Our Mathematical Senses
The Geometry Vision
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Lecture – 04
Understanding Points at Infinity

Let us start with lecture 2 in which we will try and get a handle on these points at infinity.

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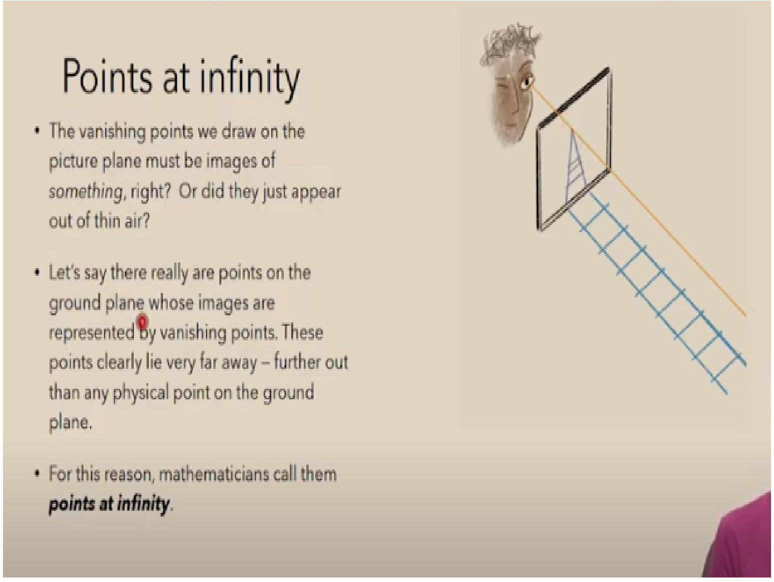

So if you remember from the previous lecture, a point at infinity or a vanishing point, is this point that appears on the picture plane when we follow a line in space further and further out as it extends away from the picture plane. This vanishing point that we end up drawing here, for example, if we are drawing railway tracks, we will end up drawing a vanishing point right there.

Well, that vanishing point must be the image of something right? If not, did it just appear out of thin air? All of these other points that we have drawn on our picture plane are the images of some point out there in space. So what is the deal with this vanishing point? Is it also somehow the image of something? And for the sake of discussion, let us just say that these vanishing points really are the image of something.

Let us say that they really are points on the ground plane, whose images are represented by these vanishing points in the picture plane. Now if that is the case, these points in the ground plane are clearly really far away, further out than any physical point in the plane, because every physical point in the plane, for example, along the side rail is already represented in this line segment here on the picture plane.

So if this is also somehow representing a point, it must be even further out than any physical point in the plane, in the ground plane. And that is the reason why mathematicians call these points points at infinity. So just maybe to be a little more precise, we refer to the point at infinity as the actual point that is on the ground plane, but further out than any actual physical point.

Whereas this image of it will refer to as its vanishing point, like we did in the previous lecture.

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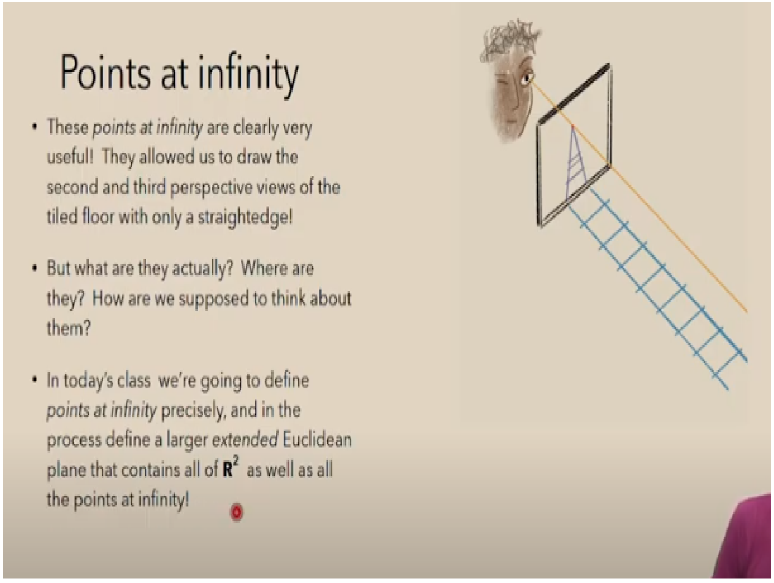

Now these points at infinity, they are clearly very useful, because we have already used them in the previous lecture, to draw the second and third perspective views of the tiled floor, using only a straight edge. So we have already seen their power there.

And we have also seen in a little, a quick tour of art history that we had at the beginning of today's lecture. We have seen what kind of variety of ways of representing our visual sphere can be realized using points at infinity. So what are points at infinity actually? What are these points at infinity? Where are they? And how are we supposed to think about them?

So in today's class, we are going to define points at infinity precisely. And in the process, we will define a larger extended Euclidean plane that contains all of the Euclidean plane R^2, as well as all the points at infinity.

