Our Mathematical Senses
The Geometry Vision
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Lecture - 16
Transformations of the Real Projective Plane

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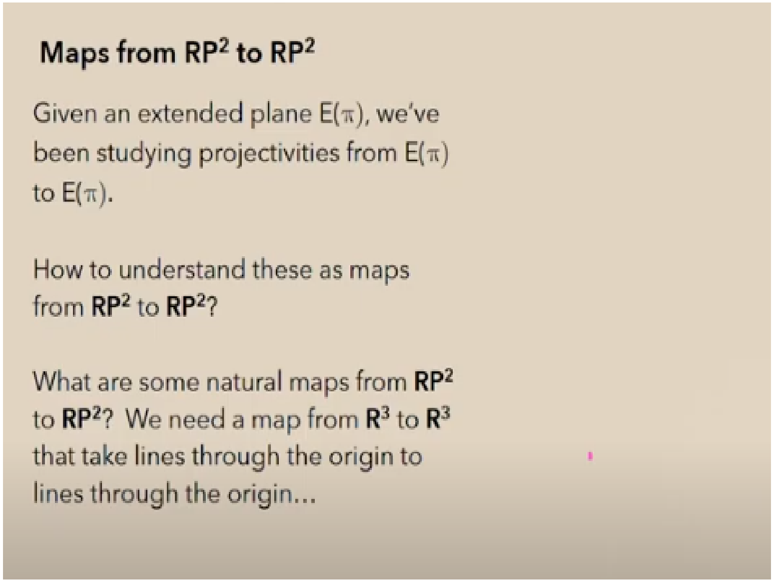

So let us talk a little bit about maps from RP^2 to RP^2. For a few weeks now, given an extended plane E(pi), we have been studying projectivities from the plane to itself, from the extended plane to itself. And now that we are kind of moving into this analytic framework, and looking at RP^2, how can we understand these maps as maps from RP^2 to RP^2?

How can we understand projectivities in this new context, with RP^2? What is a map from RP^2 to itself? What are the transformations of RP^2? Remember RP^2 is the set of lines through the origin in R^3. So if we want to have a map from RP^2 to RP^2, we need to permute these lines through the origin.

We need a map from R^3 to R^3, which takes lines through the origin to other lines through the origin. So maybe if you have taken linear algebra, you have seen something that could be a candidate. So there is a natural set of maps from R^3 to R^3, which take lines through the origin to lines through the origin, which is known as the general linear group.

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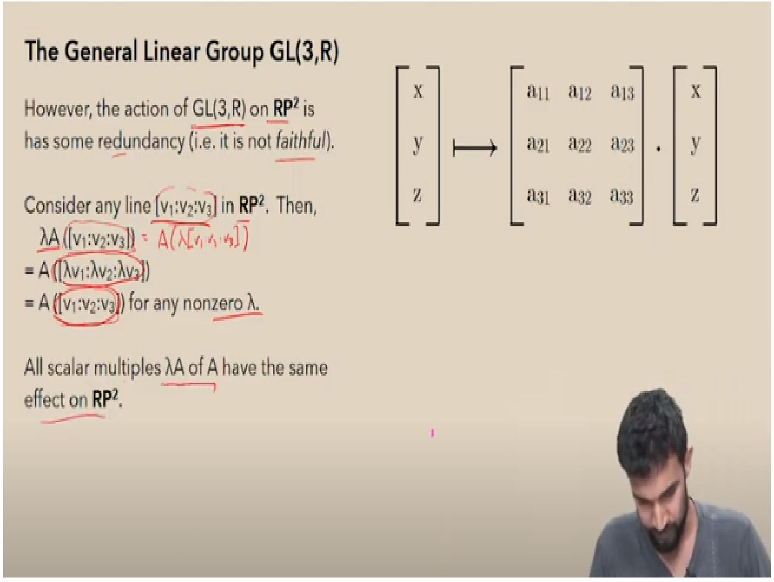

This is just the set the group of 3 by 3 matrices with nonzero determinant. And these act as maps from R^3 to R^3 via left multiplication. So for example, if A is a 3 by 3 matrix with nonzero determinant, we can see exactly how it acts on R^3 by left multiplication in this picture here, it is just through matrix multiplication. So (x,y,z) goes to the point that we get when we multiply this matrix by (x,y,z).

In particular, that is equal to a_{11}+ $a_{12}y+a_{13}z$ as the first coordinate, and so on. I am just multiplying this row by this column. I am just doing matrix multiplication to get my different coordinates. So my top coordinate is here and for my second coordinate, I would get a_{21} $x+a_{22}y+a_{23}z$ and so on.

Fine, I will just finish it. So we get $a_{31}x+a_{32}$ $y+a_{33}z$. So this is the product of this matrix, and this column vector. So this is where (x,y,z) gets sent to, via this matrix. So (x,y,z) gets sent to the point given by these expressions here. So this is a map from R^3 to R^3. This is a point in R^3. And what else can we say about this action?

I am just saying that this matrix acts on R^3. It sends points of R^3 to other points of R^3. Now the special thing is that $A(\lambda x)$ is equal to $\lambda(Ax)$, if λ is a scalar. So for example, if we put λ is here, now we are looking at λ x.

And if this is A, then $A(\lambda x)$ is the same thing as taking Ax and then multiplying by λ. Basically, every entry gets multiplied by λ in our resulting expression. So maybe I will leave that as an exercise, verify that $A(\lambda x)=\lambda(Ax)$. Since that is true, this action it takes lines through the origin to other lines through the origin.

It permutes the lines through the origin, this action of left multiplication. And therefore, it induces a map on RP^2. So it induces an action of A on RP^2. So the matrix A, it gives us a map from RP^2 to RP^2. However, the action of GL(3,R) on RP^2 has a lot of redundancy. In technical terms, it is not a faithful action.

And what that means is, there are many elements of GL(3,R), many different matrices, which are going to do the exact same thing to all the elements of RP^2. They are going to induce the exact same map on RP^2. So it is a little inconvenient to deal with for that reason, because there is all this redundancy.

So what I mean by that, if we consider any line $[v_1:v_2:v_3]$ in RP^2, then the matrix λ A, if we let that act on this line, what does it do? We just claimed is that λA times this is the same as A times λ of this, which is just multiplying each of the coordinates by λ. But these are just the same thing.

These are homogeneous coordinates, these two expressions represent the same line. We might as well write it as $A([v_1:v_2:v_3])$. And this is true for any nonzero scalar λ. So the matrix λA, applied to this projective point is the same as applying the matrix A to this projective point. And this is true for any projective point.

So λA and A have the same effect. So all scalar multiples of A will have the same effect on RP^2.

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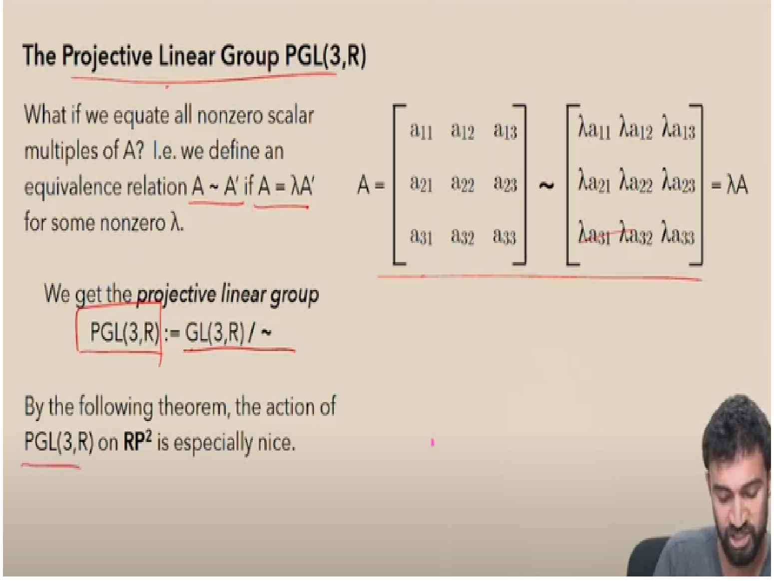

So it makes sense to talk about the projective linear group instead of the general linear group, in which we equate all the nonzero scalar multiples of A. So in other words, let us define an equivalence relation between matrices. We will see that A and λA are equivalent.

