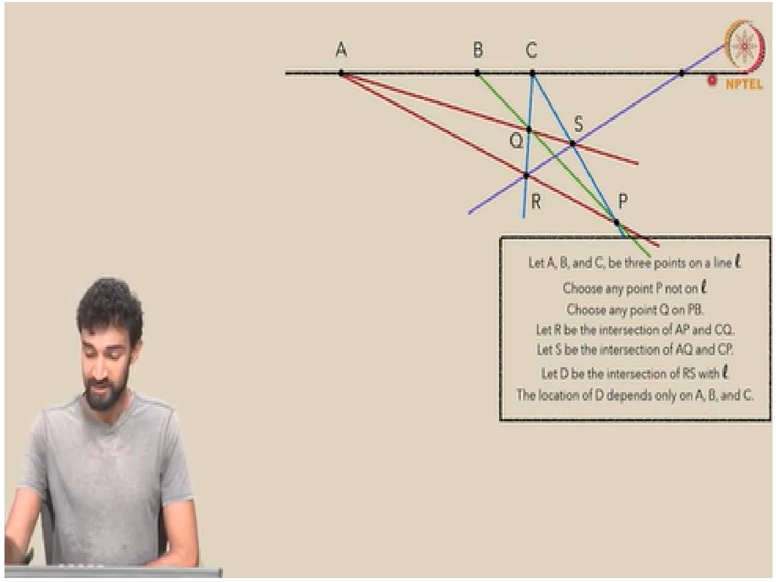

So, let us put P all the way over here and let us see what happens when we follow this instruction set. So, we connect P to B and we choose Q somewhere along this line. I am choosing it right here, but I could have chosen it anywhere else. And now we connect C to P and Q, we connect A to P and Q, we get a kind of quadrilateral here. And now we define R and S to be these two other corner points of the quadrilateral.

We connect up R and S to get this violet line here. And the claim is that the violet line will intersect our line l in the same spot and indeed it does D is in the same spot once again.

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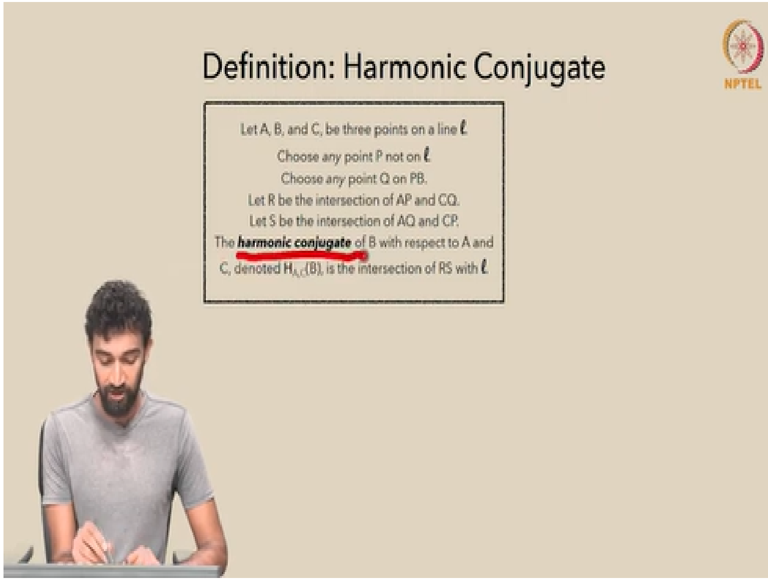

So, given that this D, this fourth point, keeps coming out in the same way, it makes sense to give it a special name and indeed it has a special name, the harmonic conjugate or sometimes projective harmonic conjugate, to distinguish it from other mathematical harmonic conjugates that come up in other fields of mathematics. So, in this class, we will just call it the harmonic conjugate and it is defined in the same way as we define D.

By the way D is the harmonic conjugate of B with respect to A and C. So, going back, D is the harmonic conjugate of B, this point, with respect to A and C. So, another way to kind of visually see that we are kind of saying that in this quadrilateral here C and A give us the sides of the quadrilateral, B and D give us the diagonals.

We will talk more about that in a second. But as a definition, the harmonic conjugate of B with respect to A and C is denoted by $H_{AC}(B)$. It is the intersection of RS with our original line l containing A, B and C. So, we can rephrase our coincidence as the harmonic conjugate theorem.

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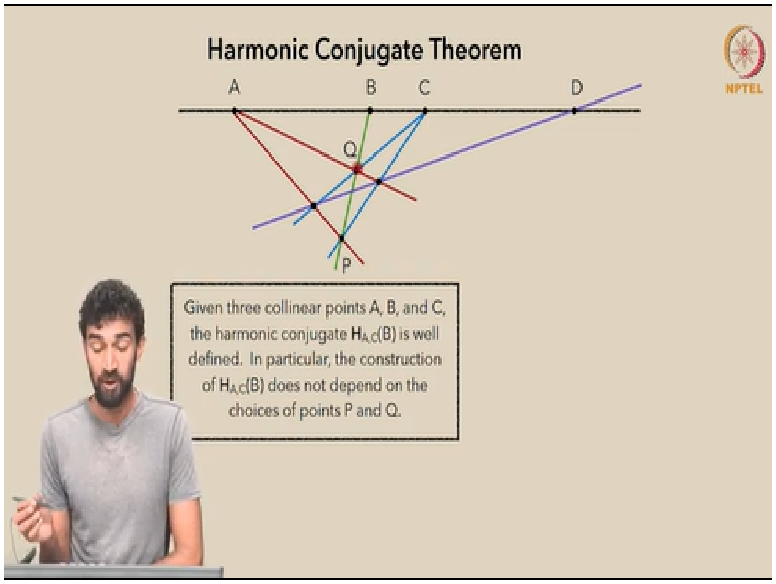

And that theorem states that given any three collinear points A, B and C, the harmonic conjugate $H_{AC}(B)$ is well-defined here. What does it mean to say it is well defined? It means that the construction of the harmonic conjugate does not depend on the choices of points P and Q. I can choose them in any way I want and I will still get the same point. So, it is well defined.

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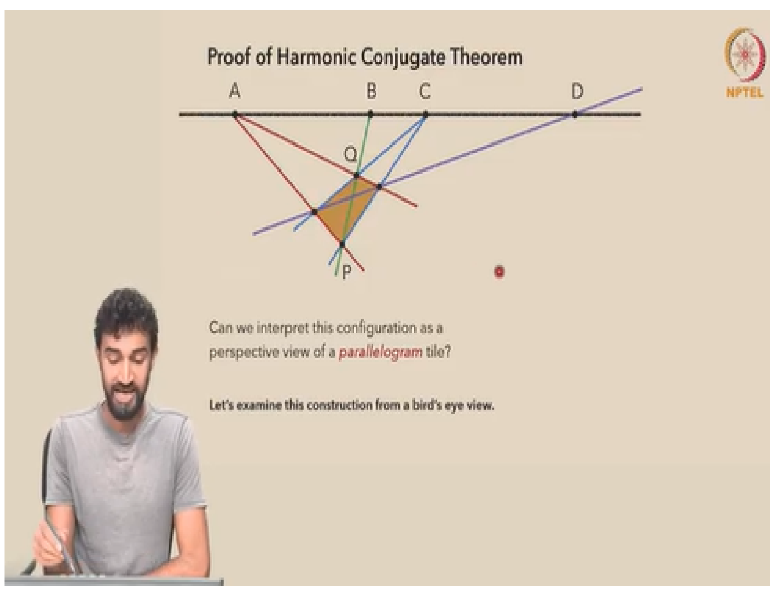

How can we prove this theorem? Well, this collection of points and lines, this configuration here, hopefully it looks a little bit familiar from the previous two lectures. In fact, it is a lot like the perspective views of the tiled floor that we have been working with since the introduction video.