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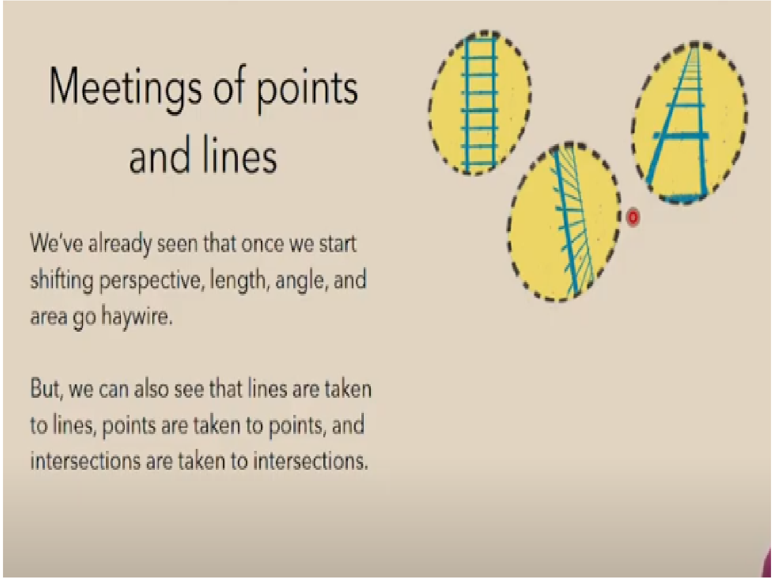

And we have already seen that once we start shifting perspective, length, angle, and area all go a bit haywire. But we can also see that lines are taken to lines, points are taken to points and intersections are taken to intersections. So something is staying the same when we shift from this perspective, to this perspective or to this perspective. At least, lines are going to lines, points are going to points and intersections are going to intersections.

So let us start with something we can hold on to. And let us work out from there in order to define points at infinity.

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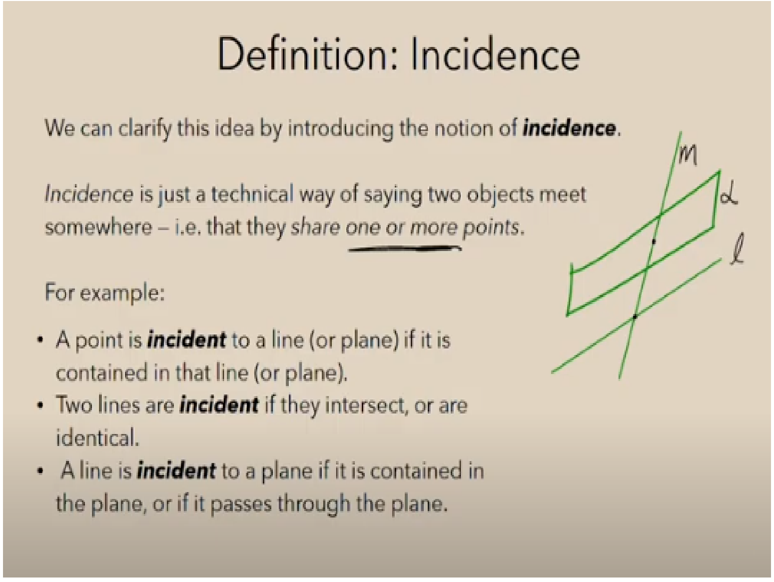

So I want to make an initial definition which is going to be very useful. This is the notion of incidence. So incidence is just a technical way of saying that two objects touch or meet each other somewhere. More precisely, it means that they share one or more points. For example, a point is incident to a line or plane, if it is contained in that line or plane.

For the case of a point, a point can only meet something if it is contained in it. On the other hand, if let us say we take two lines. Two lines are going to be incident if they intersect each other. So if this is one line l, and this is another line m, then l and m are incident, if they share one or more points, in other words if they intersect. In the case of two lines, they cannot really share more than one point without being identical.

So these two lines are incident if they intersect, or if they are identical to each other. Finally, a line is incident to a plane, if it is contained in that plane, or if it passes through that plane. Again, this is just saying that it shares one or more points with the plane. And the only way for a line and a plane to share more than one point is that the line contained in the plane.

So for example, this is a plane, let us say α. Then the line m, passes through α right there. Whereas the line l does not pass through α, let us say. So m is incident to α and l is not incident to α in this drawing that I have drawn. So that is the notion of incidents, just means that two things touch or meet, or basically that they share a point.

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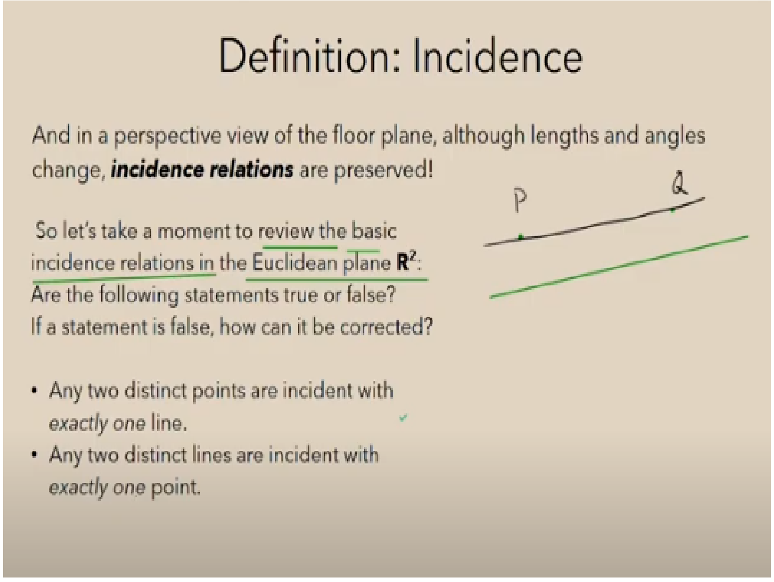

So in a perspective view of the floor plane, although lengths and angles change, incidence relations are preserved. We have already used this fact, many times for example in the introduction video, when we use that diagonal line to reveal the new corner points, which helped us draw successive horizontal lines.

So we use the fact that even if we are viewing this tiled floor from this perspective, the diagonal line, since it is meeting all of those horizontal lines, it will keep meeting those from any different perspective. Already we have seen that this very basic notion is quite useful. This basic notion that incidence relations are preserved. So let us take a moment now to review the basic incidence relations in the Euclidean plane R^2.

