Our Mathematical Senses
The Geometry Vision
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Lecture - 14
Applications of the Cross Ratio

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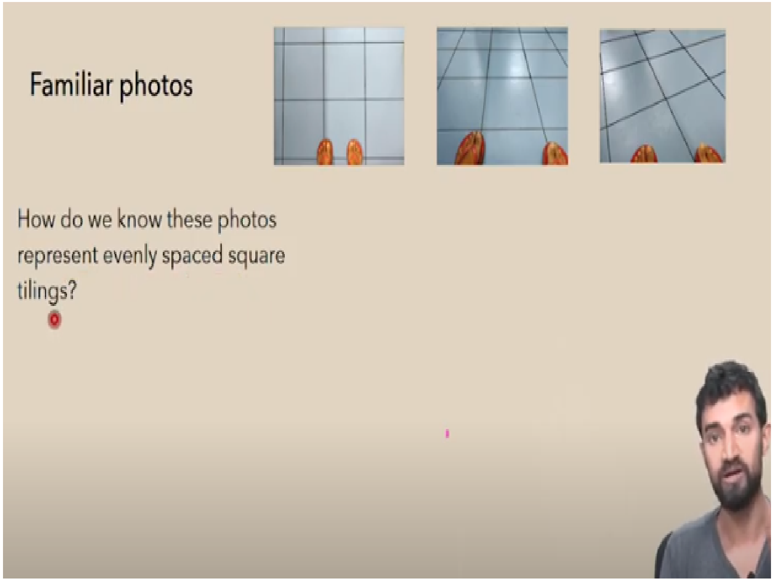

So, let us get back to some interesting applications of the cross ratio. So here are some familiar photos. At this point for all of us in this course, we have been seeing these since the intro video. These are three perspective views of a tiled floor taken with a camera from slightly different angles. And we remarked that it is kind of obvious to us when we see these photos that they are square tiled floors.

Not obvious, but we kind of know it. And the question is, how do we know that these photos represent evenly spaced square tilings? How can we tell?

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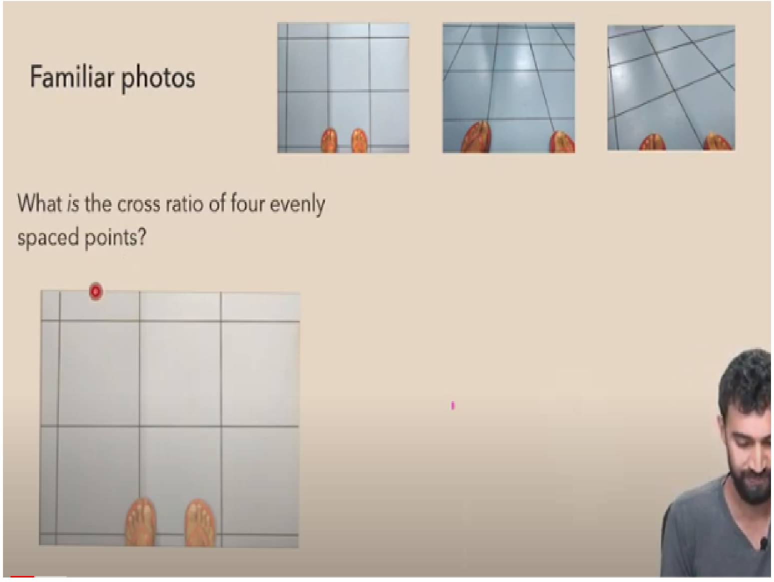

So in particular, can we use the cross ratio here? And what is the cross ratio of four evenly spaced points? What is that actually?