In particular this one looks a little bit like the third perspective view, but it looks like one of those perspective views.

So, can we somehow interpret this configuration of points and lines as a perspective view of a square tile? If so maybe then that might help us to prove this. So, let us say this is a square tile here. Can we somehow interpret this as a perspective view of a square tile sitting somewhere in space? We can interpret it that way, if this quadrilateral here is indeed a perspective image of a square.

If there is actually a way to situate a square somewhere and take a photograph of it from some angle in such a way that the square ends up looking like this. But is that possible? Is this quadrilateral actually the perspective view of a square? We do not know yet that is one of the questions that is hanging over our heads right now, which we need to prove or examine and kind of find an answer to one way or another.

But as of now we do not know what the perspective image of a square looks like. So, we cannot assert that, this is definitely the perspective view of a square tile. We can still interpret this tile as the perspective image of a parallelogram and the reason is as follows. It is certainly a quadrilateral; we can certainly interpret this as a perspective view of something.

And when we do that, if we make this our line at infinity, our horizon line, then this quadrilateral, both of these opposite edges are converging to this point at infinity here, meaning, from a bird's eye view they would appear parallel. And similarly, both of these opposite edges are converging to a pointed infinity here. Again, meaning that from a bird's eye view these would be parallel. So, it is a quadrilateral whose opposite edges are parallel to each other.

Otherwise, that is a parallelogram. So, we can interpret this as a perspective view of a parallelogram. In that case, we can also examine that parallelogram, examine this construction, this entire configuration from a bird's eye view, from a top-down view.

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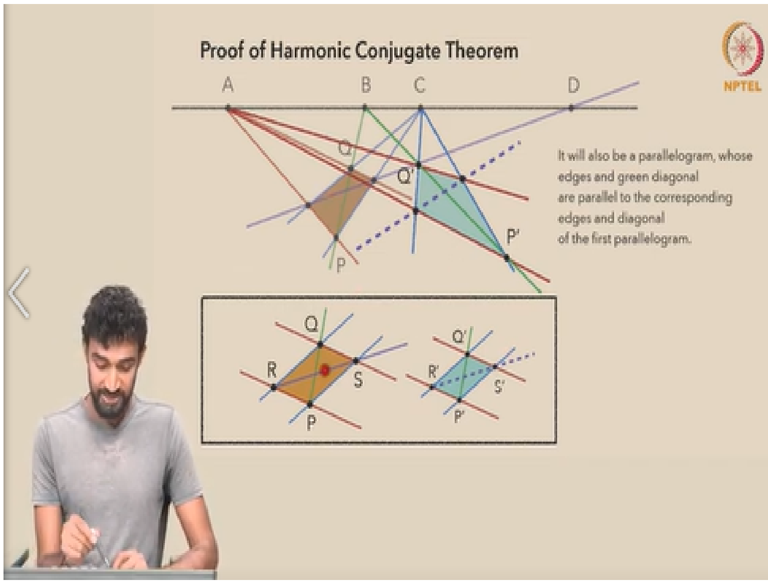

So, when we do that here, it is from a top-down view. It looks like a parallelogram, here is P, Q, R and S. I forgot to label R and S here. But you can just imagine this is R and this is S. I have retained the colours. So, we are going to colour code our points at infinity. So, because we cannot see the points in infinity from the top-down view. So, the different parallel classes, equivalence classes of parallel lines, the different families of parallel lines are colour coded.

So, the blue ones are going in this direction, the red ones in this direction, the purple diagonal here, and the green diagonal here. So, hopefully that is not too confusing. So, let us perform the construction again with a different choice of P and Q and then let us see what is going on. Let us put a new point P' here and connect it to B.

I am just following the instructions for the construction one more time. Next step is to choose a Q' somewhere on this line. So, let us choose it here. Next step is to connect up C to Q' and P' and also connect up A to Q' and P'. We get a new quadrilateral here. Next step is to draw the diagonal, the other diagonal in violet. And according to the theorem this new diagonal ought to intersect the point D here, the same point D.

So, that is what we are trying to prove. It is guaranteed to intersect the same point D and we want to know why that is. So, let us examine that second construction from the bird's eye view. It will give us another parallelogram. So, here is the quadrilateral from this view, from a bird's eye view. What will this second tile look like? And it will give us another parallelogram. Not only another parallelogram, for the same reason that the first one was a parallelogram, but this parallelogram is intimately related to this parallelogram.

Because the red, these two, edges of the parallelogram are parallel to these two edges. They are all converging to the same point at infinity A. Similarly, these blue edges are parallel to these blue edges here on our original parallelogram. In the perspective view they are all converging to this point at infinity C. The green diagonal of this new parallelogram is parallel to the green diagonal of the first parallelogram.

And from the perspective view they are both converging to this point at infinity B. So, these parallelograms are very related. This new one all of its corresponding edges seem to be parallel to the corresponding edges of the original. But we have not said anything yet about this violet diagonal here. So, in our new parallelogram, I had like to claim and assert that this violet diagonal has to be parallel to the violet diagonal here in our original parallelogram.

If that is the case then equivalently in the perspective view this dotted violet line must intersect D. So, going back to the bird's eye view the question I am asking, is R'S' parallel to RS?

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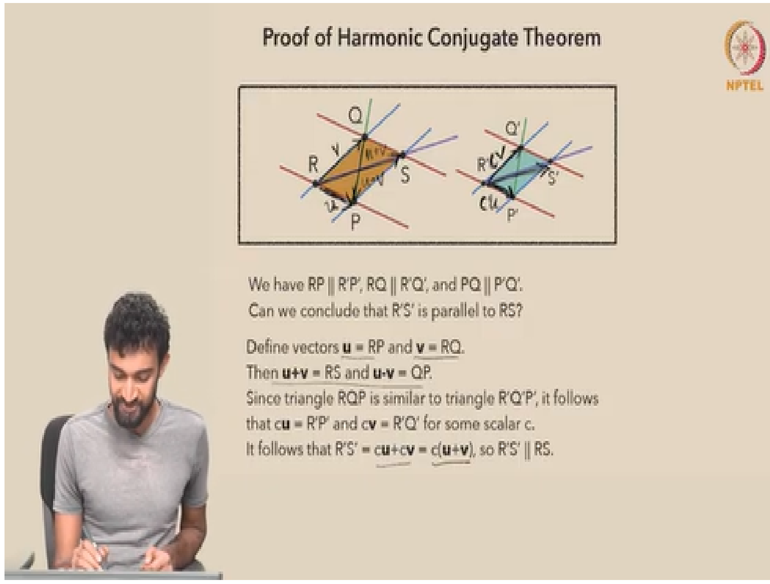

So, how can we see that indeed that is the case? Well, what do we know so far? We know that RP is parallel to R'P'. We know that RQ is parallel to R'Q' and PQ is parallel to P'Q'. What we do not know is that RS is parallel to R'S'. So, that is what we want to conclude. So, how do we do that well? Let us define some vectors. Let us define a vector u equal to RP.