So let us look at the following statements and can we decide if the statements are true or false? And if they are false how can we fix them? So the first statement is that any two distinct points are incident with exactly one line. So that is saying that any two distinct points, let us say P and Q are incident with exactly one line. And by the way, remember these are points in the Euclidean plane R^2.

So given any two distinct points in R^2, is it true that they are incident with exactly one line? Yes. We know that well from Euclidean geometry and maybe even in other types of geometry, you have studied that two points determine a line in the plane. You may have studied other types of geometry, in which that is not true.

But in Euclidean geometry, in the study of R^2, that is very much the case. So there is exactly one line running between P and Q. So that is a check. What about the second statement? Any two distinct lines are incident with exactly one point. Well, let us take this as our second line. We already have one line here. Is it the case that any two distinct lines will be incident with exactly one point?

Meaning, will any two distinct lines share exactly one point? In other words, will any two distinct lines intersect each other? Well, no. In the Euclidean plane R^2, there is an obvious counter example to the statement. Parallel lines. If two lines are parallel, they do not share any points in R^2. So no. This second statement is not true, it is definitely not the case that they share exactly one point all the time.

We can fix it though, by simply changing it to say at most one point. Any two distinct lines are incident with at most one point.

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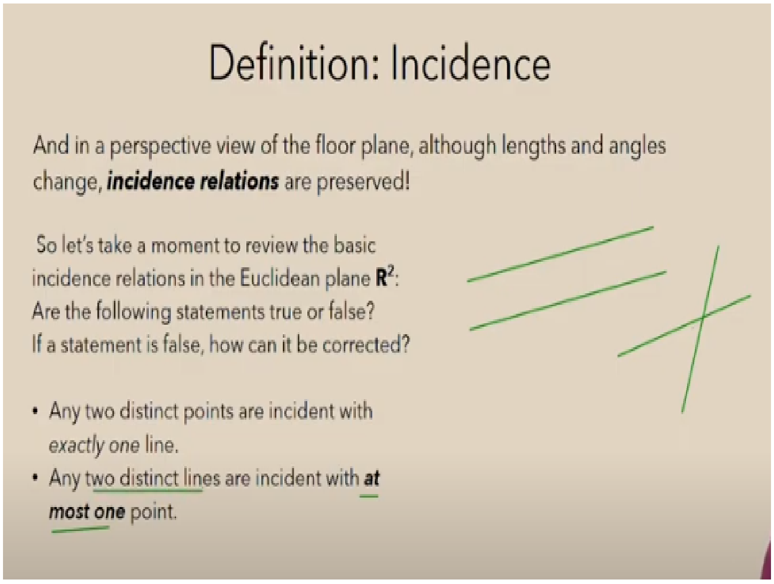

So you could have two lines like this, which are incident at zero points. Or you could have two lines like this, which are incident at one point. But you can never have two lines that are incident at two distinct points unless those lines are identical. But we have said two distinct lines here. So we have fixed the statement now by changing it to say at most one point. Okay.

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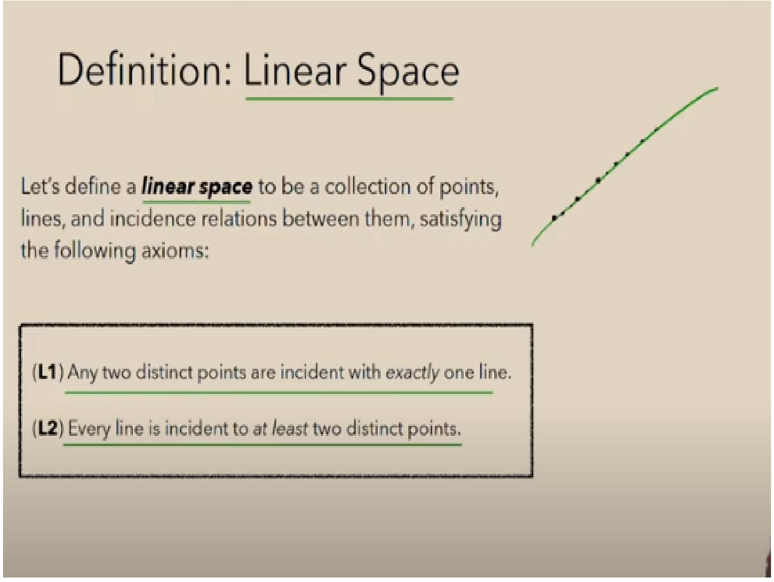

So there is another definition that I want to introduce now, which is something called a linear space. So let us define a linear space to be a collection of points, lines, and incidence relations between those points and lines, which satisfy the following axioms. So again, linear space is a bunch of lines, a bunch of points, and a bunch of incidence relations between those.

So the two axioms they have to satisfy are that any two distinct points must be incident with exactly one line. In other words, any two distinct points must determine a unique line. So we have already observed that this is the case for the Euclidean plane.

And in general for linear space, this has to be true. The second axiom is actually even simpler to satisfy. The second axiom simply says that every line is incident to at least two points. That feels almost trivially satisfied. But at least in R^2 every line is incident to infinitely many points. Given a line, there are literally uncountably infinitely many points that are incident to that line in R^2.

So this second axiom, it feels pretty silly if you come at it, from thinking about R^2 and thinking about these continuous spaces. But the reason it is there is because a linear space can also be a discrete structure. You can have a linear space with finitely many points. And finitely many lines.

And then we want to make sure that there is no singleton lines, no stupid lines, which only contain a single point, because that would be pretty meaningless. So a line should connect at least two points in our linear space. That is what that is saying.