And in other words, we say that these two matrices are equivalent, because the one on the right is just λ times the one on the left. So it is a scalar multiple of the one on the left. So we will equate scalar multiples of matrices. We think about the equivalence classes of matrices under scalar multiplication. That is what this notation here means.

We equate all matrices that are related by scalar multiplication. Then we get a new group called the projective linear group PGL(3,R). And in this group, these two matrices represent the same element of the group PGL(3,R). So it is a matrix group in which any element is represented by a whole bunch of different matrices. One for a whole family of scalar multiples of a matrix.

So by the following theorem, the action of PGL(3,R) on RP^2 is an especially nice one.

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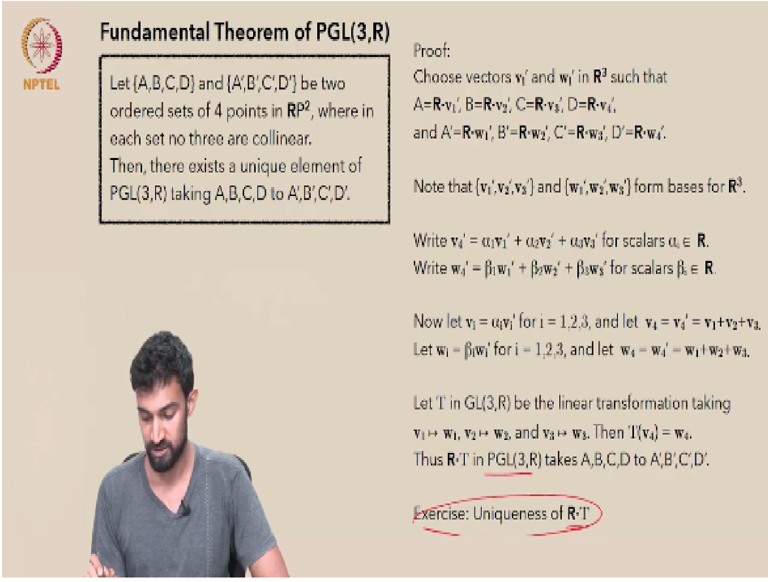

So we will call this the fundamental theorem of PGL(3,R), which is that if A, B, C and D and A', B', C', D' are two different ordered sets of four points in RP^2 and no three points are collinear in each set. Then there exists a unique element of PGL(3,R), which takes A, B, C and D to A', B', C' and D'.

So that is the fundamental theorem of PGL(3,R). There is a unique element taking any four points to any other four points as long as these four points are in general position. In each case, we do not want any three points to be collinear. So the proof of this is actually a lot simpler if we were using the machinery of linear algebra.

It is a lot simpler than a proof of a similar theorem we saw earlier, involving projectivities. How does the proof work? Well, let us choose vectors v_i' and w_i' in R^3 that represent our four points A, B, C, D and other four points A', B', C', D'. So let us choose a vector v_1' representing A. In other words A is the set of all scalar multiples of v_1' that is what I mean by the R here.

Let us choose v_2' to be a vector in R^3 representing B; v_3' represents C, v_4' represents D. Similarly, w_1' represents A', w_2' represents B', w_3' represents C', and w_4' represents D'. So we are choosing vector representatives of these eight projective points.

And let us note that $\{v_1', v_2', v_3'\}$ is a basis for R^3, they form a basis for R^3. Similarly, $\{w_1', w_2', w_3'\}$ form a basis for R^3. Why is that? Well, remember, no three points are collinear. If no three points are collinear, that means that in particular, the lines that are spanned by these three vectors are not collinear, meaning that they are not coplanar.

So they are independent. So these are three linearly independent vectors. So we can write v_4' as $\alpha_1 v_1' + \alpha_2 v_2' + \alpha_3 v_3'$ for some scalars α_1, α_2 and α_3. We can write it as a linear combination of v_1', v_2' and v_3'.

And similarly, we can write w_4' as a linear combination of w_1' w_2', w_3', where here I am using these scalars, β_1, β_2 and β_3. Now let us define new vectors, $v_1=\alpha_1 v_1'$, let us call this v_2. And let us call this v_3. Let us call this w_1. Let us call this w_2. And let us call this w_3.

So we are defining new vectors v_1, v_2, v_3, w_1, w_2, w_3, and let v_4 just be the same as v_4'. Remember that v_4' is the sum of these things. So let $v_4= v_4'$, and then v_4 is now just the sum of v_1, v_2 and v_3. Similarly, let $w_4= w_4'$. Then w_4 is just going to be the sum of w_1, w_2 and w_3. So why do we want to do that?

Well, now we can define a linear transformation, a matrix in GL(3,R). We can find a matrix and GL(3,R), which takes this v basis to this w basis. That means it takes $\{v_1,v_2,v_3\}$ to $\{w_1,w_2,w_3\}$. We can find a matrix that takes these three guys to these three guys. Let us call it T. So T sends v_1 to w_1, v_2 to w_2 and v_3 to w_3.

And where is it taking v_4? Well, it is a linear transformation. So $T(v_4)$ is equal to $T(v_1+v_2+v_3)$, which is equal to $T(v_1)+T(v_2)+T(v_3)$, since it is linear. And $T(v_1)=w_1$, $T(v_2)=w_2$, $T(v_3)=w_3$. So this is just $w_1+w_2+w_3$, which is equal to w_4. So T takes v_1, v_2, and v_3 and v_4, to w_1, w_2, w_3 and w_4. It takes the four points to the other four points.

And thus, the corresponding element of PGL(3,R), which is the equivalence class of this matrix T under scalar multiplication, that is going to take our four lines through the origin, A, B, C, and D, to these other four lines through the origin, A', B', C', D'. It is going to take these four projective points, these four elements of RP^2 to these other four elements of RP^2.

And I leave it as an exercise to prove the uniqueness of this element of PGL(3,R) and it is not that hard to show if you are okay with this proof. If you have enough linear algebra to understand this proof, I would encourage you to try and prove the uniqueness of this transformation R.T.

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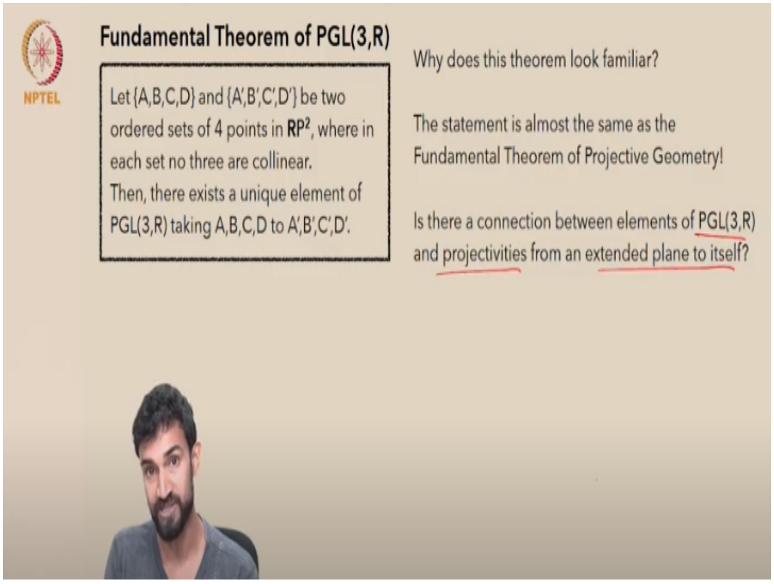

But I will leave that aside for now. Why does this theorem look familiar? The statement is almost the same as the fundamental theorem of projective geometry. So that is kind of easy, and hopefully not so hard to see. In both cases, we are saying there is a unique element of some group, which takes four points to four other points, assuming that no three of these points are collinear.