So, that is a vector here and let us define a vector v equal to RQ. So, here is our vector v, here is our vector u. And in that case notice that u+v, if we add u and v, we get this diagonal vector. So, this is u+v and similarly u-v is the vector going from the point Q to point P. So, we actually capture all of the sides and diagonals through those vectors u and v.

Now we know that the triangle RQP is similar to the triangle R'Q'P' because all of their corresponding sides are parallel to each other. So, they have to be similar triangles since their corresponding sides are parallel. Since they are similar triangles, it follows that R'P' is equal to cu for some scalar c. Anyway, we can say it is equal to cu, for some scalar c because it is parallel to RP.

But since these triangles are similar that means that R'Q' is equal to cv for that same scalar c. So, this vector here from R' to P' is cu and this vector here, from R' to Q' is cv. And from that it follows that R'S' as a vector is equal to cu+cv. But we can just factor out the scalar c, so that is c(u+v). And therefore, R'S' is parallel to RS. So, therefore the harmonic conjugate D or $H_{AC}(B)$ is indeed well defined from what we have now seen from that top-down perspective.

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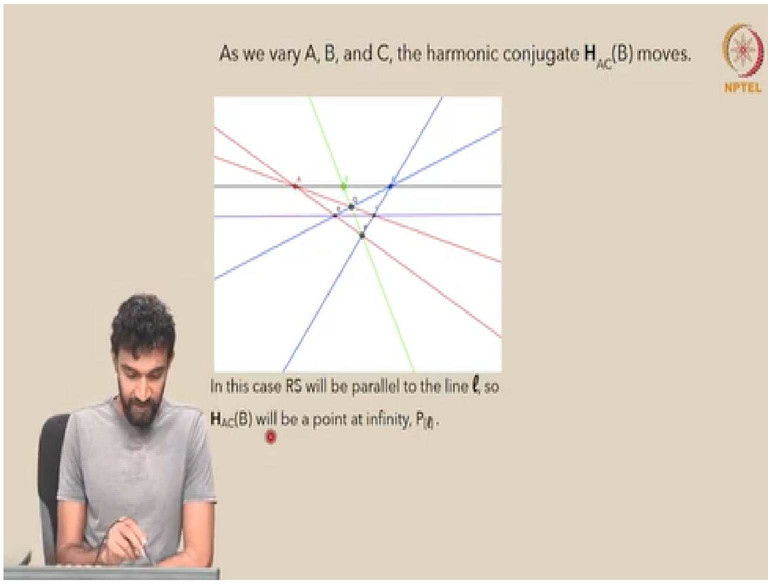

So, we have a well-defined quantity called the harmonic conjugate, given three points on a line we can define a fourth point based on those three and it is known as the harmonic conjugate. Of course, as we vary A B and C the harmonic conjugate will move around on that line. So, here are some screenshots from GeoGebra. It is a free software. You can download it and try it yourself if you want to try and create this construction and play around with it.

It is a very useful tool for exploring plane and geometry, projective geometry and Euclidean geometry. So, let us just see a few screenshots. So, I have put A, B and C here and D came out over here. But of course if I move P and Q around, D will stay the same. That is what we have seen so far. So, D is a well-defined function of A, B and C. But what if we move A, B and C. So, let me move B around and see how that affects the location of D.

So, if I move B to the right, D moves a little closer to C, they move together a little on the other end. If I move B closer to A, D also moves closer. So, you can see how moving B around will affect the location of D. But what happens if I evenly space A, B and C? So, if you space A, B and C evenly along the line then we get a kind of special configuration.

And the point D is nowhere to be seen; it actually goes off to infinity in that case. And you can see that because the horizontal line connecting R and S, that violet line which is supposed to meet our original line at D. It is actually parallel to our original line containing A, B and C. So, they will only meet at infinity in this case. So, in this situation, this configuration, the harmonic conjugate of B with respect to A and C actually lies off at infinity, it is a point at infinity.

So, the point at infinity is $P_{[l]}$, remember l is our original line. So, the harmonic conjugate of B will be the point at infinity $P_{[l]}$.

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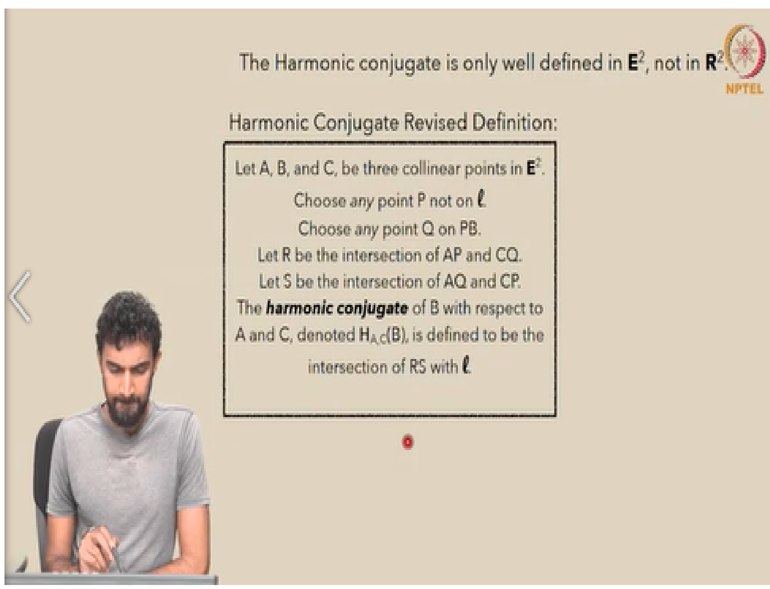

So, in other words the harmonic conjugate is only well defined in E^2, the extended Euclidean plane, not in all situations in R^2. So, let us revise our definition a little, keeping that in mind. So, it is the exact same construction but I have just added an E^2 here just to clarify that. We are picking our points A, B and C to be three collinear points in E^2. And in that case, we have a well-defined harmonic conjugate. But now that we are in E^2, is there anything else we have to check?