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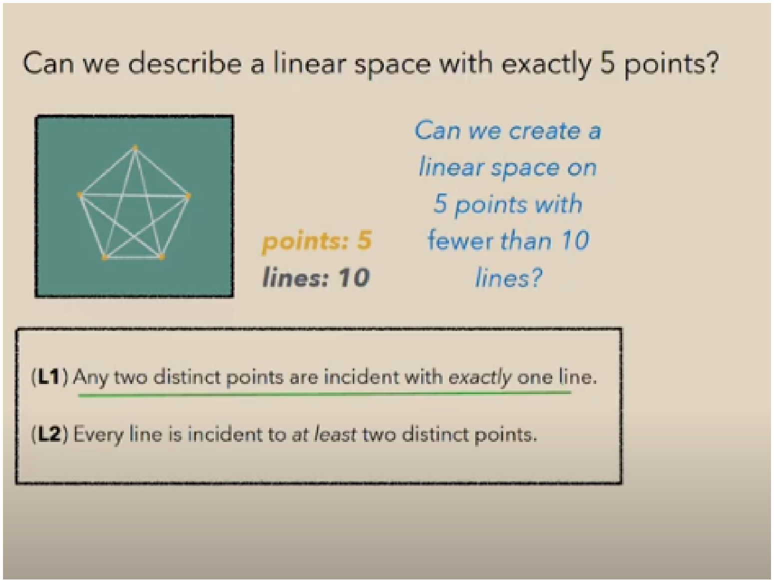

So can we look at some very simple example of linear space, a discrete example of a linear space with exactly 5 points? Yes, we can actually do that. We can just draw five points, say in a pentagon, and then connect every single pair of those points, and call that out and have a line for every pair of points. So in this case, we will have 5 points.

And we will have 10 lines, because there are 10 distinct pairs of points among these 5 points. And every one of those we connect with a line. Another way of saying that is 5 choose 2 is equal to 10. So this is an example of a linear space with 5 points and 10 lines. And does it satisfy the two axioms?

Well, clearly, any two distinct points are incident with exactly one line because we literally put in one line for every pair of points. And secondly, every line is incident to at least two distinct points. We do not have any singleton lines, we do not have any lines that are not serving any purpose. So both of these are satisfied. So this is a legitimate linear space, 5 points and 10 lines. So here is a question.

Can we create a linear space on 5 points with fewer than 10 lines? Clearly we cannot put more than 10 lines here because in that case, we would have a problem with our first L1 axiom, any two distinct points are incident with exactly one line. Already, every pair of points has a line running between them. So we cannot add any more lines.

But can we have fewer lines? Can some of these lines connect multiple points? That is another way of saying it. And yes of course, all we need to have three or more collinear points. And then we can reduce our number of lines.

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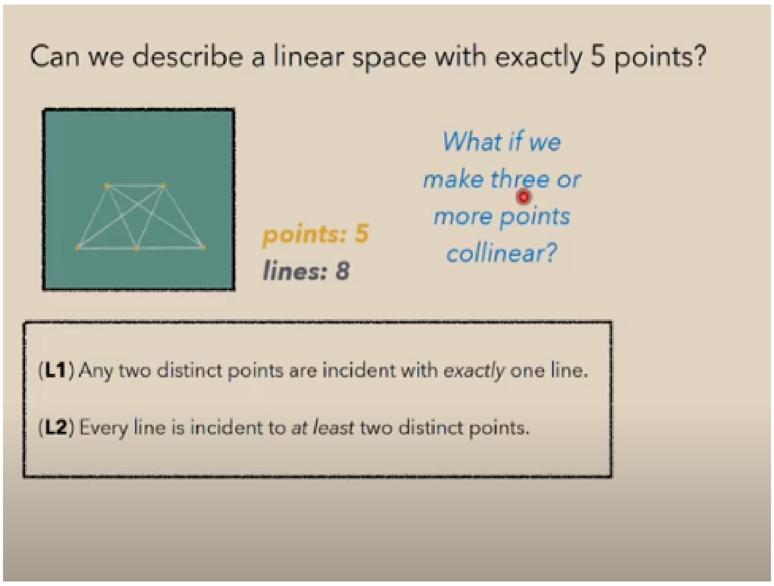

So what if we make three or more points collinear like this? Here, we have these three points collinear. And now how many lines are there? Well, there is one line running between these three collinear points now. There is another line between these other two points. So we have 8 lines in this linear space. And does it satisfy the axioms? Well, any two distinct points are incident with exactly one line.

We can see that from the picture we have drawn. Any two distinct points have exactly one line running between them. The second axiom is that every line is incident to at least two distinct points. And yes, that is again, true, no line is wasted. No line is just containing only one point. So are there any more linear spaces besides these? Well, what if we made more than three points collinear?

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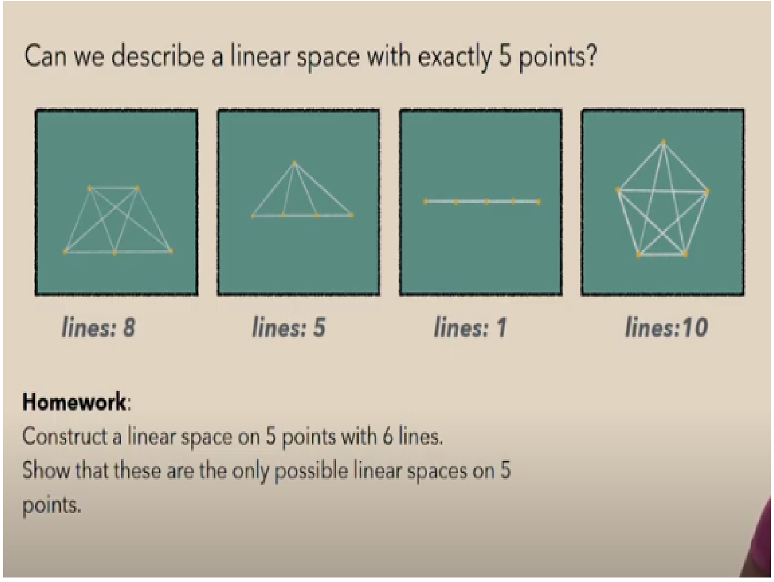

For example, we can make 4 points collinear, like in this case here. And in that case, we will have exactly 5 lines. Or we could just put all 5 points in one line as a collinear set. And then we have just one line. So these are four different examples of linear spaces with exactly 5 points. So as a homework exercise, can you try and construct a linear space on 5 points that has 6 lines.

