**Our Mathematical Senses**
**The Geometry Vision**
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**Lecture - 15**
**The Real Projective Plane**

Hi, welcome back to the Geometry of Vision. This is lecture eight, the final lecture in which we will talk about the real projective plane and its transformation group.

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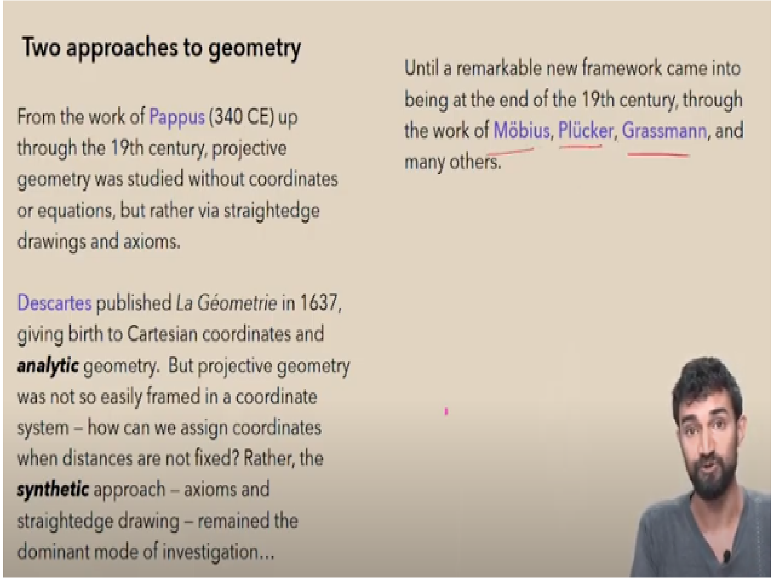


So there are two approaches to geometry and projective geometry in particular. So from the work of Pappus, the mathematician whose theorem we proved, and who actually discovered the cross ratio, or at least who wrote it down. In fact, it is quite likely other mathematicians before him had also noticed and thought about the cross ratio. But, we have explicit writing from him on it, from 340 CE.

And from the work of Pappus, up through the 19th century projective geometry was studied without coordinates or equations, but rather with drawing, with straight edge constructions, and with axioms. So we have seen quite a few straight edge constructions at the beginning of the class. And most of what we have done so far with the exception of the cross ratio can be done with a straight edge, can be explicitly drawn.

For example, a projectivity taking three points to three other points can be explicitly drawn with a pencil and straight edge. So using straight edge constructions and basic axioms, how many different people who were exploring these questions got a handle on projective geometry and geometry in general. That was kind of the overall frame of thought.

In 1637, Descartes published La Geometrie in which he gave birth to Cartesian coordinates in some sense and analytic geometry, which is a totally different way of approaching geometry, using explicit coordinates, numerical coordinates, which allowed for exploring geometric objects through equations. But actually, projective geometry was not so easily framed in a coordinate system like that, because distances are constantly changing when you are considering a single object.

We might be considering an object in projective geometry, but we look at it from one way or another and all the distances and angles and areas are shifting. They do not stay the same, even though we are still looking at the same object. So we cannot easily assign coordinates in a meaningful way.

So for that reason, maybe not just that reason, but mathematicians preferred the synthetic approach, when it came to projective geometry. So they preferred this approach of axioms and straightedge drawing over the analytic coordinate equation approach of Descartes. So that kind of remained the dominant mode of investigation for projective geometry through the 19th century.

But there is a remarkable new framework that came into being towards the end of the 19th century through the work of many mathematicians, including Mobius, Plucker, and Grassmann, just to name a few.

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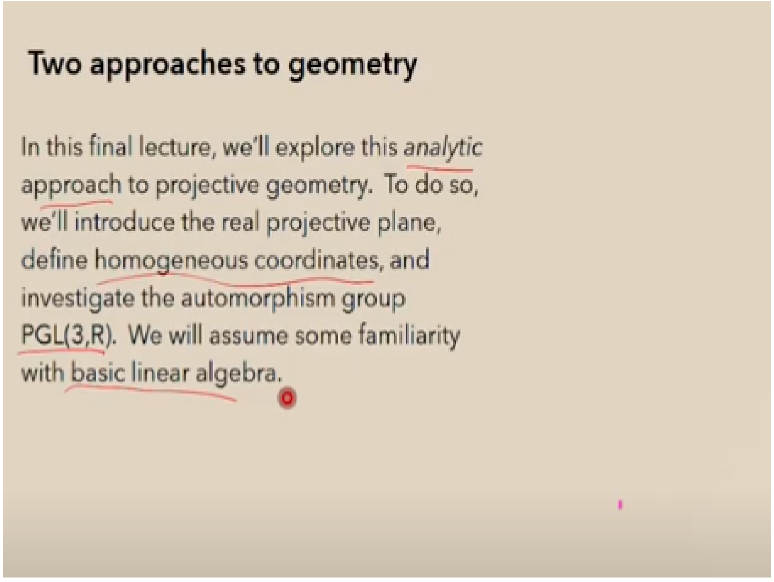


So we are going to explore that framework, which allows us to bring the analytic approach to projective geometry. And in order to do this, we are going to introduce the real projective plane and define homogeneous coordinates, which is a coordinate system, which allows us to talk about objects in projective geometry, independent of distance in a way.