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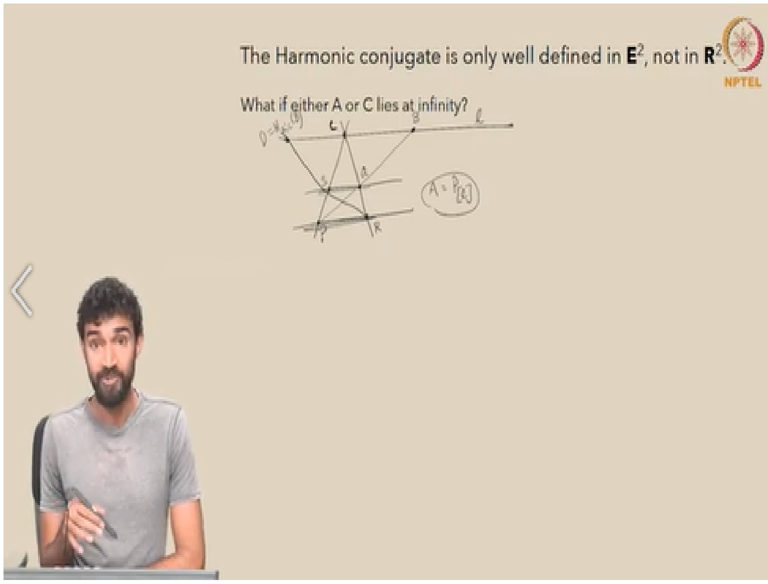

What if one of these points, for example, lies at infinity? For example, what if A or C lies at infinity? Well that is something to think about. But very roughly we will still get a well-defined harmonic conjugate and the harmonic conjugate in that case probably will not lie at infinity. But if A or C lies in infinity our picture will look a little bit different. So, in our picture here if A or C lies at infinity then we can still imagine this as a perspective view of a tiling.

But it will not be the exact same picture; it will be in one point perspective. So, imagine that P lies at infinity and say this is the point C. So, A is a point at infinity, let us say $P_{[l]}$. Then we will end up getting a quadrilateral that looks something like this. For example, let us put our B here, we picked our P here, our Q here, here is our R and here is our S and we will end up getting a point D, a harmonic conjugate over here.

So, this would be the situation if A lies at infinity. So, this pair of opposite sides of our quadrilateral would end up being parallel. So, this would resemble that one point perspective view of this square tiled floor that we looked at back in the introduction.

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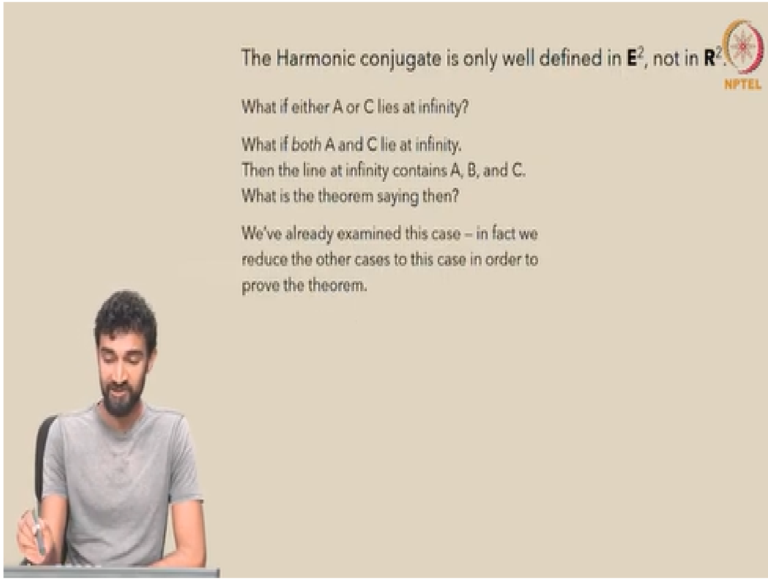

What if both A and C lie infinity? Well, in that case the entire line l is the line at infinity. It will contain A, B and C. So, what the theorem says is that in that case we have already examined and proved this case. This is the bird's eye view that we actually reduced all other cases to, in order to prove the theorem. So, we have actually dealt with that already. So, indeed the harmonic conjugate is well defined in E^2.

And you can try and convince yourself by just going through the definition again, going through the construction again and checking that there is nothing that we have left out. And there is actually maybe a little hint there is something a bit sneaky I have done which you can see if you can figure it out. There is a small hole in my proof that I have given so far. So, I will admit that right now. See if you can figure it out, we will address it and patch it up in lecture four.

But for the time being I will pretend that everything I have done is fine. If you can find the gap in my proof then good for you. If not, do not worry about it too much. I will come back and patch it up tomorrow in the next lecture. But I want to leave that gap now because patching it up requires something that we still need to build. If you are understanding what is going on so far, you are still in good shape for what is coming next.

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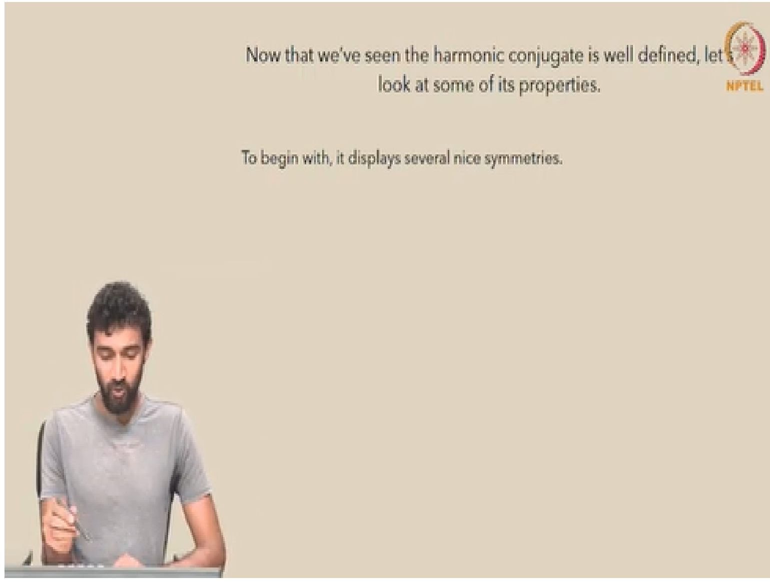

So, now that we have seen that the harmonic conjugate is well defined. Let us look at some of its properties. Why have we introduced this kind of strange construction and this fourth point? What is it all about? So, to begin with the harmonic conjugate displays some very nice symmetries. So, let us take a look at those.