So this does not exhaust the number of situations that we can create. There is actually one more, there is another linear space on 5 points with 6 lines. But it turns out that that is the last one. So as the second part of this homework, can you show that these are the only possible linear spaces that one can create on a set of 5 points. So that is a homework, and I will let you think about that.

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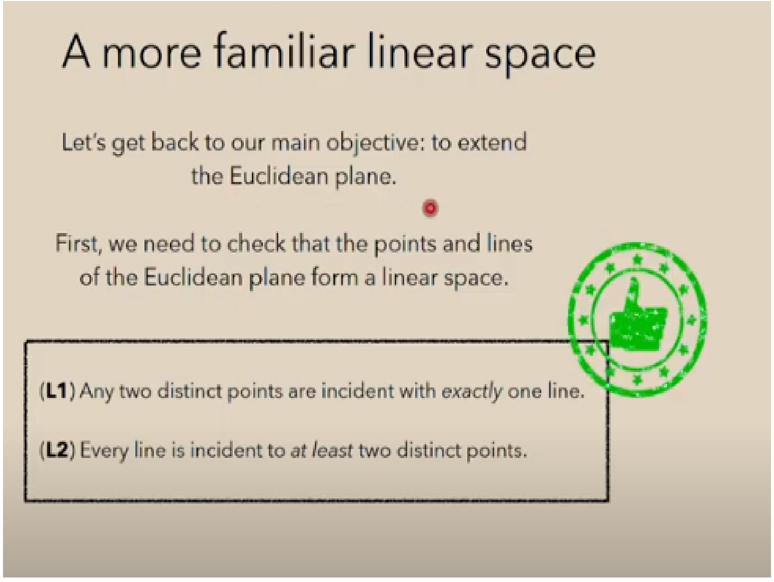

This discussion about discrete linear spaces, that is just to give some practice with the idea of a linear space. Now I want to return to the Euclidean plane, which is not at all discrete, which is uncountably infinite in terms of the number of points it contains. And I want to return to this Euclidean plane.

And I want to get back to the objective I described earlier, which is to extend the Euclidean plane so that it contains these points at infinity, so that we can make them mathematically precise. So to do that, the first thing is to check that the points and lines of the Euclidean plane actually form a linear space. So remember, I am just putting the axioms up here again, so you can see them.

And we have to check that any two distinct points are incident with exactly one line. Well yes, clearly that is the case, we know that. And two, every line is incident to at least two distinct points. And again, we have already said that that is true. Every line has uncountably many points in it. So it clearly has at least two distinct points. So since that is all that is needed to be a linear space, the Euclidean plane passes the test. It is very much a linear space.

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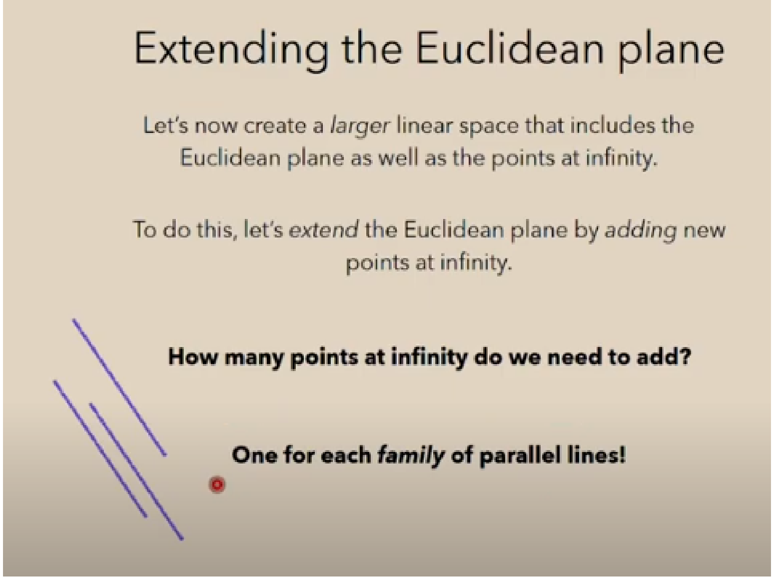

Okay, so it is linear space. Now we have to extend it, but we want to extend it carefully in order to maintain that property of being a linear space. We do not want to destroy that, because a linear space captures some of the most basic properties that we would want in a configuration of points and lines. So for losing that, everything is going to go haywire.

So when we are extending the Euclidean plane, we have to make sure we preserve the property of being a linear space. So let us create this larger linear space that includes the Euclidean plane, as well as all the points at infinity. We want to throw in all the points at infinity. And how are we going to add these points at infinity? Well, first of all, how many points in infinity do we need to add?

The answer is one for each family of parallel lines. We have seen in many examples now that distinct families of parallel lines converge to distinct vanishing points. Within a given family, all the lines converge to one shared vanishing point. But different families of parallel lines converge to different vanishing points. So we are going to need one vanishing point for every single family of parallel lines.