And finally, we will investigate the matrix group PGL(3,R), which governs the transformations of the real projective plane. So we will assume some familiarity with basic linear algebra, in order to do this. So for those of you who do not have some basic linear algebra, it is okay. Still listen to the lecture and the homework will not be that heavy on linear algebra.

So, here and there, we might use a few terms that you are not so familiar with. But hopefully you can think of that as a teaser for linear algebra itself and why it might be nice to take a course on that. And more than linear algebra, we are really just going to use vector calculus, which hopefully you have seen a little bit of in high school when you studied calculus.

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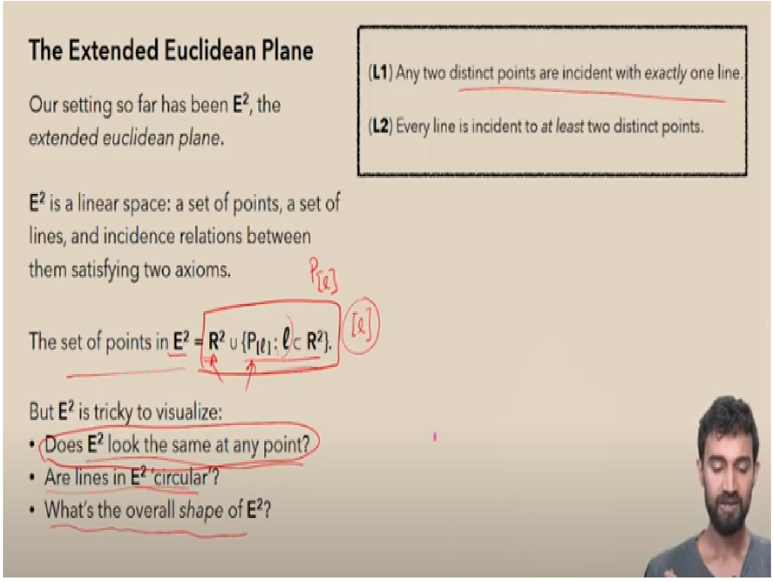


Our setting so far, for projective geometry, has been the extended Euclidean plane $E^2$. And $E^2$ is a linear space remember. What is a linear space? It is a set of points, a set of lines, and a set of incidents relations between the points in the lines which satisfy two very simple axioms, namely, any two points should be incident with exactly one line. In other words, any two points should determine a unique line.

And the second axiom is that every line should contain at least two points. We should not have lines that are empty sets, devoid of points, and we should not have lines that have just a single point, then they are kind of meaningless. So every line should be incident to at least two points. And the major thing is just that any two points determine a unique line. That is what makes the space a linear space.

So the set of points in $E^2$, what is it? Well, it is just equal to the set of points in $R^2$ along with these extra points at infinity that we added. The notation we used was $P_{[l]}$, what is this funny [l]? It is the equivalence class of all lines parallel to a given line l; l is some line in $R^2$. This is the set of all lines parallel to l. And we defined a point at infinity corresponding to that equivalence class.

We defined $E^2$ to have a set of points consisting of $R^2$ along with all these points at infinity. That is how we defined $E^2$ to begin with. So it is defined very abstractly that way. It is just all the points of $R^2$ plus all these abstract points at infinity. And they turn out to be kind of hard to visualize. Because, first of all, does $E^2$ look the same at any point?

If you remember, we saw that, from the properties of points and lines, from the incidence relations, they were kind of the same at every point. It was the case, not only was it a linear space, but we had an additional property that any two lines determine a unique point. Any two lines are incident to a unique point. And that property is the same.

That is true, whether you are taking ordinary lines or lines of infinity. Similarly, any two distinct points are incident with exactly one line. That is true for any two points. It does not matter if they are ordinary points or points at infinity. But it is still hard to see why that is the case, because it is defined so abstractly. Why does $E^2$ look the same at every point? Actually, we do have a distinction.

We have these ordinary points of $R^2$ where we have these abstract points sitting off at infinity that we defined abstractly. So $E^2$ is a little tricky to visualize. Another problematic point that we encountered, problematic issue we encountered is that, for a given family of parallel lines, we introduced one vanishing point. So it is the same vanishing point in one direction and the same vanishing point in the other direction.

And we had to do it that way for it to be a linear space. But that had the strange consequence, that if you go off in this direction, that is the same as going off in that direction, infinitely far. Meaning that some of the lines connect up with each other at a pointed infinity. Meaning the lines are kind of circular in a way. And what does that, what do I mean by that? Are the lines really circular?

Can we think of them as circles? Or what is the deal with that? How do we visualize lines in $E^2$? And there is a third confusing thing here, which is a little more vague, but just what is the overall shape of $E^2$? How does it all stitch together? We have all these individual lines that are connecting up like circles, how do they all stitch together? How do we imagine $E^2$ as a whole?

So these are questions that are hard to answer if we are just thinking of $E^2$ abstractly as this extension of the Euclidean plane by these points at infinity. And this line at infinity connects the points at infinity.