They are almost the same statement. So is there any connection between PGL(3,R) and this set of projectivities from an extended plane to itself? They are both occupying very similar roles in these two different theorems. So how are they connected to each other?

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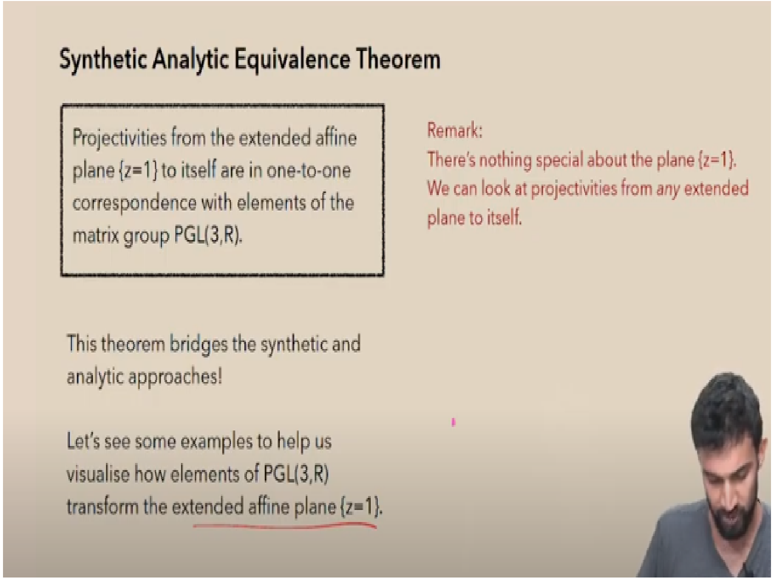

And that is the content of this last theorem, the synthetic analytic equivalence theorem is what I am calling it. And it states that projectivities from the extended affine plane z=1 to itself, so projectivities, which we defined earlier, you know is a composition of perspectivities, which go from z=0 to itself. These set of projectivities to itself are in one to one correspondence with the elements of the matrix group PGL(3,R).

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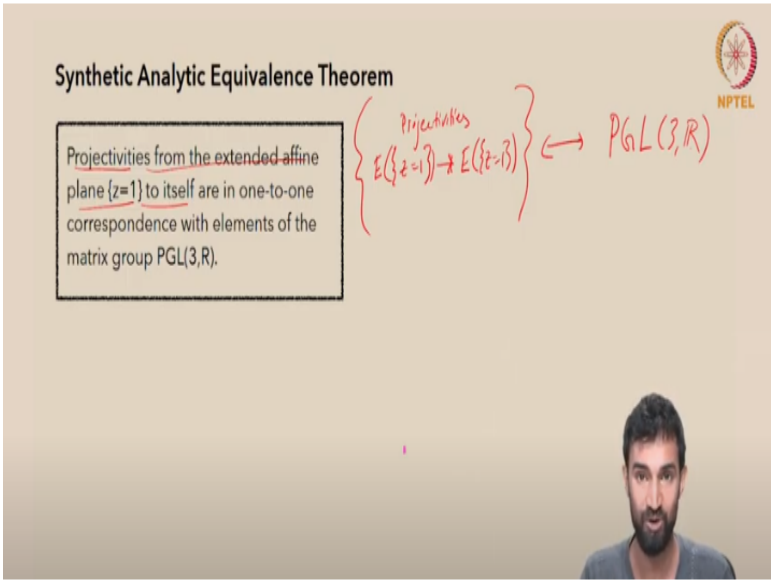

So basically projectivities mapping via the extended plane z=1 to the extended plane z=1. This collection of projectivities is in one to one correspondence with the elements of PGL(3,R). What does that mean?

It means we can actually think of projectivities as elements of PGL(3,R), which gives a lot more structure to the set of projectivities, which otherwise, it is hard to wrap our heads around, it is hard to get a handle on. Because projectivities are defined in a kind of complicated way.

A projectivity is a sequence of perspectivities of arbitrary length, involving any number of centers of perspectivity and any number of planes in space, that start at z=1 and end at z=1 in this case. But that seems like a crazy complicated set of maps. So this theorem is pretty significant, because it is saying that a crazy complicated set of maps is actually in one to one correspondence with, and this is a nice one to one correspondence with this matrix group PGL(3,R).

So this theorem, it bridges the synthetic and the analytic approaches to projective geometry. Projectivities arise from the synthetic approach of drawings and straightedge constructions and axioms. This matrix group comes out of the analytic approach of looking at RP^2, lines through the origin and maps between those.

So I just want to mention that this theorem bridges these two historically very different approaches, and shows that they are equivalent. So it is a very significant theorem in that way. And I just want to mention, there is nothing special about the plane z=1. I could replace this plane z=1 with any other extended affine plane, and the same theorem would hold.

But I am just using this because it is especially nice to visualize the connection. So we are not going to prove this theorem today. But the last thing I want to do in today's final lecture is to see some examples of this theorem. So let us see some examples to help us visualize how elements of PGL(3,R) can act as projectivities or contract rather how they can transform the extended affine plane z=1.

So how can we actually visualize a connection between these elements of PGL(3,R), and projectivities from z=1 to itself? So let us take this affine plane z=1 and let us try and visualize what the transformations we get via elements of PGL(3,R), what do they look like?

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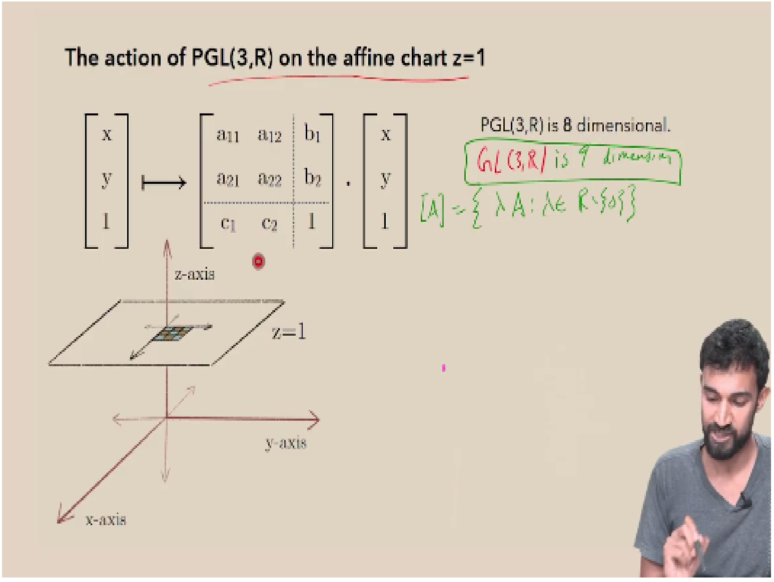

So the action of PGL(3,R) in other words, on this affine chart, z=1, that is what we want to consider. And I want to just mention first, that PGL(3,R) is an 8 dimensional space. GL(3,R) is how many dimensional? It is the three by three matrices with nonzero determinant. But that does not actually reduce the dimension.

Because the set of matrices with determinant zero is a zero dimensional subset in the full set of matrices. So we are excluding that, we are not reducing a dimension. So GL(3,R) is 9 dimensional. And PGL(3,R) is 8 dimensional, because we are equating one dimensional families of matrices, given any matrix A, we are looking at all scalar multiples of A for all nonzero real numbers.

This is an element of PGL(3,R). We can see that is the equivalence class of A. So we are losing a dimension, because we are equating full one dimensional families in GL(3,R) and making those into points of PGL(3,R). So PGL(3,R) is 8 dimensional. And we can represent elements of PGL(3,R) like this one here, by just choosing one coordinate, which will make it into 1.

Remember, we are allowed to scale our matrix without changing, without moving to a new element in PGL(3,R). So if this is an element in PGL(3,R), we can just choose a representative, which has a_1 in this entry. So that will be a convenient thing to do. So from now on, I want to just always look at representatives of PGL(3,R), which have a_1 in this lower right entry.