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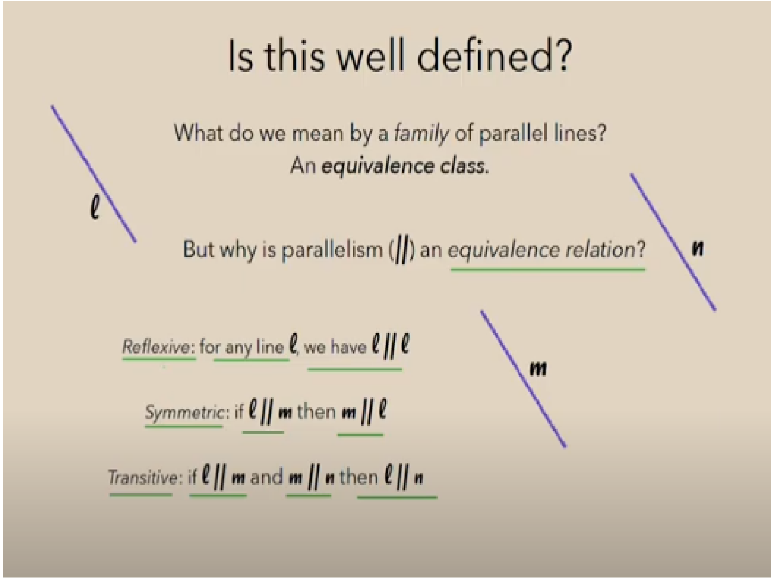

Okay, so is this well-defined? What we are saying about throwing in a vanishing point or a point at infinity for every family of parallel lines? In particular, what do we mean by a family of parallel lines? Well, what we mean is an equivalence class. If you are not fully familiar with an equivalence class or if you do not fully remember what it is, we are going to review it in just a second.

So even if you have seen it, why is parallelism an equivalence relation? Why is the property of two lines being parallel? Why is that an equivalence relation between lines? So the answer is that it satisfies the three necessary conditions for an equivalence relation. The first property of parallelism is reflexive, that means for any line l, we have that l is parallel to itself. By definition, any line is parallel to itself.

Secondly, it is symmetric. If a line l is parallel to a line m, then the line m must be parallel to the line l. So it is symmetric. Finally, parallelism is transitive. If a line l is parallel to a line m, and a line m is parallel to a line n, which we can see right here l is parallel to m, m is parallel to n, well then l must be parallel to n.

So it is transitive in that sense. So in general, an equivalence relation is any relation that satisfies these three properties. So parallelism is therefore an equivalence relation.

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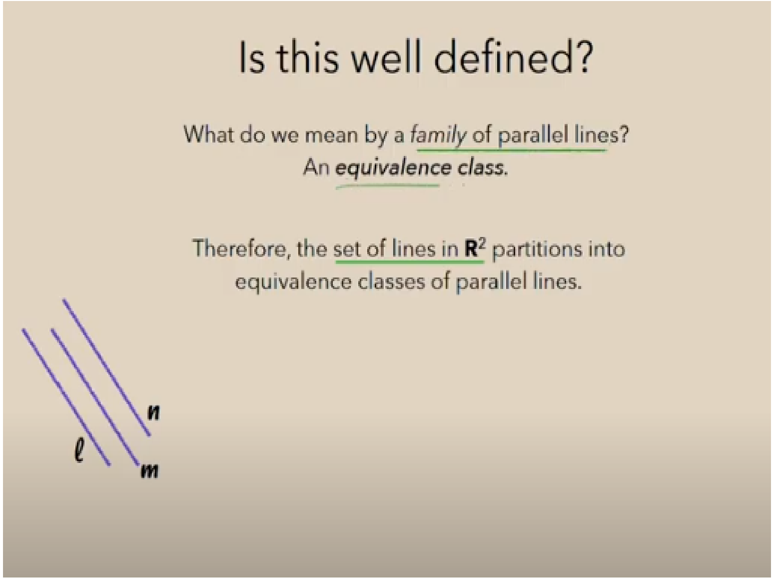

And what that tells us is that the set of lines in R^2 is partitioned into equivalence classes of parallel lines, disjoint equivalence classes of parallel lines. So when we say a family of parallel lines, we are referring to one of these equivalence classes.

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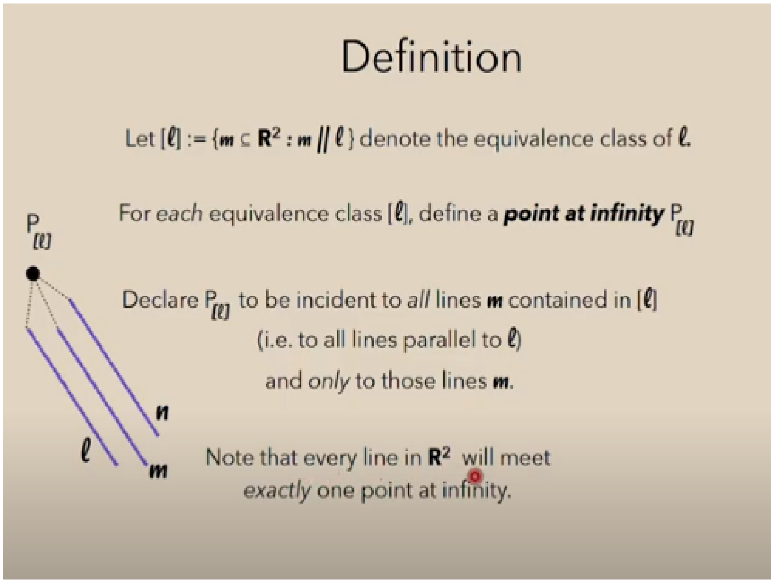

Okay, so that is what we mean by a family of parallel lines. And we have seen that the set of all lines naturally partitions into these disjoint families of parallel lines. So let [l] denote the set of all lines m in R^2, which are parallel to l. In other words, let this denote the equivalence class of l. So for each equivalence class, [l], let us define a point at infinity $P_{[l]}$.