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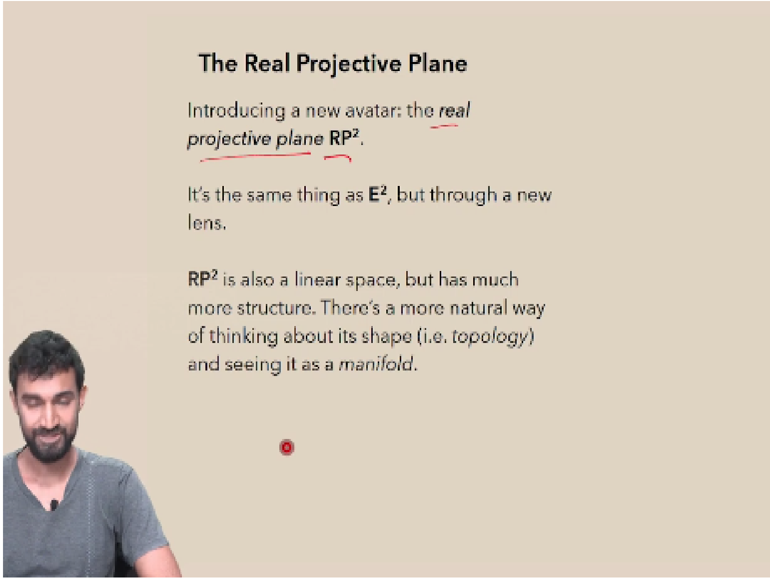


So I want to answer these questions now by introducing a new avatar of the projective plane known as the real projective plane, $RP^2$. So it is really the same thing as $E^2$, but through a new lens in a way. It is also a linear space. $RP^2$ is also a linear space, but it has much more structure. For one thing, there is a natural way of thinking about its shape, its topology, and seeing it as a manifold.

And do not worry, if you do not know these terms. For people who do know these terms, $RP^2$ is a prime example of a topological manifold. And we can very clearly think about its shape and answer those questions.

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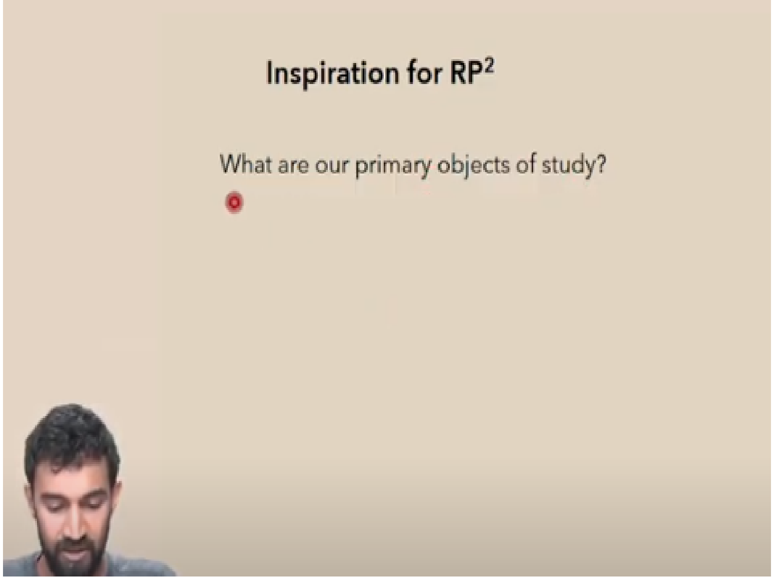


So let us first give some inspiration and motivation for where $RP^2$ was going to come from. So all this time, what are our primary objects of study in projective geometry?

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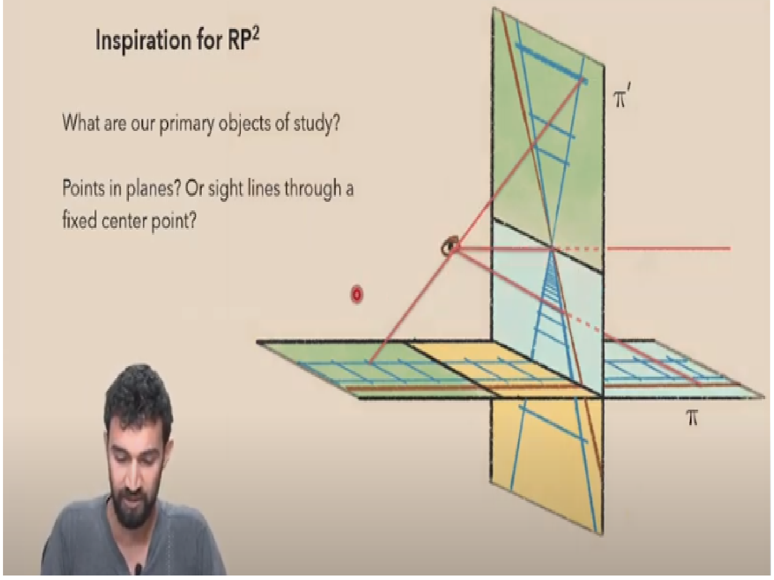


Well, remember that we have been looking at lots of points in planes under perspectivities. But does it really make sense to think of points as the primary objects? Or maybe it makes sense to think of sight lines through a fixed center point as the primary objects. And the reason for that is that we have seen how, if we are just looking at this plane $\pi$, let us say, we have a bunch of points on it.

And we can imagine seeing those points. In fact, the sight lines which see those points, and taking those points to this plane, $\pi$', that was that perspectivity that we defined. However, we run into problems with the points at infinity. $\pi$ has a whole bunch of points at infinity, one for every family of parallel lines. And we cannot see those, but those also correspond to sight lines through the center point.