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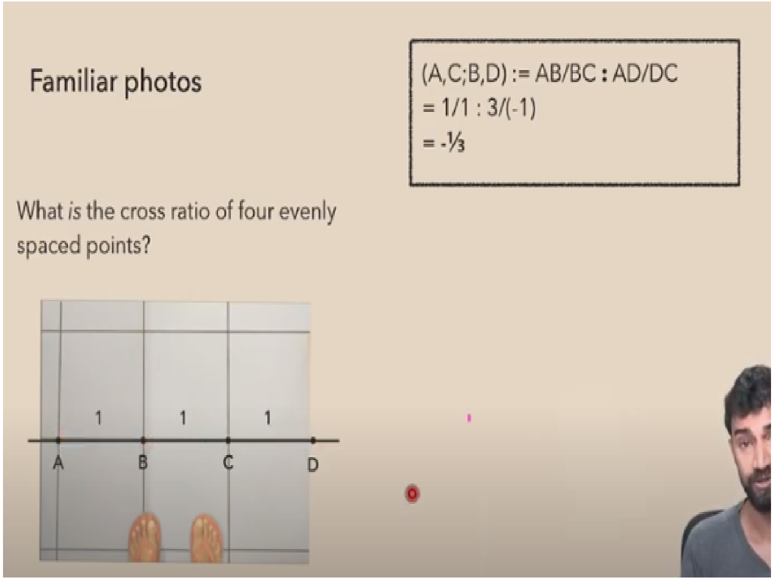

So here is my definition of cross ratio. And let us draw our four evenly spaced points A, B, C and D. Let us just say that there is one unit between any two of them. The distance from A to B is 1, B to C is 1 and C to D is 1. So what is the cross ratio (A,C;B,D) in this case? Well, it is (AB/BC)/(AD/DC). Well, what is that? AB is equal to 1, BC is equal to 1, AD is equal to 3, and DC is equal to -1.

So we get (1/1)/(3/-1), which is just -1/3. So the cross ratio of these three evenly spaced points is -1/3.

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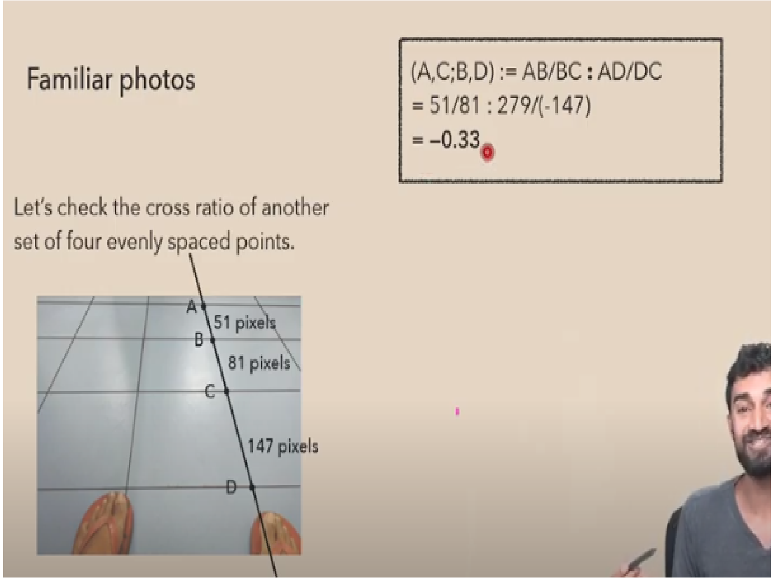

By our invariance of the cross ratio theorem, for another four evenly spaced points, or if we take another photograph of these four evenly spaced points, we should get the same cross ratio. So let us just check that. Let us check the cross ratio of this other set of evenly spaced points. Well, let us check what it is. So again, I have put this photograph onto my computer.

And I have downloaded a program, which lets me measure the number of pixels between any two points. So I measured that this is 51 pixels, this is 81 pixels, and this is 147 pixels. I actually did this measurement, it is not that I somehow cheated or something or cooked up these numbers, I actually did these measurements to see how it works.

What is the cross ratio now of these four points taken in this order A, B, C, D? Well, it is (AB/BC)/(AD/DC). AB is 51. BC is 81. AD is 51+81+147, which is 279. And finally, DC is -147. So plugging this into your calculator, you can try this yourself, you get -0.33. I was actually pretty surprised, but it worked out so nicely.