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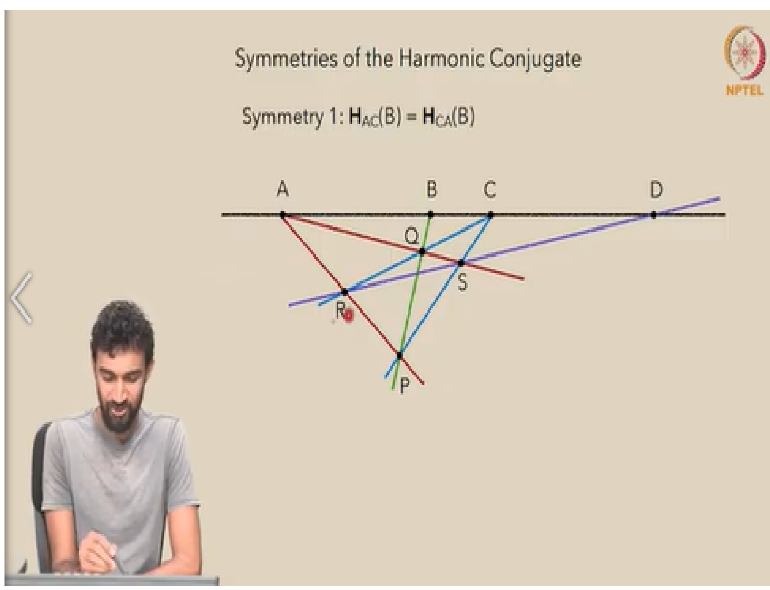

And the first one is very simple; it is just that $H_{AC}(B)$ equals $H_{CA}(B)$. So, what is that saying? Remember, in our construction we actually do not distinguish between A and C. So, it makes sense that there is a symmetry like this. And in our construction, remember, after picking our P connecting it to B, picking our Q the next step was just to connect A and C to P and Q. Drawing these four lines in any order does not matter.

So, in that way A and C are symmetric, there is nothing to distinguish them from each other. So, it makes and from that point onwards R and S also, it does not matter which is R which is S. We are just drawing a line connecting R and S and seeing where it intersects D. So, in this construction the roles played by points A and C are perfectly symmetric. So, it makes sense that this symmetry holds. So, that is our first symmetry, fairly simple.

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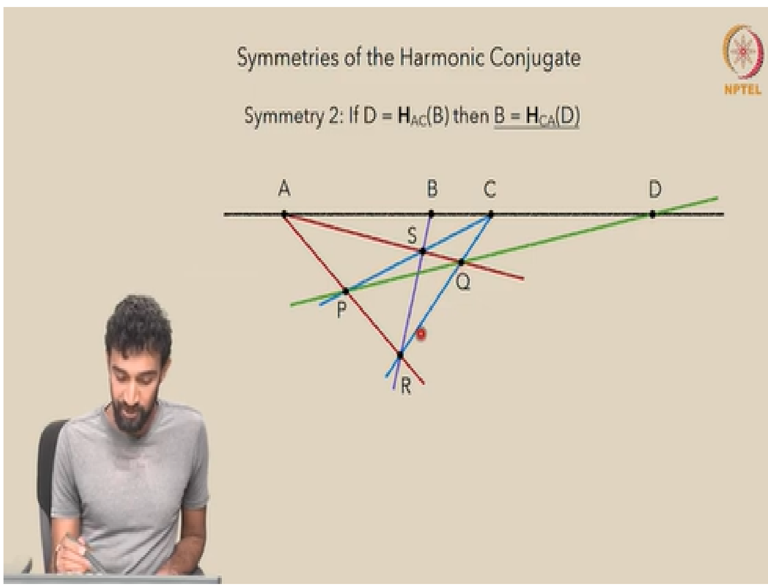

Our second symmetry is a little bit more interesting. Let D be defined to be the harmonic conjugate of B with respect to A and C. Then B is the harmonic conjugate of D with respect to A and C. I wrote that as CA I could have written it as A C. We have already seen that it does not matter but it is a little confusing. But I am not talking about this symmetry, I am talking about this, the symmetry that if D is equal to $H_{AC}(B)$, then B is equal to $H_{AC}(D)$.

So, let us just look at the diagrams for these two situations. So, if D is equal to the harmonic conjugate of B with respect to A and C, we have a quadrilateral like this that kind of represents this harmonic relationship. This diagram shows that if we can pick our P and our Q and get our

D. And in order to see the second relationship we needed a different diagram. But I argue that we can actually alter this diagram a little bit to build our second one.

So, this relationship is saying that after picking P and Q we have drawn our quadrilateral and our diagonals of our quadrilateral are actually hitting the line at B and D. So, what if we actually want to draw a diagram that represents the second situation. The exact same diagram will work if we just switch the labelling a little bit.

So, in particular I have just switched my labels a little. I am now starting with A, C and D and I am starting my construction by picking my P here, connecting my P to D because D is what I am taking the harmonic conjugate of. Now I am picking my Q somewhere along this line, I am picking it right here. Now I have my line segment PQ, I am connecting A and C to P and Q. Connecting A to P and Q, connecting C to P and Q, that gives me this quadrilateral here.

Now we already have one diagonal, let us look at the other diagonal RS of that quadrilateral. And that diagonal if we extend it hits our line l at B. Why does it hit it at B? Because it is the same diagonal that was green in our previous situation. Basically, we have created the same quadrilateral here. We have just switched the two diagonals. Since we know that the harmonic conjugate is well defined, we can pick our P and Q anywhere we want.

So, we can pick our P, where in the places that used to be R and S. Why do not we just pick those to be P and Q? That will result in the same quadrilateral that we had earlier except with the diagonals exchanged, showing that B is the harmonic conjugate of D with respect to A and C. So, this is a way of seeing the second symmetry.

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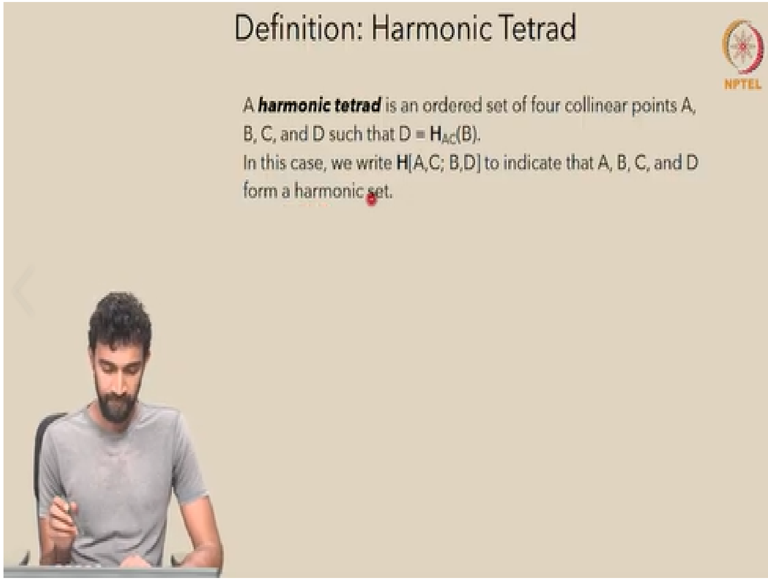

So, not only can we switch A and C but somehow B and D are playing a symmetric role. So, it will be useful to give another definition rather than just talk about harmonic conjugates. Let us talk about something called a harmonic tetrad, tetrad just means four things. Harmonic tetrad is an ordered set of four collinear points A, B, C and D such that D is the harmonic conjugate of B with respect to A and C.

And in this case, we will introduce this notation H[A,C:B,D] to indicate that A, B, C and D form a harmonic tetrad. So, this notation that is actually a statement when I write this, we can unpack that as a statement saying A, B, C and D form a harmonic tetrad. So, this is not a quantity as such, it is actually a statement.