So for example, this was a point at infinity. In fact, if we keep looking further and further and further and further along, our sight lines eventually converge to this sight line here, which is no longer a point on the plane. But it is an abstract point at infinity. So maybe we should be thinking about the sight lines, rather than the points on the plane. So maybe we should think about sight lines through a fixed central point.

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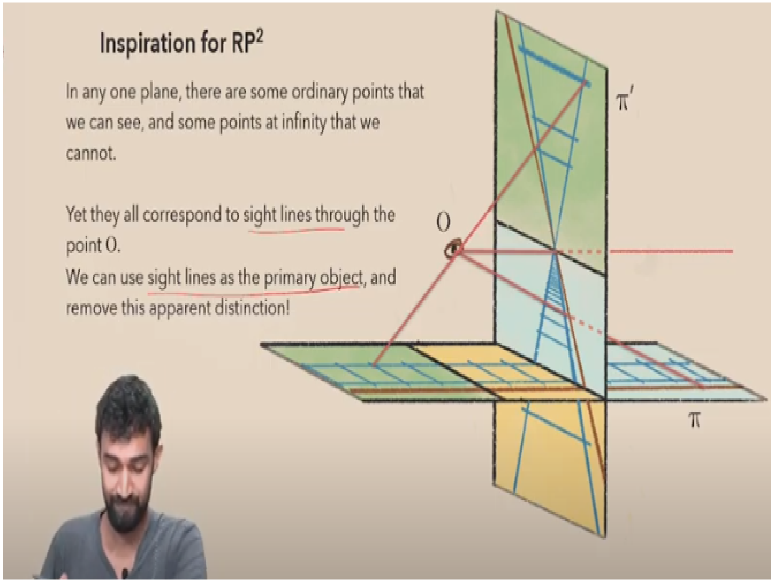


And as I just said, in any plane, $\pi$ or $\pi$', it does not matter which plane we pick, there is always going to be some ordinary points that we can see and some points in infinity that we cannot. For example, with $\pi$' here, we have points in infinity, corresponding to lines like this. Of course, we have other points, ordinary points that are also corresponding to sight lines, lines through this point.

But then we also have this point at infinity, these many points at infinity that do not. So maybe, these sight lines, these lines through the center point are the things to consider. Everything and everything corresponds to one of them. So let us try and use sight lines as our primary object and remove this apparent distinction between ordinary points and ideal points. Points should just be points and their geometry should be kind of the same.

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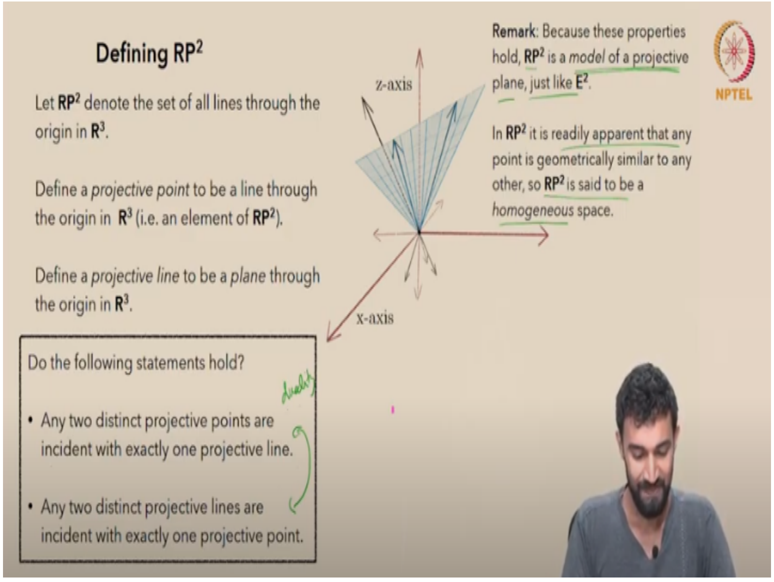


So let us define the real projective plane. So let $RP^2$ denote the set of all lines through the origin in $R^3$. So, the center of my sight lines is going to be the origin. And instead of sight lines, I am just going to call them lines. Let $RP^2$ denote the set of all lines through the origin in $R^3$.

So here are just a few of them. But of course, we have many more lines through the origin in $R^3$. Let us define a projective point to be a line through the origin in $R^3$. In other words an element of $RP^2$. So this is a projective point here. This is a projective point here. This is a projective point here. I have drawn three projective points, right here. So it is a little confusing, but we will get used to it.

A projective point is a line through the origin in $R^3$. And expanding on that, let us define a projective line to be a plane through the origin in $R^3$. So here I have drawn a projective line, which is the plane. I have drawn a plane that connects this projective point and this projective point, and that gives us a projective line. And it keeps going and going. I have just kind of cut it off here so we can visualize it.

But this is a projective line. It is just a plane through the origin in $R^3$. Now here is a question for you. Do the following statements hold? The first statement is any two distinct projective points are incident with exactly one projective line. Any two projective points, in other words, any two lines through the origin are incident with exactly one projective line, meaning a plane through the origin.