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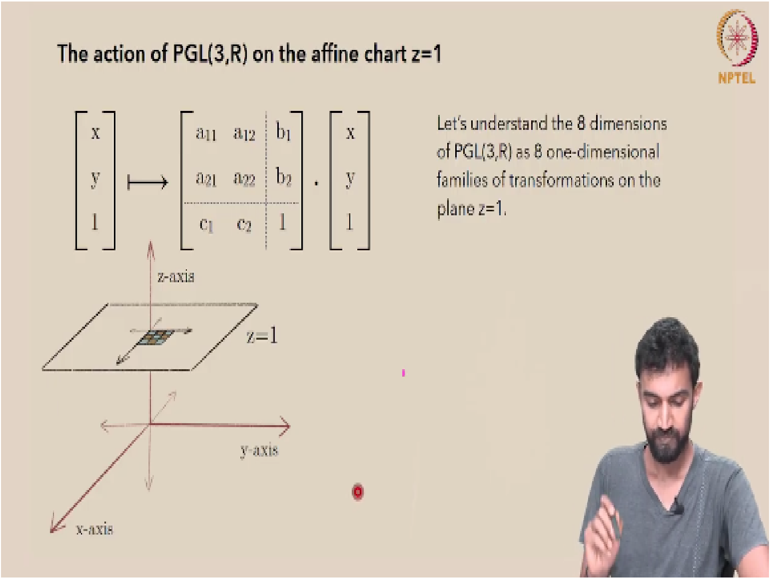

And I want to now understand the 8 dimensions of PGL(3,R) as 8 different one-dimensional families of transformations of this affine plane z=1.

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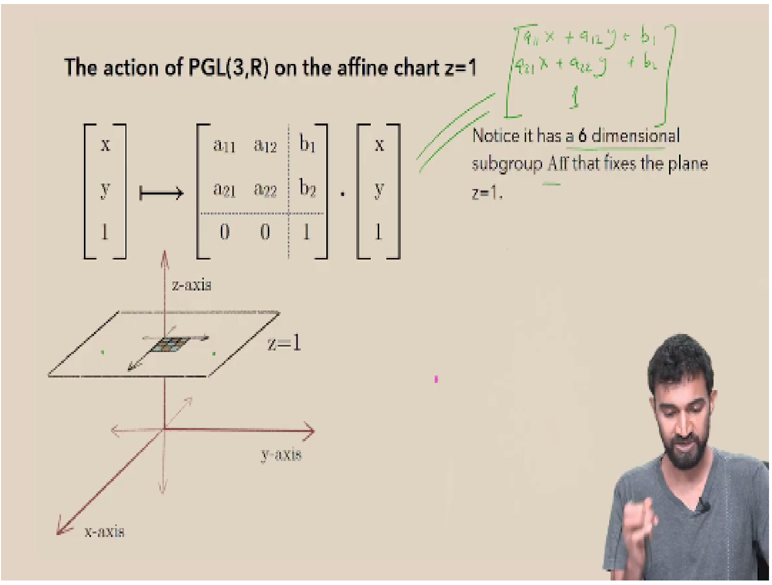

So it might be useful to notice that there is a 6 dimensional subgroup of PGL(3,R), which I have drawn right here, in which these six entries can freely move about, they can freely range over the reals. And these bottom entries are 0, 0, and 1, those are fixed. So there are 6 degrees of freedom. It is a 6 dimensional subgroup, and it is called Aff, affine. Aff is for affine.

But notice that this matrix, it fixes the plane z=1. We can see that very quickly. So here we have an element of the plane z=1. And when we multiply these two matrices, what do we get? Well, we get a_{11} x+a_{12} y+b_1×1. Here is the top entry as the first entry. Then we get a_{21} x+a_{22} y+b_2×1.

And then we get 0×x+0×y+1×1, which is just 1. So this image remains in the plane z=1. So x, y, 1, which is some point in the plane, z=1 is mapped to this point here. So this is a transformation that is fixing z=1.

It is taking points in z=1 and taking them to other points in z=1. So there is this 6 dimensional subgroup Aff, which just transforms the plane directly.

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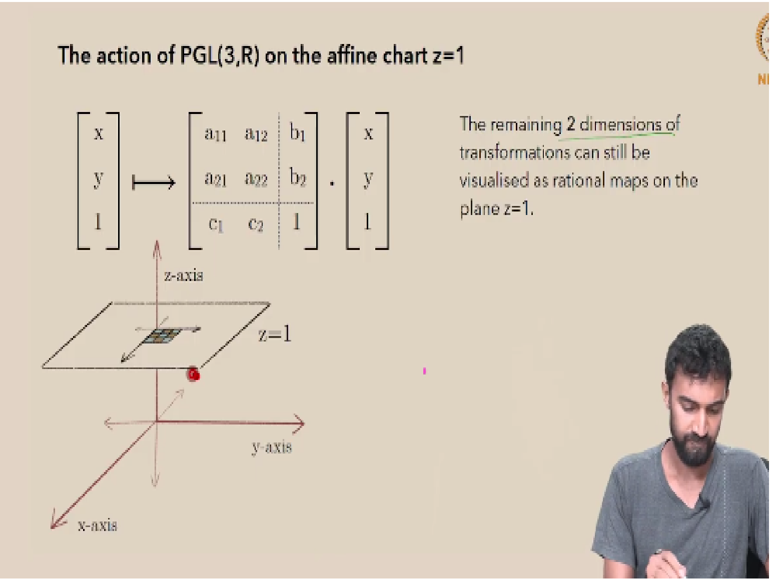

The remaining two dimensions of PGL(3,R) come in when we allow these two entries to freely range over R. Now we no longer purely fix the plane z=1. But we can still visualize these transformations as rational maps on the plane z=1. So we now have to bring in the fact that we are in RP^2.

So when we have this matrix act on this vector, we might get a vector that is outside of z=1, but we can find a representative for it. Since we can use homogeneous coordinates, we can find a representative for it on z=1. And we still have a rational map from z=1 to itself that way. So let us try and visualize all 8 dimensions via their action on the plane z=1.

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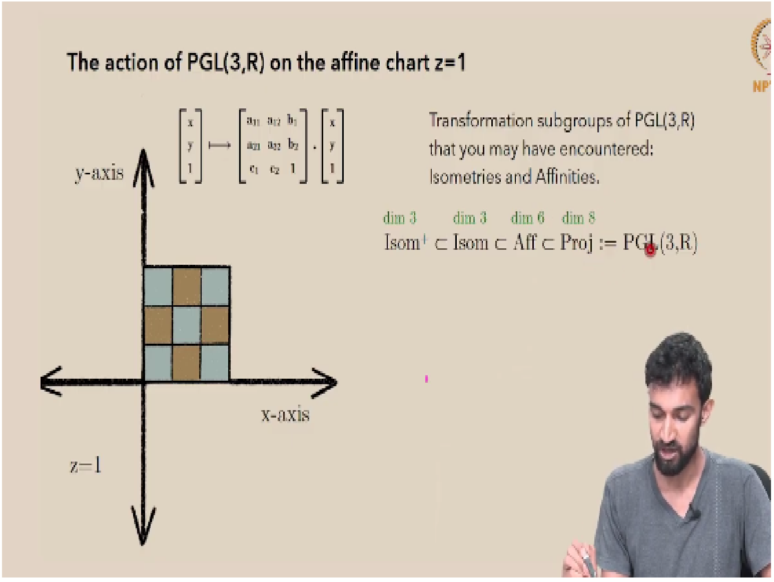

That is the final thing I want to do in this class. So I just mentioned there are some transformation subgroups of PGL(3,R) that you may have encountered before, isometries and affinities. So in particular, we have orientation preserving isometries and arbitrary isometries which consist of rotations, translations, reflections. These are rigid motions of the plane. That is what those groups are.