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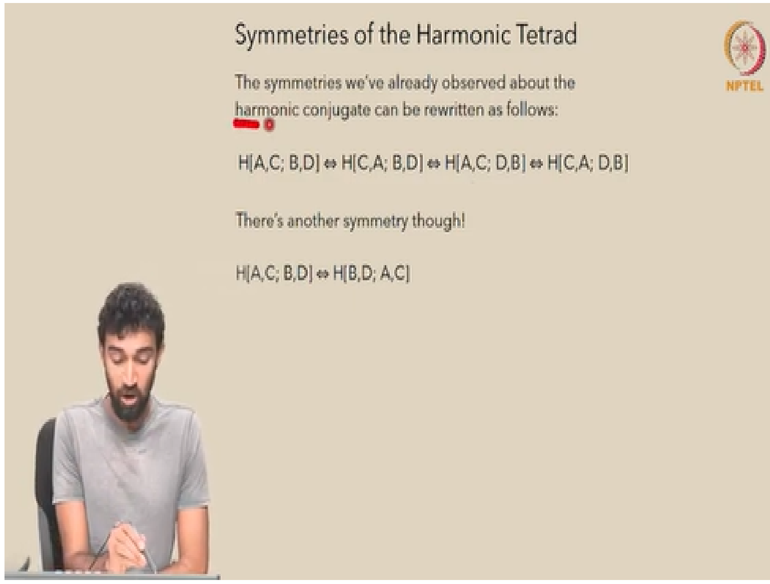

So, let us look at the symmetries of the harmonic tetrad rather than just the conjugate. So, using the symmetries that we have already observed about the harmonic conjugate can be translated into symmetries of the harmonic tetrad in this way. We have already seen that H[A,C:B,D] if and only if H[C,A:B,D] In other words, we have already seen that A B C D where A and C are the vanishing points of the sides of the quadrilateral and B and D are the vanishing points of the diagonals of the quadrilateral.

We have seen that that is true if and only if we can basically switch A and C. So, this is also a harmonic tetrad. Similarly, we can switch B and D that is what we saw in our second symmetry and this is still a harmonic tetrad. Finally, this is also a harmonic tetrad, we can switch A and C, we can also switch B and D.

So, basically when I write this notation, I am saying that this points A, B, C and D form are harmonic tetrad in which the things on the left are the vanishing points of the sides of the quadrilateral, the things on the right are the vanishing points of the diagonals of the quadrilateral. That is really what this notation means. And you can freely permute the things on either side. But that is what we have shown so far with our previous two symmetries.

But there is another symmetry which is a little bit more interesting which is that we can also permute the set of sides with the set of diagonals, we can switch them. So, here A and C are the vanishing points of the sides, B and D are the vanishing points of the diagonals. Over here B and D are the vanishing points of the sides, and A and C are the vanishing points of the diagonals. So, we have switched these. So, this one is more interesting and also a little bit harder to see.

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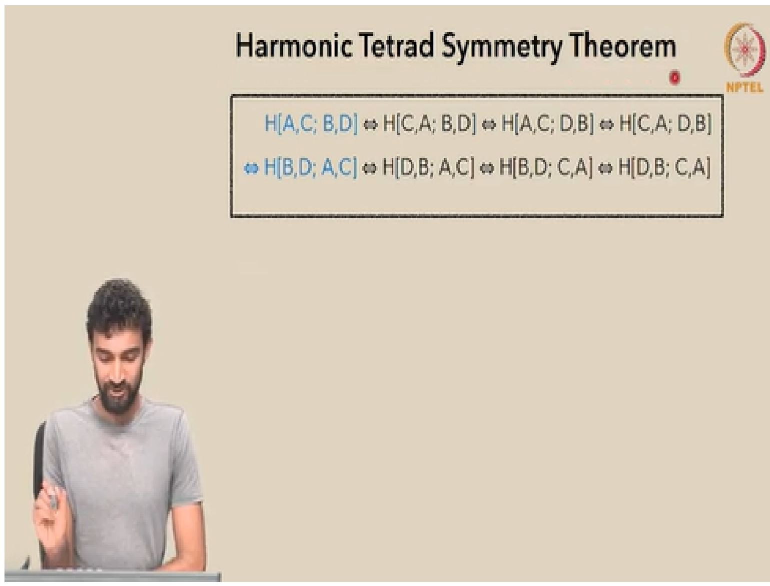

So, let us write this as a theorem the harmonic tetrad symmetry theorem. So, it is really coming down to showing this because all of these equivalences we have already seen in our previous two symmetries. So, we need to show this.

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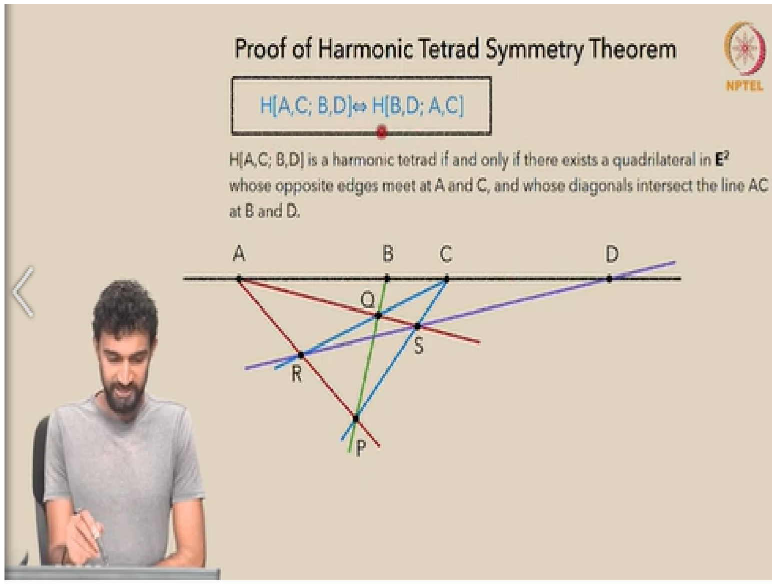

So, this is a harmonic tetrad if and only if this is a harmonic tetrad. So, again H[A,C:B,D] is a harmonic tetrad if and only if there exists a quadrilateral in E^2 such that the opposite edges meet at A and C. In other words, this edge and this edge are opposite edges which meet at C, this edge and this edge are opposite edges which meet at A. So, the pairs of opposite edges meet at A and C respectively and whose diagonals intersect the line AC at the points B and D.

So, that is a way of characterizing our harmonic tetrad. So, let us construct a quadrilateral in which the pairs of opposite edges meet at B and D respectively and whose diagonals intersect that line B D at A and C. So, I am just saying the exact same thing. Since this is a characterization of a harmonic tetrad let us prove that this is a harmonic tetrad by constructing the appropriate quadrilateral whose edge vanishing points are B and D and whose diagonal vanishing points are A and C.

So, to do that we can actually do it as a straight edge construction. So, let us try and construct, this is something you can actually do with a straightedge. Let us try and construct this associated quadrilateral. So, the first step is to draw lines from A and C to this centre point here, the centre point of the tile. Then that actually gives us four different points, midpoints of the sides of this tile.

Connect these four points perfectly to pre existing vanishing points. In particular, these two connect to D and these two connect to B.