Are any two lines through the origin incident with exactly one plane through the origin? Yes, they are. Any two lines at the origin will determine a unique plane through the origin. So yes, this holds. The second statement is that any two distinct projective lines are incident with exactly one projective point. So is that true?

Is it true that any two projective lines, any two planes through the origin are incident with exactly one line through the origin? Well, any two planes intersect in a line. And if these are both planes through the origin, they are going to intersect in a line through the origin. So again, yes. Any two distinct projective lines will determine exactly one projective point, exactly one line through the origin.

So both these statements are true. So both these are kind of the most fundamental axioms in a way of a projective plane. And these both hold, which means that since these properties hold $RP^2$ is a model of a projective plane. That is how that is the term some people use, meaning that it satisfies these two properties. Just like $E^2$, $E^2$ also satisfies these two properties.

So like $E^2$, $RP^2$ is a model of a projective plane. It has this kind of nice duality of points and lines. And in $RP^2$, has a slight advantage, which is that it is readily apparent that any point is geometrically similar to any other point. We do not have this apparent distinction between points at infinity and ordinary points. So in that, for that reason, $RP^2$ was said to be a homogeneous space, homogeneous means same everywhere.

The geometry at this point is the same as the geometry at this point. There are no special points in that sense. So this is just the definition of $RP^2$. And we have seen that it is a projective plane and satisfies these basic properties, incidence properties of points and lines.

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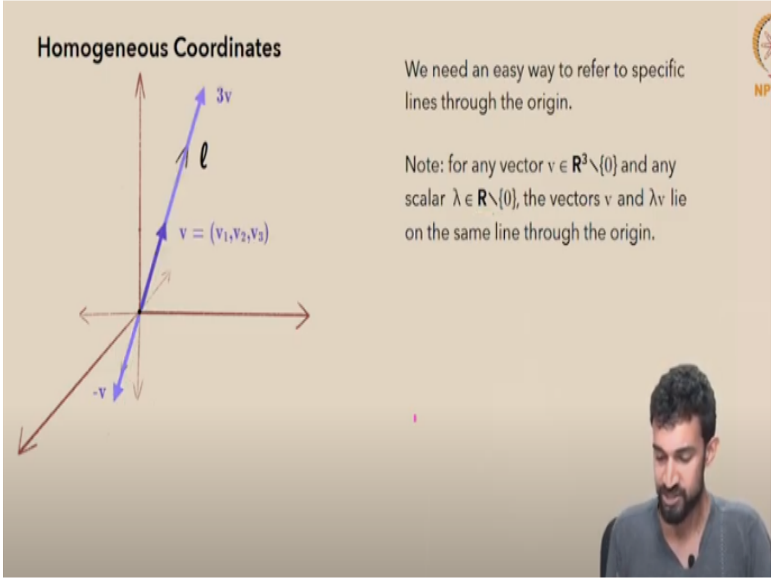


Okay, it will be nice to have an easy way to refer to lines through the origin. Now that we are in $R^3$ which has coordinates, can we find a way to refer to lines to the origin using coordinates? Well, we have coordinate vectors. So you know we can always choose a vector v lying on the line l, $v = (v_1, v_2, v_3)$ where these are all real numbers; $v_1$, $v_2$ and $v_3$ are three real numbers.

And for any nonzero vector v and any nonzero scalar $\lambda$, notice that v and $\lambda$v lie on the same line l through the origin. So v lies on this line l, any scalar multiple of v will also lie on the line l, for example, 3v will also lie on l. Or -v will also lie on l. And any scalar by any nonzero real number will continue to lie on l. Actually, even the scalar by 0 will lie on l, but it gets confusing if we do that.

So I would not explicitly use that one. So all of these scalar multiples of v lie on l.

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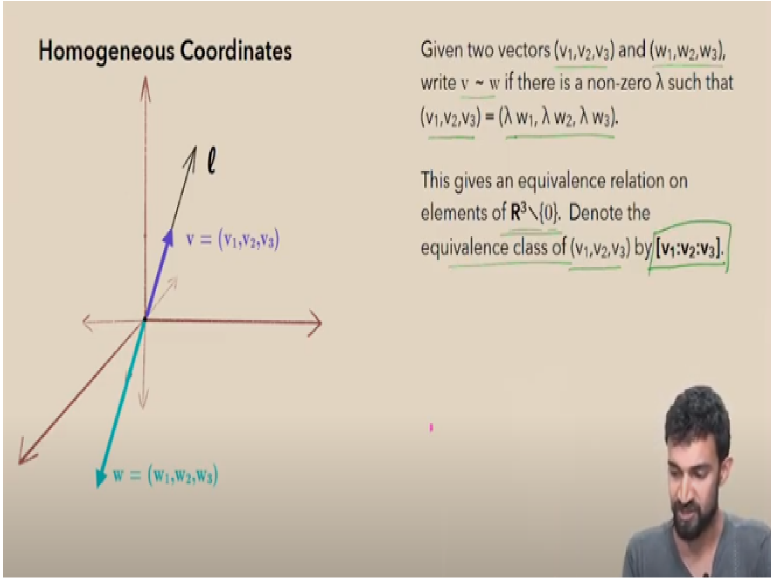


And I want to make a quick definition here. Given two vectors, ($v_1$, $v_2$, $v_3$), and ($w_1$, $w_2$, $w_3$), let us write v~w, if they are related by a nonzero scalar, i.e., $v_1$, $v_2$, $v_3$ are a scalar multiple of $w_1$, $w_2$, $w_3$. And it is easy to check that this gives an equivalence relation on the elements of $R^3$-{0}. Basically, any two vectors are equated. They are related if they are scalar multiples of one another.