So we know that these equivalence classes are well defined. So for each one of them, we can just create a point at infinity, which will denote $P_{[l]}$. And we are adding it to our linear space. So in addition to adding the point, we have to say, which points, which lines, our point is incident to. So let us add the following incident's relations.

Let us declare that $P_{[l]}$ is incident to all lines m that are contained in [l]. In other words, all lines that are parallel to l, and only to those lines. So $P_{[l]}$, this new point at infinity here, is incident to all of the lines in this family of lines which are parallel to l. And it is only incident to that family. It is not incident to any line, which is not parallel to l. So that is our point at infinity $P_{[l]}$. We have defined it.

Note that every line in R^2 will meet exactly one point at infinity. Why is that? Well, every line in R^2 lies in some family of parallel lines. It lies in some equivalence class. So an equivalence class corresponds to one particular point at infinity, which we have added. So any line in R^2 will meet that point at infinity, which represents its parallelism class, its equivalence class.

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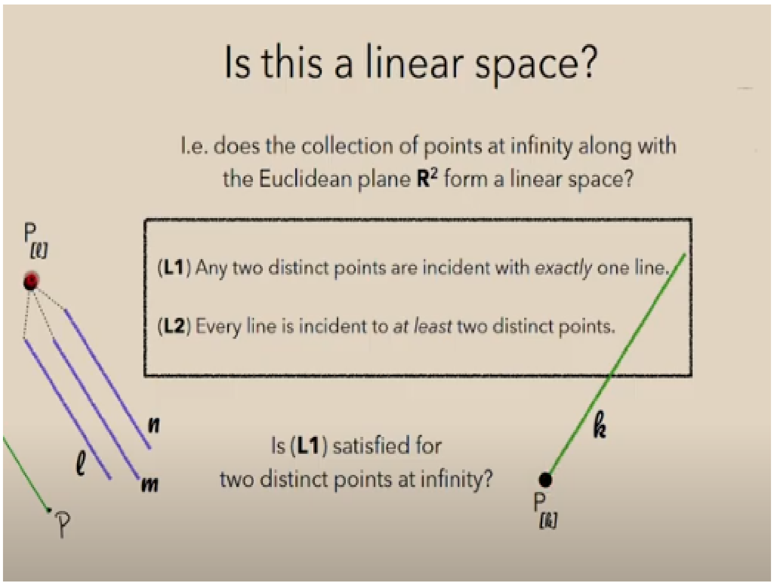

So we have taken R^2, we have added a bunch of points at infinity, one for each family of parallel lines. Is this a linear space? In other words, is it the case that any two distinct lines are incident with exactly one, any two distinct points determine a unique line? Is it the case that any two distinct points determine a unique line?

And L2, is it the case that every line is incident to at least two distinct points? Well, clearly every line is still incident to at least two distinct points, because we have not added any new lines. So the only lines are the old ordinary R^2 lines and they will always contain many points. So L2 we do not have to worry about. But L1, let us pause and think about. Is L1 satisfied for any two distinct points?

It is still satisfied for any two ordinary points in the plane, because given two points in the plane, say P and Q, they determine a line in the plane, an ordinary line. And there is only that one line that runs between them, because we have not added any new lines. So that is fine.

But what about an ordinary point P and a point at infinity $P_{[l]}$? Between these two points, is it the case that they determine a unique line that they are incident with? Yes. And what is that line? Well, it is simply the line running through P, which is in the same parallel family to l.

For example, let us look at this picture here. If we add any point P here, then there is a unique line running between P and $P_{[l]}$. And it is simply the line in this parallel family beginning at P. All the lines in that family will eventually be declared are kind of defined to be incident to $P_{[l]}$, including this line here that we are drawing starting at P. So that is the unique line running between P and $P_{[l]}$.

So that case is also fine. But what about the case of two distinct points at infinity? Imagine that we have $P_{[l]}$ over here and $P_{[k]}$ over here. So in that case, is L1 satisfied? Is there a unique line running between this point at infinity $P_{[l]}$ and this point at infinity $P_{[k]}$? And the answer is no, because we saw a second ago that every line in R^2 will meet exactly one point at infinity.

Every line in R^2 is in exactly one family of parallel lines, one equivalence class, so it will only meet one point at infinity. So there is no line running between or connecting $P_{[k]}$ and $P_{[l]}$.

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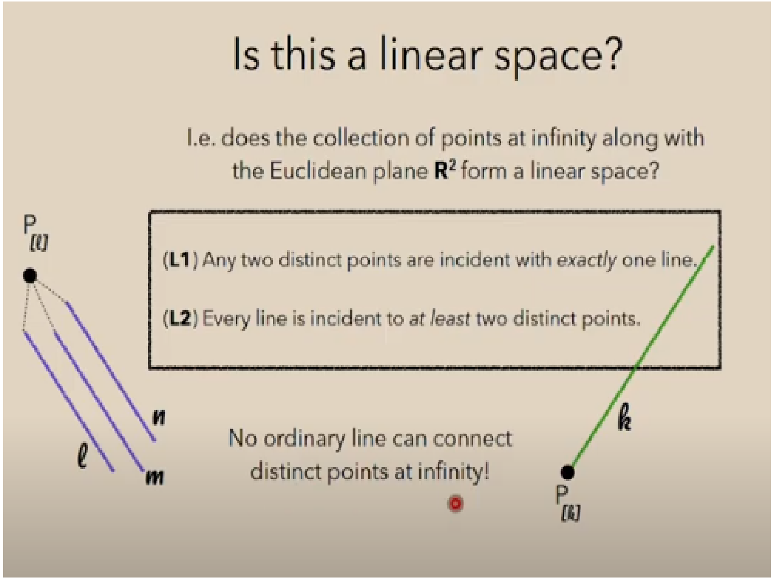

So at least no ordinary line, there is no ordinary line connecting these distinct points at infinity.