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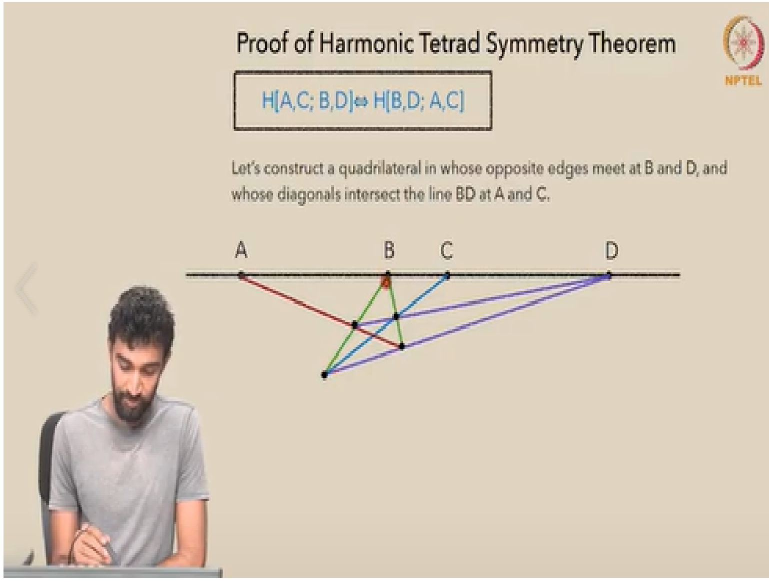

So, we have done it. We have created a new quadrilateral whose side vanishing points are B and D and whose diagonal vanishing points are A and C. So, I will just show you the construction again. We use that centre point to get these midpoints, connected up the midpoints and erased everything else and that is all we had to do. So, why does this work? So, here you can imagine, this is P and this is Q now. And it might be easier to understand why this worked by looking at a bird's eye view again.

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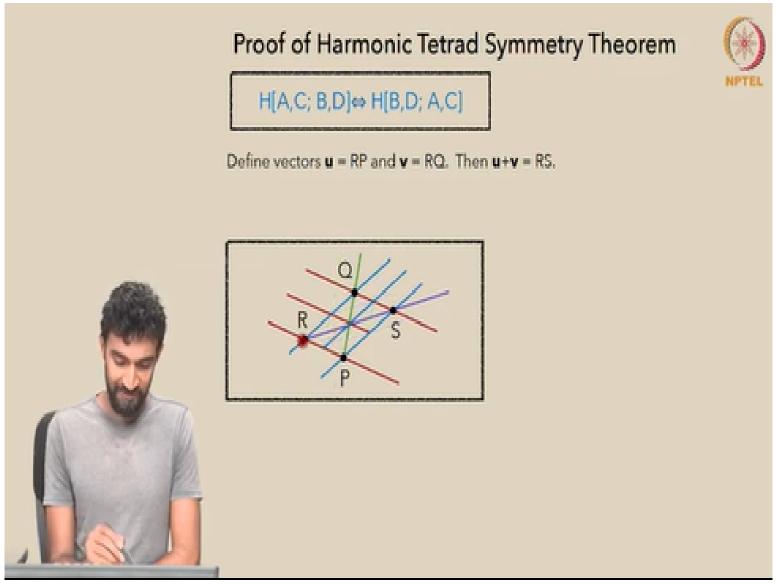

So, here is the bird's eye view of a parallelogram associated with the original harmonic tetrad, H[A,C:B,D]. Here is its parallelogram and let us define vectors u equal to RP and v equal to RQ. So, u plus v is then equal to RS. Now like we did earlier, I am going to draw lines through the centre point that are parallel to the edge vectors. So, that splits my parallelogram into four mini parallelograms and it also bisects each of the sides.

Now I can connect up those new midpoints and get my new parallelogram. So, what am I doing? Well, joining the midpoints of a parallelogram yields a new parallelogram whose diagonals are now u and v. u and v were my side vectors. u was RP, this red one here, v was this blue RQ.

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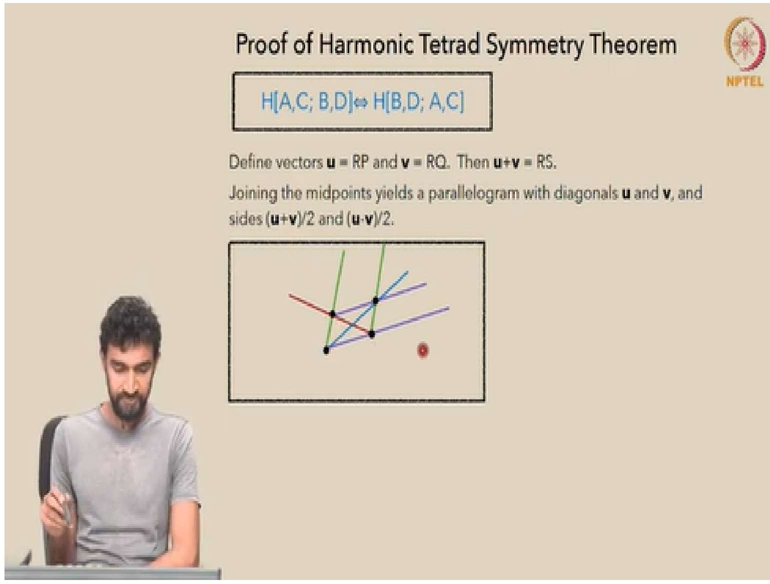

But in the new one those become diagonals and meanwhile, the sides of this new mini parallelogram are (u+v)/2 and (u-v)/2. You can verify that. So, clearly all of the parallel relations that we want to hold, I mean u and v are just the same as u and v. Clearly (u+v)/2 is parallel to u+v. (u-v)/2 is parallel to u-v. So. Now the sides are parallel to what were earlier the diagonals.

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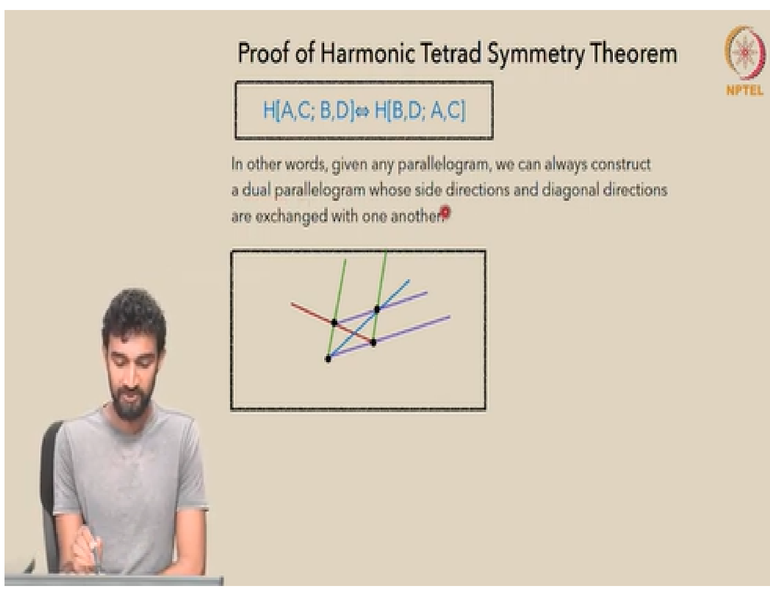

So, in other words given any parallelogram, we can always construct a dual parallelogram whose side directions and diagonal directions are interchanged with each with one another.