So all of the vectors along this line l will be related to one another, under this relation. So let us denote the equivalence class of ($v_1$, $v_2$, $v_3$) by [$v_1$:$v_2$:$v_3$]. This is a useful notation. And the colons just emphasize that, we are now looking at ratios in a way.

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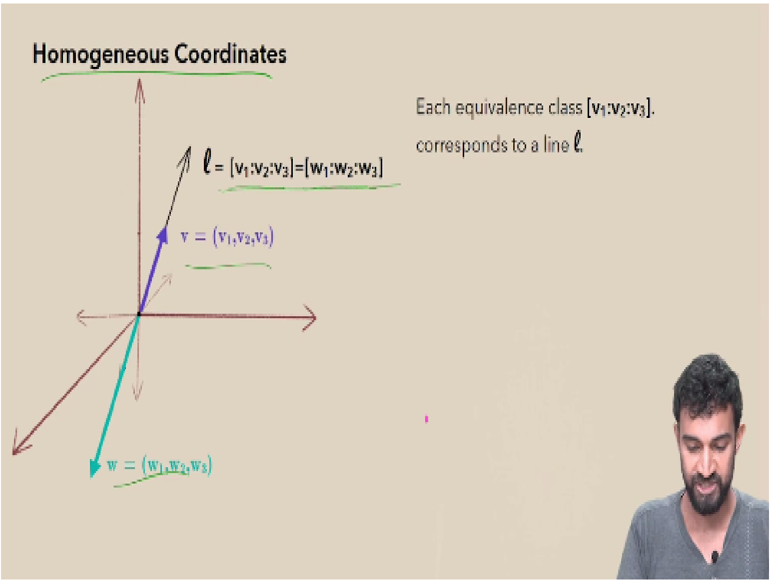

So each equivalence class now corresponds to a line. All the vectors along a line l, like v and w here and all their scalar multiples will be related to each other under this equivalence relation. So they will be equated with each other. And l corresponds to their equivalence class, which we can write as $[v_1:v_2:v_3]$ or we could equally well write it as $[w_1:w_2:w_3]$. These mean the exact same thing and they both refer to that line l.

So this is our handy notation for referring to lines through the origin. And these are known as homogeneous coordinates.

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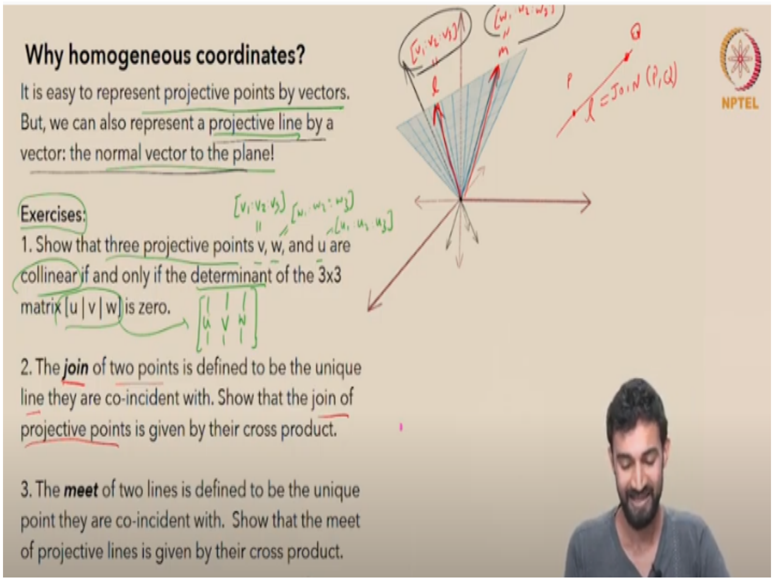


Why do we want homogeneous coordinates? Well, first of all, obviously it gives an easy way to represent projective points by vectors. But we can also represent projective lines by vectors. So a projective line, which is a plane through the origin can also be represented by a vector. In particular, it can be represented by the normal vector to it.

So if you have a normal vector to this projective line, we can think of that as referring to that plane through the origin, that projective line. And in fact, that is well defined, because any scalar multiple of this normal vector will continue to be normal to this plane here. So any scalar multiple of it in any direction, in both directions, is going to keep being normal to that plane.

So that entire equivalence class of that normal vector, we can associate with that plane. So homogeneous coordinates not only give us projective points, but interestingly, they also give us projective lines, which is kind of cool. So some exercises to see how useful they are. The first one I want you to look at is to show that three projective points v, w, and u, remember these are projective points, so there are lines through the origin.

So we could think of them as being written in homogeneous coordinates like this. v is $[v_1:v_2:v_3]$, w is $[w_1:w_2:w_3]$, and u is $[u_1:u_2:u_3]$. So can you show that three projective points, v, w, and u are collinear if and only if the determinant of the three by three matrix that they form is zero. We take them as column vectors. So what is this matrix here? I am just writing, the order does not actually matter.