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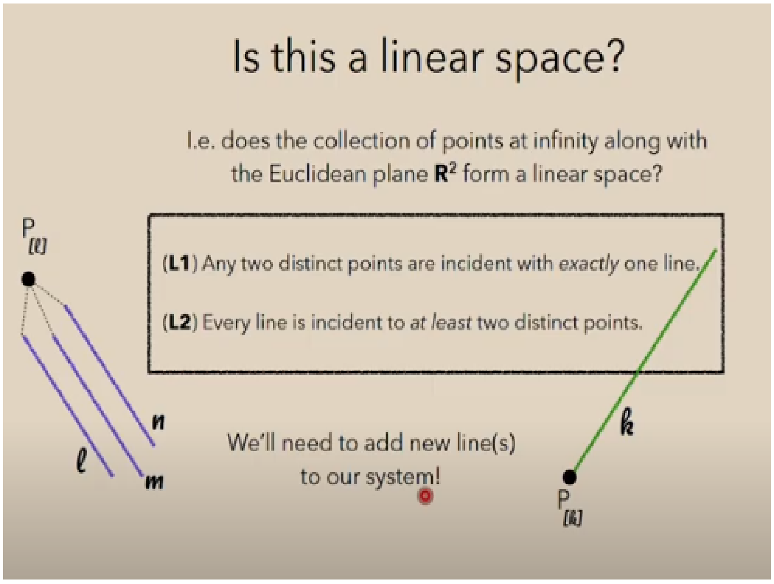

Which means we have to add new lines to our system. This is not yet a linear space, we need to do a little bit more work.

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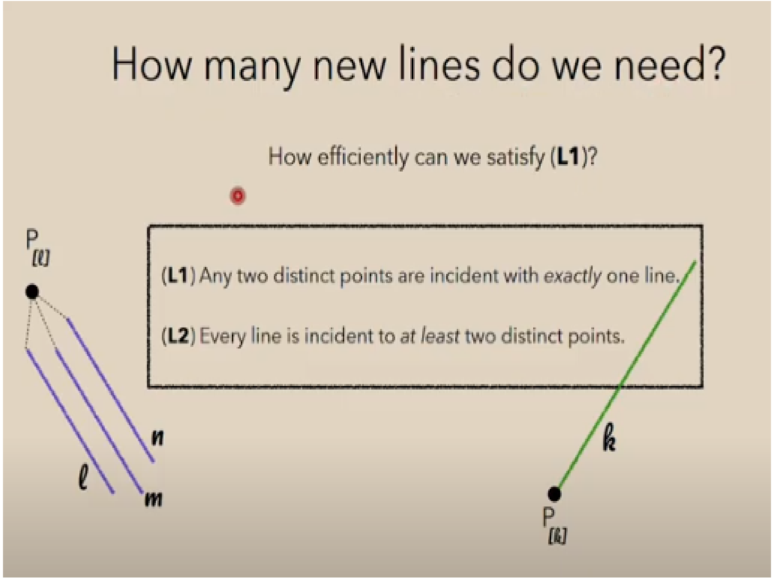

So how many more lines do we have to add in order to satisfy this first axiom?

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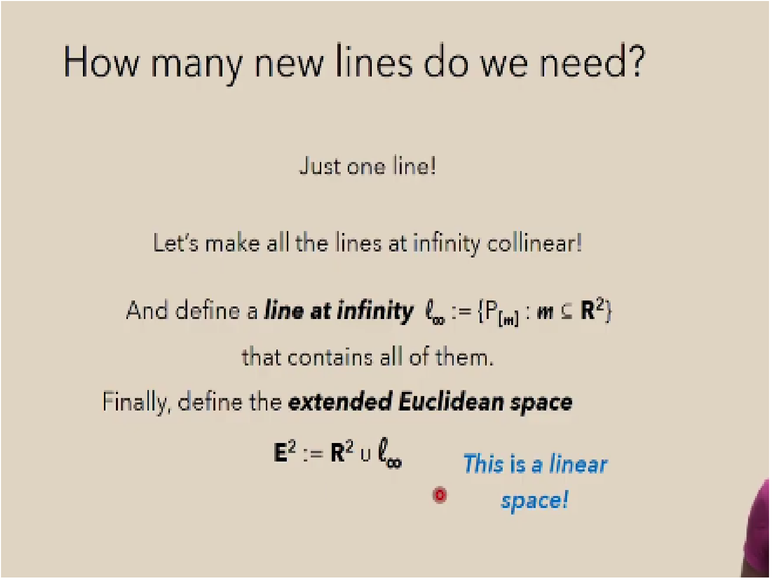

Well, just one line. It turns out we can do it quite efficiently. And the strategy is we can just make all the points at infinity collinear. We can just declare all of these points we have added, these points at infinity to be collinear to one another. In other words, we can define a new line at infinity l_∞, which just contains all the points at infinity.

If that line at infinity is added, any two points at infinity are connected by a unique line, namely the line at infinity. Which means both of these axioms are now satisfied. So we have our nice linear space, let us define it, let us pin it down and let us give it a name. We will call it the extended Euclidean plane. And we will denote it as E^2.

And it is just defined to be R^2 union the line at infinity with all of these incidence relations we have added and this is a linear space that we just discussed.