Affinities is a slightly larger group, which you study in linear algebra, which includes scaling and shearing or slanting. These are basically the maps that we get when we look at general linear maps and combine that with the ability to translate in the plane. And isometries form 3 dimensional groups. Affinities, we bump up the dimension to 6, like we saw the affinities are simply the things that fix the plane z=1. And finally, our entire set PGL(3,R) is dimension 8.

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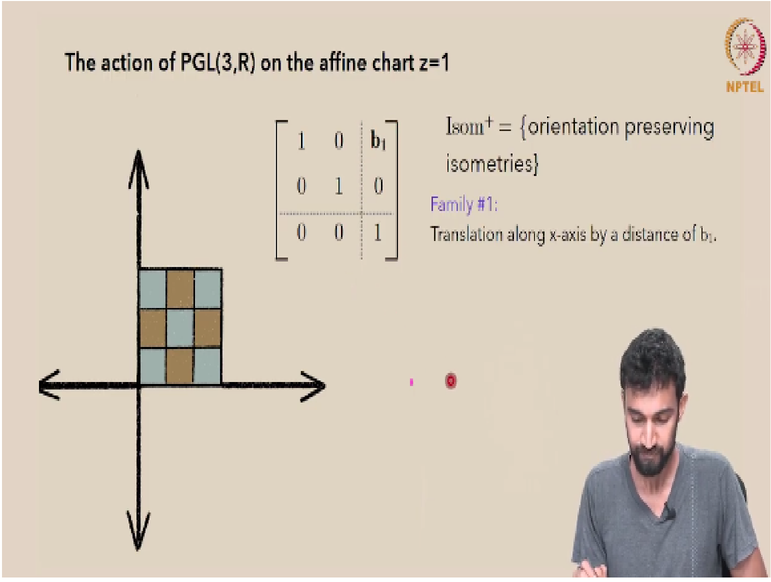

So let us start seeing how these actually operate. So first, let us look at the first family, which is an orientation preserving isometry. And we get it by just letting this entry here vary along the reals b_1. And this corresponds to translation along the x axis by a distance of b_1. So in this case, I am just translating this square along the x axis by a distance of b_1. That is my first family. And it corresponds to this entry here in PGL(3,R).

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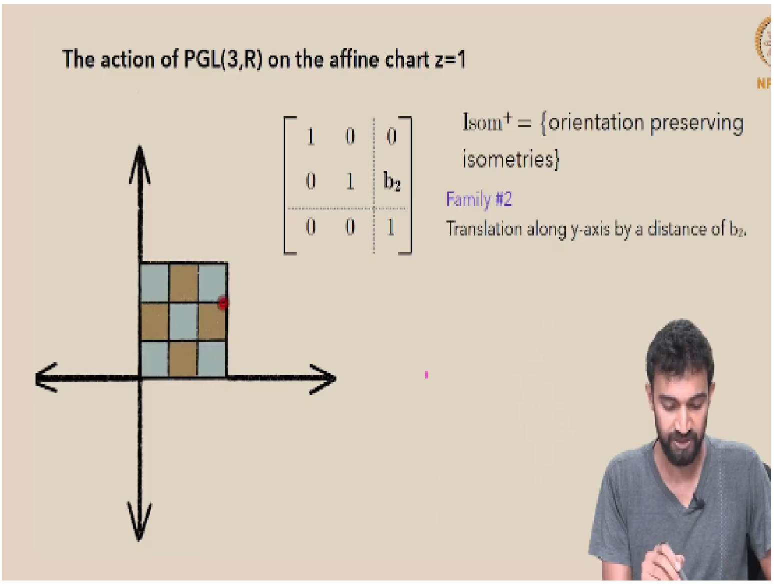

Okay, the second family you want to consider is translation along the y axis by a distance of b_2. And we are now looking at this entry, and that translates along the y axis, this square here.

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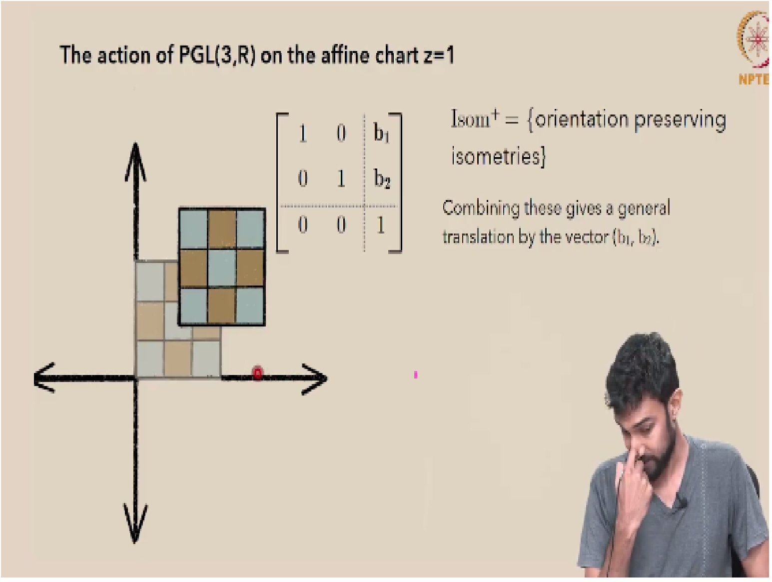

Combining b_1 and b_2, we can get general translations by a vector (b_1,b_2). Here, I am going by the vector (b_1,b_2) and translating.

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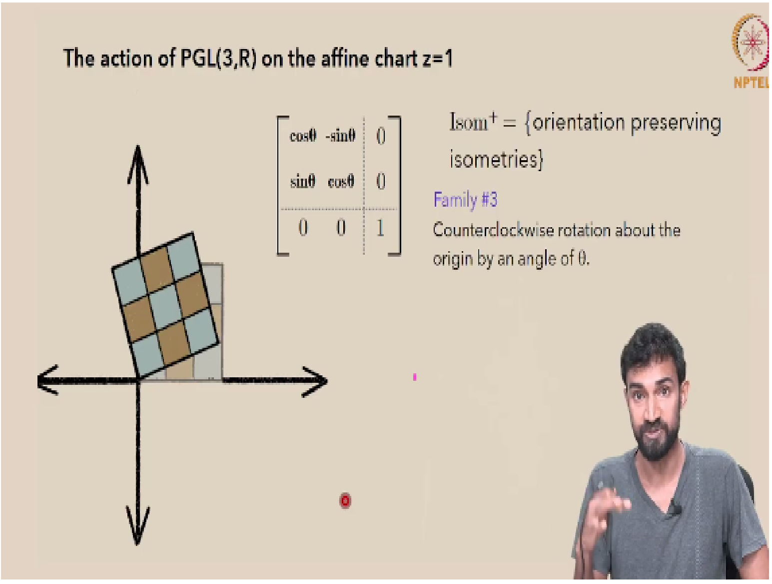

Okay, there is another type of orientation preserving isometry, which is known as a rotation. And this is given by this kind of matrix. And it does, and this matrix accomplishes counterclockwise rotation about the origin by an angle of θ. So you can see exactly what it is doing in the picture here. And you can even though it is using four entries, there is just one parameter θ. So it is adding one dimension. So far, we have looked at three different dimensions.