I mean, I figured it would be something near there. But I did not expect to get it on the nose like that. Because I am just measuring these pixels, there could be an inaccuracy somewhere, I was not sure how good my measurements were. But it actually worked out to be -0.33, which is pretty cool.

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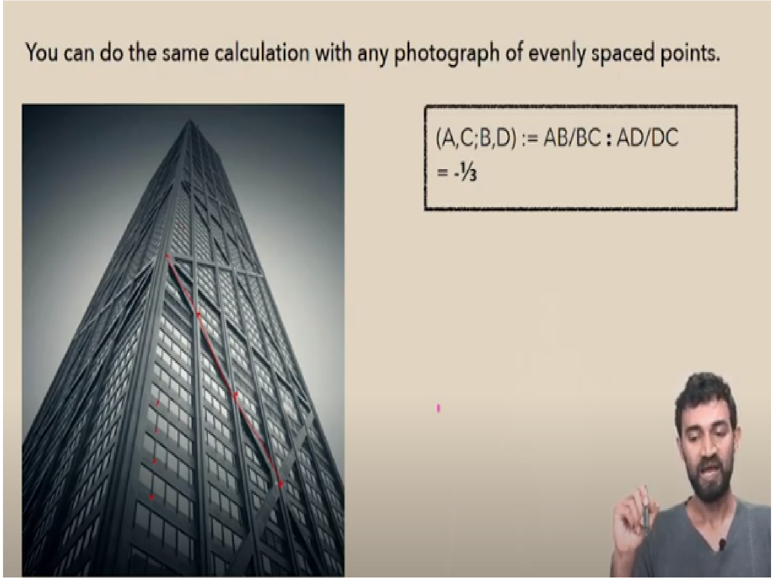

So you can do the same calculation with any photograph of evenly spaced points. There are lots of different sets of evenly spaced points in this skyscraper. You could look at the windows in this direction. Or you could look at windows in this direction. And you could find lots of examples of evenly spaced points. And you could check the cross ratio of them. You could do something like this.

Anyway, you get the idea. Or maybe something like this is more interesting, can be diagonal. And you can check that any of these evenly spaced points, any set of them, if you take the cross ratio, you will get -1/3 unless they are not actually evenly spaced.

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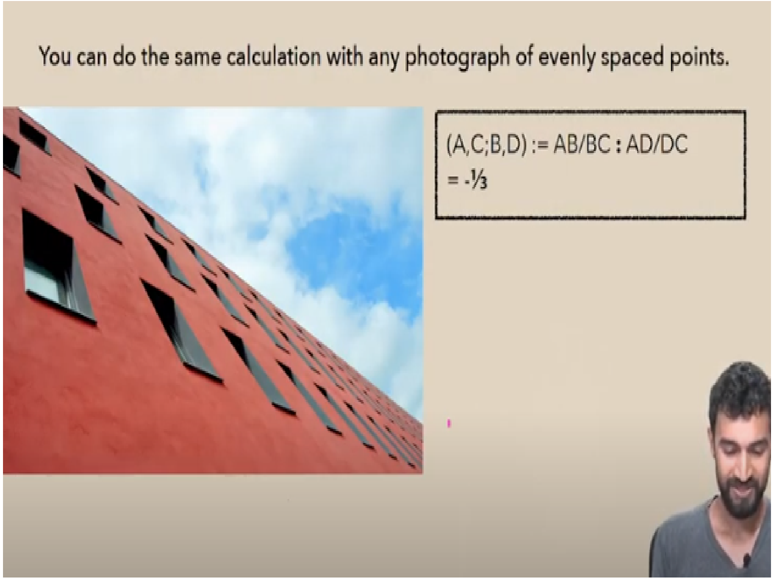

So here is another photograph. You can just try it yourself. You can try it with any photograph.