Let me write it this way, u, v, w. In fact, I could also equally write it as row vectors, it does not really matter. You could take any three representatives for these points, and create a matrix of them and check its determinant. And the claim is that these three projective points will be collinear if and only if the determinant of this three by three matrix is 0.

So that is a nice exercise. And as a hint, remember that a matrix has determinant 0, if and only if the column vectors are linearly dependent. The column vectors are linearly independent if and only if the determinant is nonzero. It is another way of saying that.

So what does it mean for three projective points to be collinear? Well, projective points are lines through the origin. If they are collinear, if three lines through the origin are collinear, they must lie on the same projective line. They must lie on the same plane through the origin. So this picture here, we have one projective point here, another projective point here.

And maybe if we take this projective point, that is another projective point, which is collinear with the other two, because it lies on the same plane through the origin.

Maybe if this is v and this is vector w, this could be vector u. So this is a picture of u, v and w being collinear. They all lie in the same plane through the origin.

So maybe you can see how to do the first exercise. They are collinear if and only if their determinant is 0. So I will let you finish that exercise. But that is but do try and try it out and work that out yourself. Okay, so the second exercise to look at is that the join of two points is generally defined to be the unique line that they are coincident with. So this is true in $RP^2$ or in general just points in the plane.

Given two points in the plane, P and Q we see that their join is the line that they determine, the line that joins them. The join of two points is defined to be the unique line that they coincide with. So this line here l in this picture is the join of P and Q.

And can you show that the join of two projective points is given by their cross product, or what I mean really is the cross product of any two vectors representing those two points. So given two projective points l and m. Let us say that, here is one projective point l and here is another projective point m and maybe we have some vector representations for these.

So maybe $l = [v_1:v_2:v_3]$ and $m = [w_1:w_2:w_3]$. This exercise is to show that the join of these two projective points, which is defined to be the unique projective line containing them, which we can see here is that it is this plane through the origin, that is equal to the cross product of any two vector representations, any two vectors representing these lines.

So the cross product of v with w. So can you show that that is true? And there is not that much to do more than what I have just said. It is just the cross product of v and w, we know is a vector that is orthogonal, that is normal to the plane spanned by v and w. If you have studied the cross product, you will know that which is normal to this plane here.

So it represents that projective line. If we think of a projective line as represented by its normal vector, it gives us that. So I let you think about this last one, which is very cool, because we can similarly define a dual motion of the meet between two lines. So given two lines in a projective setting, they determine a unique point. And we call that point their meet, the place where they meet is, it is a meeting place.

So any two lines have a meet, a unique meet. And the claim is that the meet of two projective lines is again given by their cross product. So take two projective lines, there are planes through the origin. They are represented by their normal vectors. And the claim is that, if we take the cross product of those normal vectors, we will get a vector representing their meet, representing the projective point that is their meet.

So I will let you think about that one. But there is an incredible amount of duality going on here in how we represent these.

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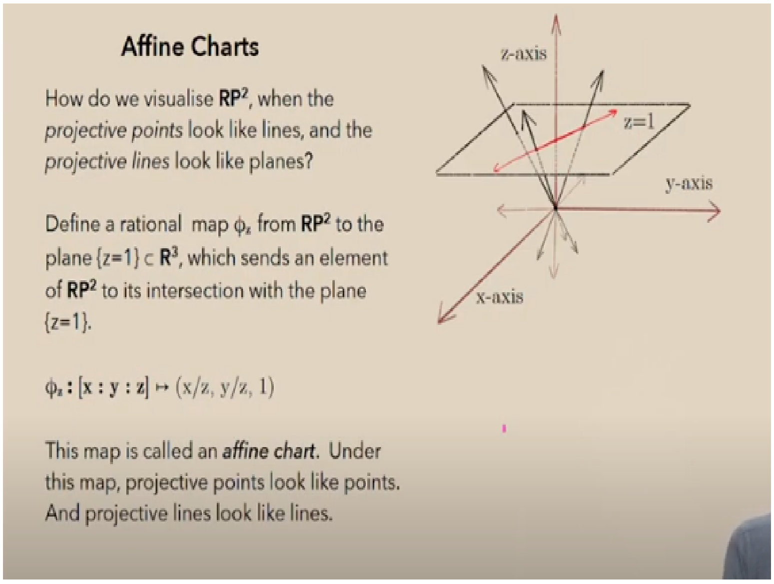


And this framework here is due to Grassmann who was a mathematician in the 19th century. So now how do we visualize $RP^2$? When projective points look like lines, we have just seen in this discussion here, it gets a bit confusing, because projective points are actually lines through the origin and projective lines are actually planes through the origin.

So it gets a bit cumbersome to talk about and to keep going back and forth, especially when you are first exposed to that. So you are totally forgiven if you are finding this a bit confusing at first, because it is confusing to keep talking about projective points being lines through the origin and projective lines being planes through the origin.

So the question is, how do we visualize $RP^2$, this real projective plane, when projective points look like lines and projective lines look like planes? Well, here is a way, there is a nice way to do it, which is to define a rational map. I will call it $\phi_z$, from $RP^2$ to plane z=1 in $R^3$.

The way I will define this map is that I will have it send any element of $RP^2$ to its point of intersection with the plane z=1. Remember, an element of $RP^2$ is a line through the origin. And it is going to intersect that plane z=1 in a unique point to that plane. So this map simply sends an element of $RP^2$, a line through the origin to its intersection with that plane, z=1.