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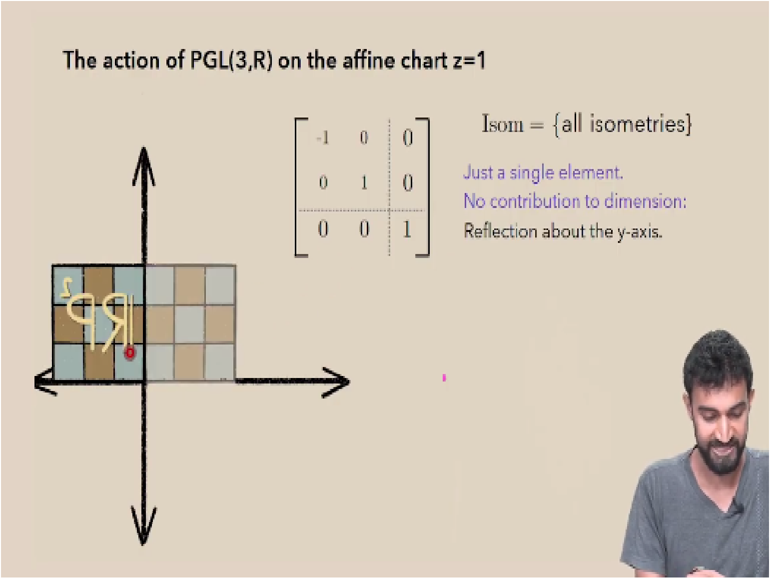

Now if we move to the group of isometries that are not necessarily orientation preserving, we add another element here, which is reflection. This is just a single element. It does not contribute to the dimension, but it is important to consider. We get reflection about the y axis that is what this element gives us.

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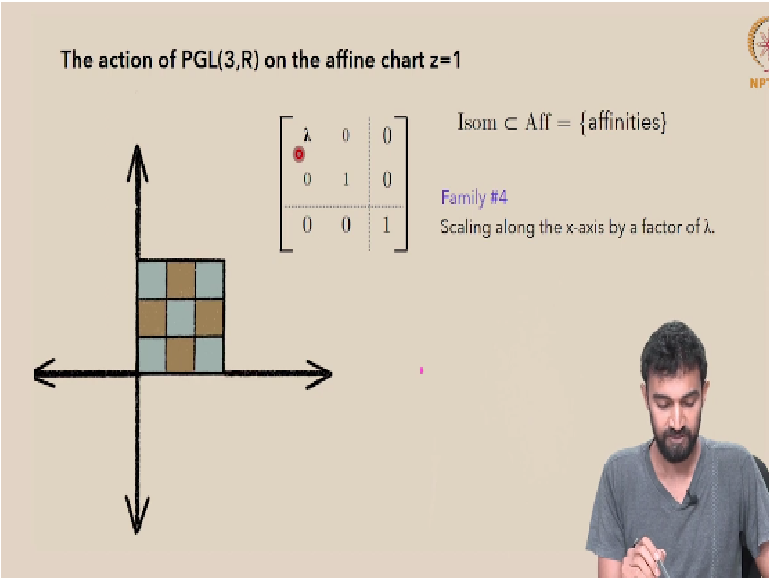

So my fourth family, that I want to consider is scaling along the x axis by a factor of λ. So if I put a λ here, in this entry, I can think of that as scaling or dilating along the x axis.

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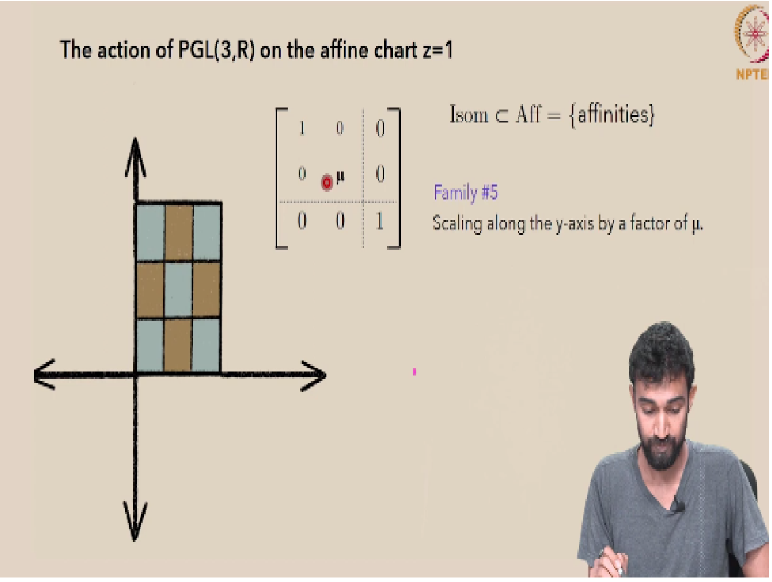

Similarly, a fifth family I get from this entry μ here, this is another one dimensional family that comes from scaling along the y axis, in this time by a factor of μ. And of course, you can combine those and do arbitrary scalings. You can do uniform scaling by letting $\lambda = \mu$.

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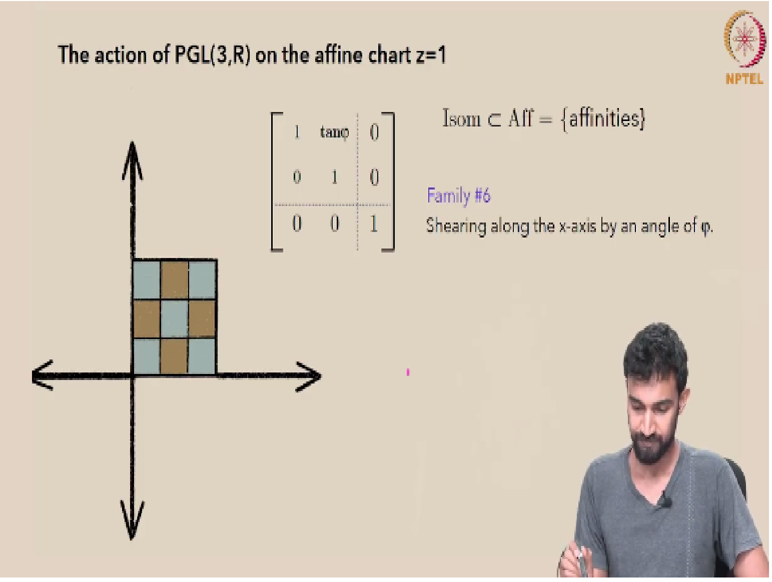

Okay, the sixth family I want to talk about is shearing. This is a map that happens in linear algebra, which will slant or shear a square into a parallelogram. So for example, shearing along the x axis by an angle of φ is given by this matrix with a $\tan(\varphi)$ here. And in this case φ refers to this angle here. So this is a shearing map, which shears along the x axis by an angle of φ.

So now we have looked at six different one-dimensional families. And all of these together give us the entire 6 dimensional subgroup affinities. So far, we have only looked at maps that take z=1 to z=1. These entries here have remained 0. So we are fixing the plane z=1, and we are getting all of the affine maps of z=1 this way. And that is a 6 dimensional set of maps, of the plane z=1 to itself.

But now things get a bit interesting. Now we see why PGL(3,R) is actually adding something new that we never encounter in linear algebra.

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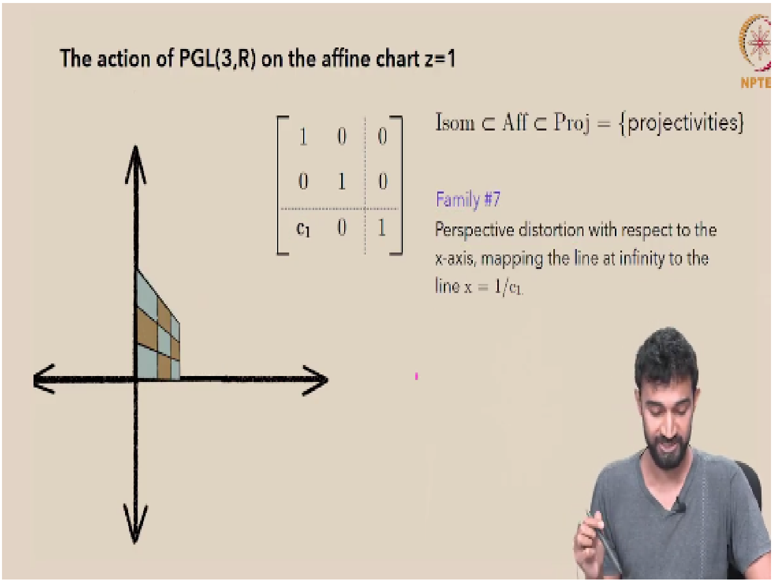

So the seventh family I want to consider comes from letting this entry vary. We no longer fix it at 0. We let it be a real number c_1. And what does this do, it gives perspective distortion with respect to the x axis in this way. And in particular, it maps the line at infinity to the line $x=1/c_1$. Here is the line $x=1/c_1$.