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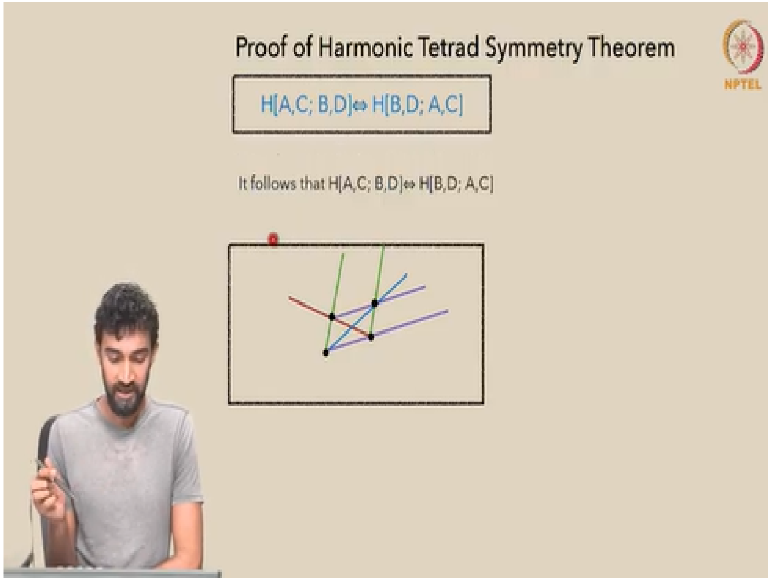

And it follows that we get this symmetry of harmonic tetrads.

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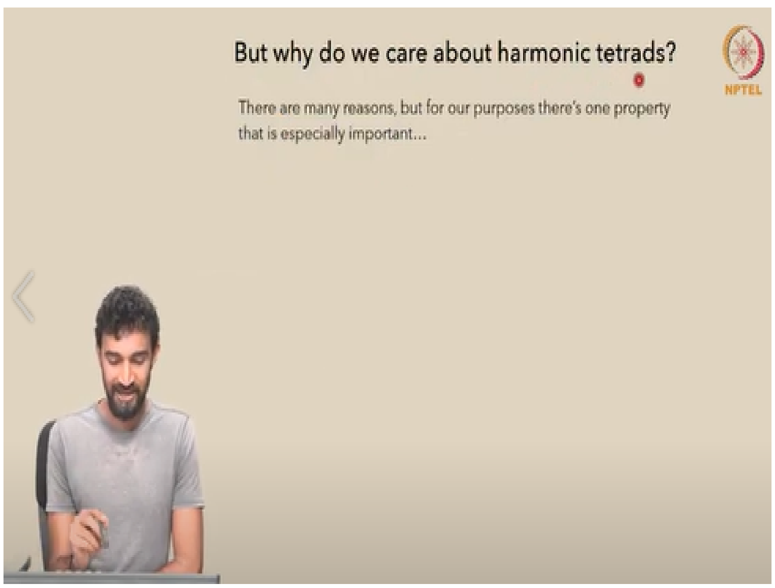

So, we still have not really seen why we care about harmonic tetrads? There are many reasons too. But for our purposes, there is one property that is going to be especially important.

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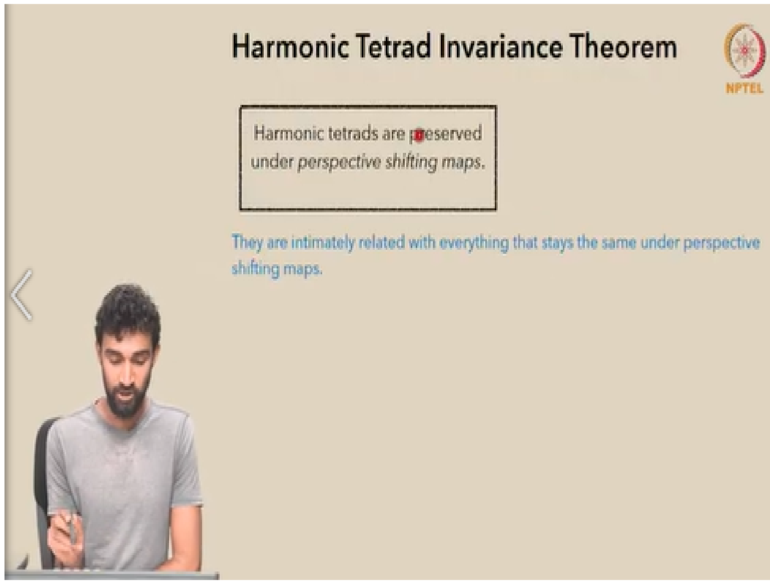

Which is that harmonic tetrads are preserved under perspective shifting maps. So, they are going to be key to understanding what stays the same when we shift from one perspective to another. Which is one of the big questions that we have been trying to answer.

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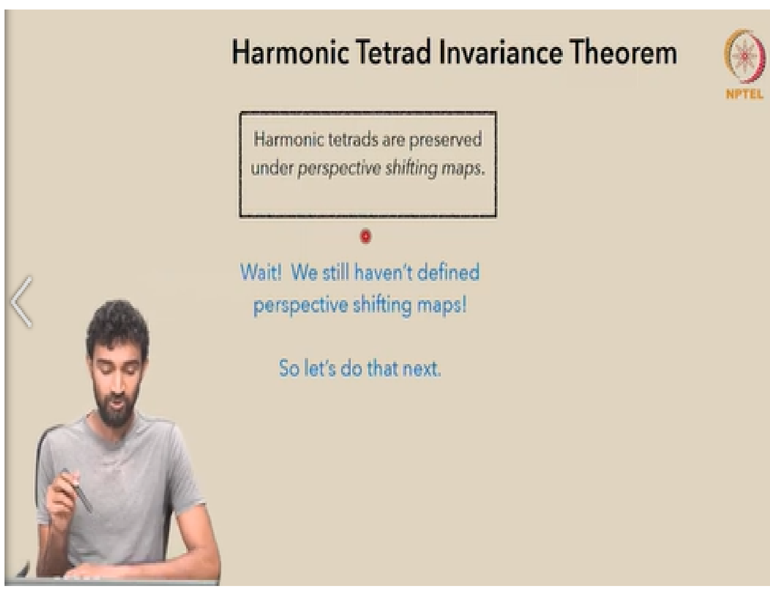

So, the problem is we still have not defined what a perspective shifting map is precisely. So, before we can show that harmonic tetrads are preserved under changes in perspective, we have to precisely define what we mean by a change or a shift in perspective. So, that is our next task and that is what we are going to do in lecture four. So, stay tuned and see you then.