And you can see it illustrated right here. So here is the plane z=1. And you can see that this projective point, this line through the origin, gets mapped to this point here. This one gets mapped to this point here, and this one gets mapped to this point here. So we can actually write out in homogeneous coordinates exactly what this map is doing.

It is taking an element of RP2. Maybe this element here is [x:y:z]. And it is sending that to this point here, which in this case is the point (x/z, y/z, 1), right? This z coordinate is 1, because it is in the plane z=1. And clearly, if you know these homogeneous coordinates, we can scale by any scalar, and get a different vector along this line. So what we are doing here is we are just scaling this by 1/z.

So this is equal to [x/z:y/z:1]. That is the same, that represents the same, that is just another vector along the same line, maybe if we call this line l, it is given by this vector and this vector also lies on it. And this vector has the advantage that it ends on the z=1 plane. So we are basically just going to a particular vector representative of this class.

And the particular representative we are going to is the one where z=1. And we get there by scaling all the coordinates by 1 over z. So this gives a map which sends an element of $RP^2$ to the plane z=1. And I mentioned this is a rational map, because it is not defined everywhere. Not every element of $RP^2$ can be plugged into this function to get a point on z=1.

Can you see a point of $RP^2$ that does not go to a well-defined point on the plane z =1. Remember, we are taking a line through the origin to its intersection with z=1. Well, which lines through the origin do not intersect, fail to intersect z=1? How about the x axis, or the y axis, or any other points on the xy plane. So we have some places where it is not defined, that is why it is a rational map.

Now this map is called an affine chart. That is the technical term for it. And under this map, we see that projective points look like points. All these lines through the origin, which are my projective points, actually look like points under this map. And projective lines look like lines. If you imagine a plane through the origin, it will actually look like the plane that we saw earlier, which connects these two lines, it is going to look like a line.

So a plane through the origin will intersect z=1 in a nice line. So in that way, it is a very nice way to visualize $RP^2$.

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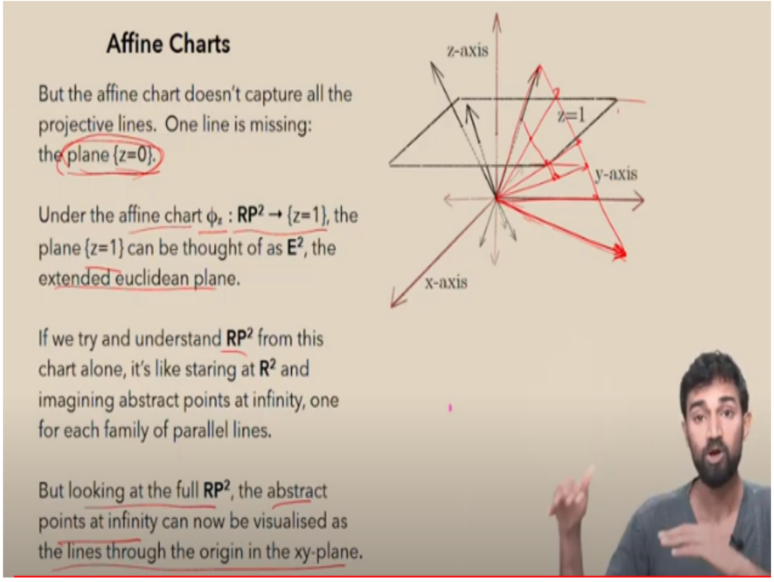


But like I said, it does not capture all of the projective lines, the plane z=0 is missing, the xy plane. One projective line I should say, is missing. And notice that under this affine chart $\phi_z$, which sends $RP^2$ to the plane z=1, actually it feels a lot like $E^2$. It feels a lot like an extended Euclidean plane.

Because now all of our points look like points, all of our lines look like lines, except we have these points on the plane z=0. And those ones, let us say this point here, or like this line through the origin here, that actually is similar to a point at infinity. It does not appear on this plane z=1. But we can imagine it. It is kind of a limit. So we can imagine, maybe going further and further out.

This way, maybe you are going further and further and further out along the plane. And as you do that, maybe you are moving from this line towards this line. And as you do that, you are looking further and further and further out along the plane, like we have seen before, in perspective drawing. And finally, in the limiting sight line in a way you get to here, but that does not appear on your plane.

So these lines on the xy plane are kind of like points at infinity for $E^2$. And the xy plane itself, this plane z=0, is a projective line. And it is kind of, it corresponds to a line at infinity for our extended Euclidean plane. So if we try and understand $RP^2$

from this affine chart alone, it is a lot like staring at $R^2$ and imagining abstract points at infinity, one for each family of parallel lines.

And maybe you can convince yourself that just as we have seen before, in this setup as well, if we follow any two parallel lines along, they will actually converge to a single point at infinity. They will converge to a single projective point which is lying on this plane z=0.

So the nice thing, however, is that since we are looking at the full $RP^2$, and we are looking at all of these lines through the origin, the abstract points at infinity, can now be visualized easily as lines through the origin just like all the others. So all of our points are lines through the origin. And the points at infinity are lines through the origin in the xy plane. They are just lines through the origin that happened to sit in the xy plane.