And you can see that all of our lines are converging to this point on this line at infinity. So it is doing a perspective distortion with respect to the x axis. If you apply this matrix to this square here, the image you will get will look like this. So it is a perspective distortion. It is like tilting it away from us.

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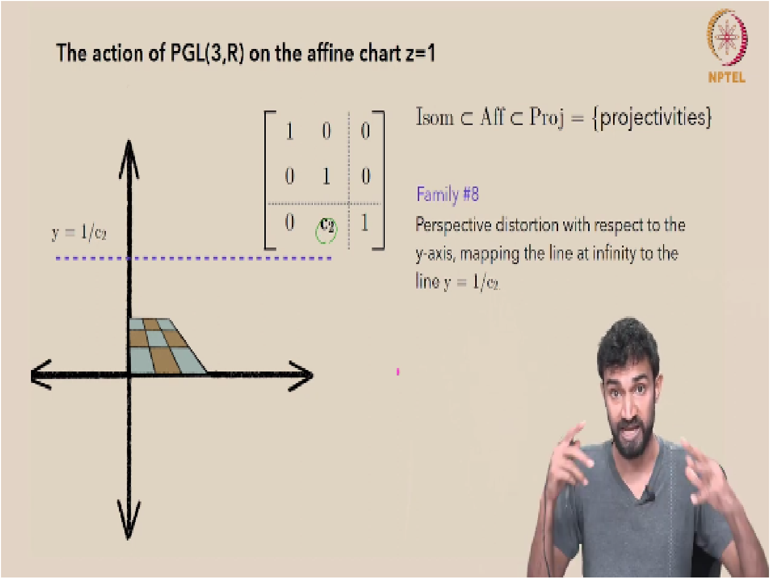

Finally, the last family I want to consider is perspective distortion with respect to the y axis. So that is mapping the line at infinity to the line $y=1/c_2$. So that is the eighth and final family. So this is a way of thinking of the 8 dimensions of PGL(3,R) as 8 different one-dimensional families of transformations.

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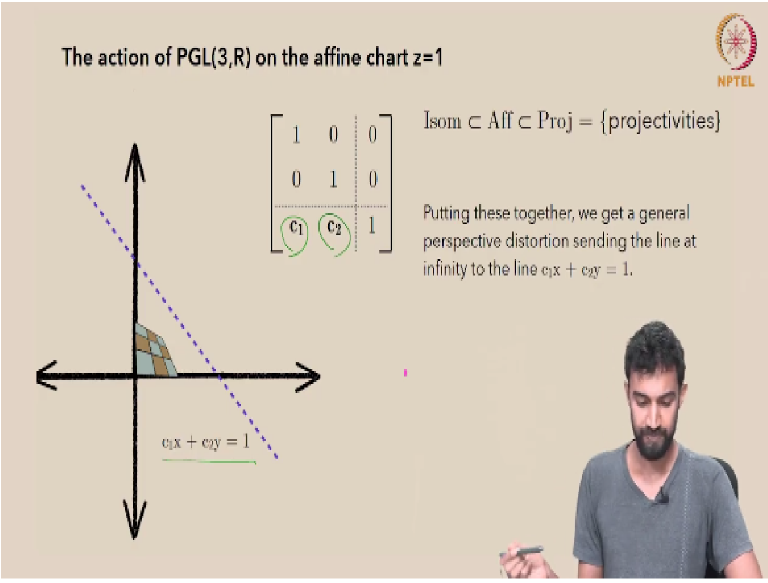

I will just mention that we can also put together these two types of perspective distortions and get arbitrary ones. So for example, here is a perspective distortion, which goes to the line at infinity $c_1x+c_2y=1$. So that is a more general perspective distortion you can get by playing with both of these entries together.

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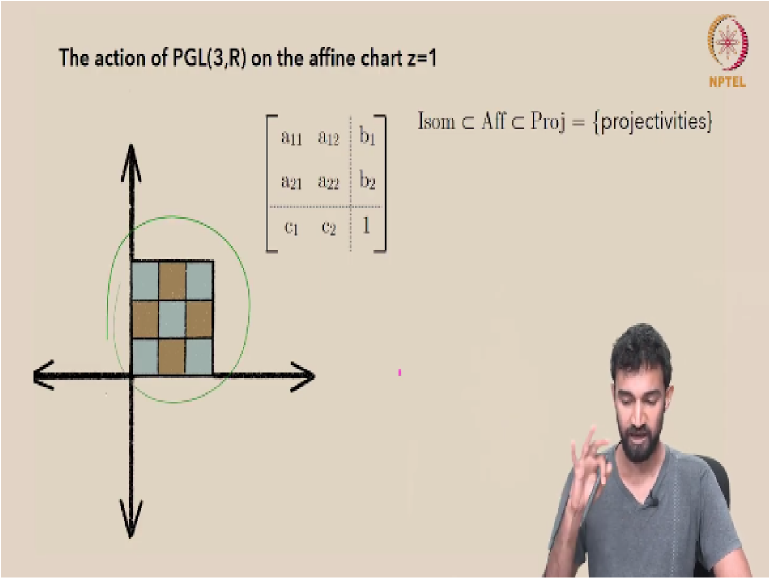

So taken together this 8 dimensional set of maps gives us a whole bunch of different transformations of this plane.

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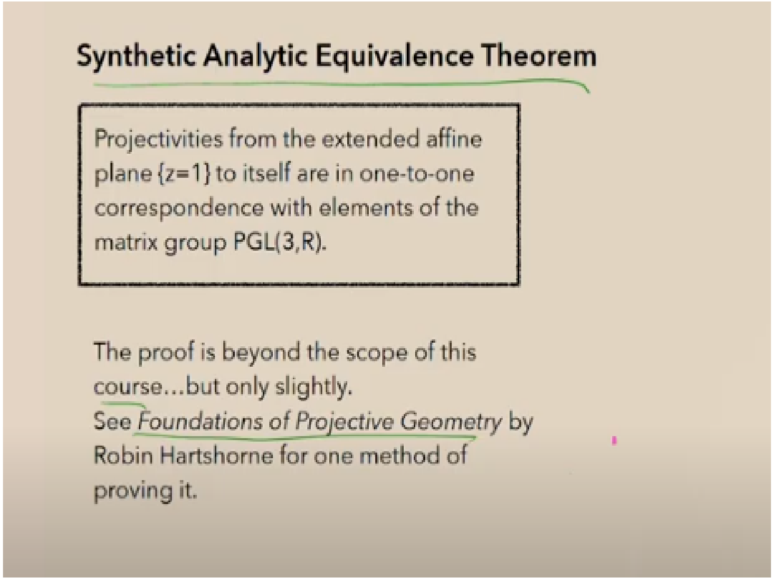

And it gives us a sense of how we can think about this synthetic analytic equivalence theorem. So all of these transformations we have seen here, all of these different transformations can be realized as projectivities in the old fashioned sets and synthetic framework from the plane z=1 to itself. They can all be realized as a sequence of perspectivities. But they can also be realized as concrete matrices.

So the proof of this, but these are very different things, matrices and sequences of perspectivities. So it is not at all clear how to relate them at a first glance. So the proof is beyond the scope of this course. It is actually not as hard as you might think. So if you are interested, you can check out the book Foundations of Projective Geometry by Robin Hartshorne, for one method of proving it.

But for the sake of our course, I will leave it there. So we have our synthetic approach. But we also have our analytic approach, which lets us write down concrete matrices, and get a handle on what these many different projectivities look like, and how many of them are there.