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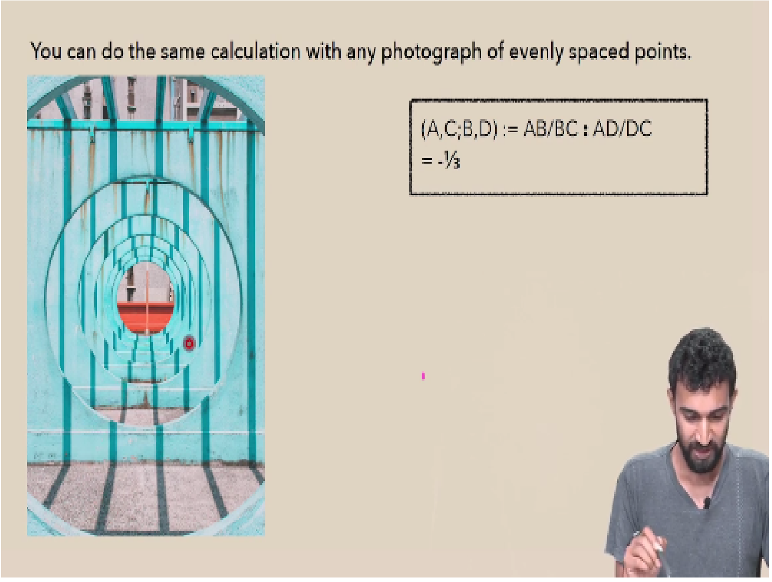

Here is another one, you could take points in any direction actually.

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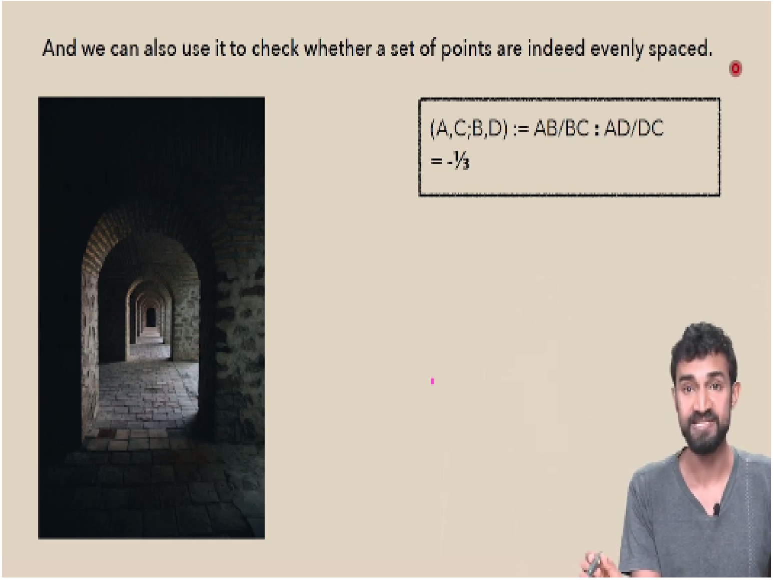

And you can even use it to check whether a set of points is indeed evenly spaced. Maybe we do not know if this set of points in real life are evenly spaced or not. And all we have is a photograph. We can check whether they are evenly spaced or not by taking the cross ratio of these four points. If we get -1/3, then they are evenly spaced.

If we do not get -1/3, then sorry, they are not evenly spaced. So that is a way to verify. So that is a very cool and actually potentially useful application of this cross ratio.

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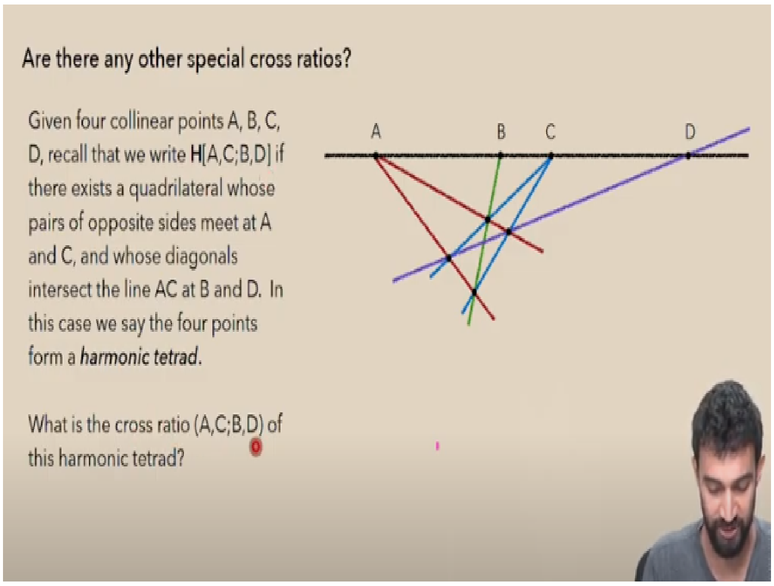