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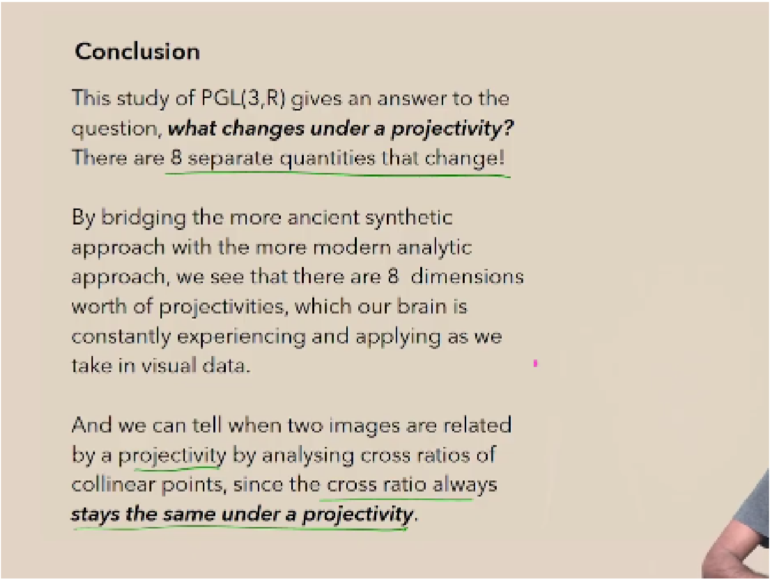

And it also gives us an answer to the question, what changes when we shift perspective. What changes under projectivity? And in a sense, there are actually 8 separate quantities that change, there are 8 dimensions of change, there are 8 degrees of freedom in the set of projectivities, from a plane to itself.

So by bridging the more ancient synthetic approach, with the more modern analytic approach, we see that these 8 dimensions worth of projectivities are actually, we are actually experiencing them constantly when we are viewing, viewing images from multiple perspectives, and kind of combining and bringing these different perspectives together.

So our brain is somehow constantly experiencing and applying these as we take in visual data. And as we saw in the previous lecture, we can see when two images are related by a projectivity by analyzing the cross ratio of collinear points in the respective images.

And since cross ratios are always the same under a projectivity, we have an answer to our other question of what is staying the same when we shift perspective. So between the cross ratio and PGL(3,R) we have some answers to these questions of what changes and what stays the same as we shift perspective.

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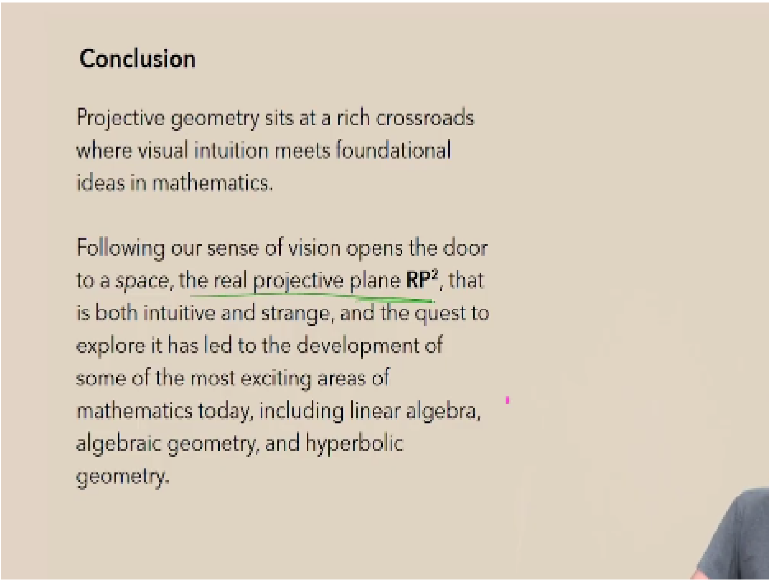

And the nice thing about projective geometry that we have kind of opened up is that it really sits at an interesting crossroads between our visual intuition and many foundational ideas in mathematics. So by following just our basic sense of vision, we have encountered this space, the real projective plane. And although it is both, it is strange, it is also kind of intuitive.

And historically, this attempt to understand perspective, starting from perspective drawing to the basic ideas of projective geometry and the real projective plane has been really instrumental in the development of some of the most exciting areas of mathematics today. Linear algebra, algebraic geometry, and hyperbolic geometry all have their roots in the study of projective geometry in the 19th century.

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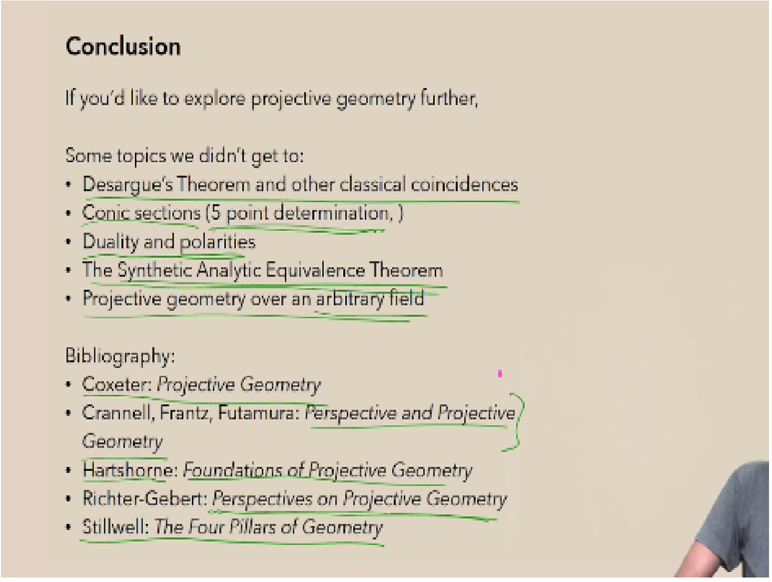

So if you would like to explore projective geometry further, there is a bunch of topics that we did not get to which I would have liked to get to, but there was simply not enough time in this semester, or in a four week course. But there is many other classical coincidences you could explore, like Desargue's Theorem to start with.

There is conic sections, which involve curves, but still have very nice definitions coming out of projective geometry. For example, five points determine a conic. And there is a purely straight edge construction, in which if you are given five points, you can generate the entire conic, you can generate as many points as you want, just using a straight edge.

We barely touched on duality, which is a very rich subject, because from duality, we actually get maps which interchange points and lines in the projective plane. And these maps are known as polarities. So there is a lot of interesting geometry there. We did not get to the proof of the synthetic analytic equivalence theorem, the fact that the synthetic approach is equivalent to the analytical approach, that projectivities are the same as elements of PGL(3,R).

So proving that means delving more deeply into the structure of these projectivities. And that is certainly something you can look into if you are interested. Finally, for those of you who are inclined to do so you can study projective geometry over an arbitrary field. It does not have to be over the reals, you can study it over the complex numbers or finite fields.

So some references if you want to study more are and these are the books that I used to design this course. Firstly, Projective Geometry by Coxeter, Perspective and Projective Geometry by Crannell, Frantz and Futamura, which is by the way, the most elementary of these different books. It only assumes High School algebra and nothing beyond that.

The book Foundations of Projective Geometry by Hartshorne, which does not assume that much either, but does require some mathematical maturity. The book Perspectives on Projective Geometry by Richter Gebert, which is maybe appropriate for when you have some mathematical maturity, you have had a few more classes, a few classes involving proofs.

And finally the book, The Four Pillars of Geometry by Stillwell, which is an excellent book for undergraduates, and contains two very nice chapters on projective geometry. So that about sums it up. So I hope you had an interesting time in this class, and I hope that you are somewhat enticed to look further into projective geometry and explore RP^2 a little bit more.

And also see this connection between our senses, in this case our sense of vision, and a huge amount of very deep mathematics that comes almost directly from that. So mathematics does not have to come purely from abstraction, it can also come from our own sensations. And this course is also an attempt to explore that idea. So I hope you enjoyed it and I will see you during office hours. Thanks.