Okay, so we have seen the cross ratio of evenly spaced points. Are there any other special cross ratios? So there is one that is definitely worth mentioning, because it is quite special. So given four collinear points, A, B, C and D in our usual setup.

Recall that we can write H[A,C;B,D] if there exists a quadrilateral like this whose pairs of opposite sides meet at A and C.

So this is one pair of opposite sides meeting at A. This is another pair of opposite sides meeting at C. And whose diagonals meet the line connecting A and C, this kind of horizon line connecting A and C at points B and D. The diagonals meet our horizon line, which we get by connecting A and C. Our diagonals meet that line at B and D.

So if we have a quadrilateral, that is relating our points A, B, C and D in this manner, if we have a quadrilateral whose four vanishing points coming from its sides at its diagonals are actually equal to A, B, C and D, then we say that A, B, C and D form a harmonic tetrad. And we had a few other definitions related to that.

We said that D is the harmonic conjugate of B with respect to A and C. Similarly, B is the harmonic conjugate of D with respect to A and C. Or we can say that A is the harmonic conjugate of C with respect to B and D. Basically that is why they are grouped like this. Anything on this side is basically these two guys are harmonic conjugates with respect to these guys.

And these two guys are harmonic conjugates with respect to these two guys. So that is a quick review of harmonic tetrads. And why am I bringing them up? Well, perhaps they are. It would be interesting to see what the cross ratio of a harmonic tetrad H[A,C;B,D]. Well, one thing we could do now is just measure the number of pixels here, here and here and calculate what it is.

But that is not that elegant, and it is kind of a pain and it kind of involves a calculator. So is there a way we can do it that does not involve a calculator, and that is a bit easier for lazy people.

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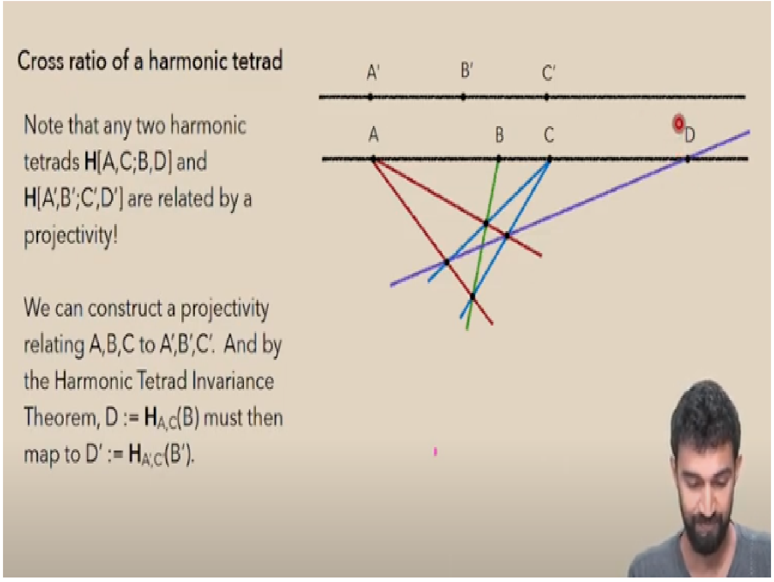

So in fact, there is. Notice that any two harmonic tetrads are related by a projectivity. If we have a harmonic tetrad H[A,C;B,D] and another harmonic tetrad, H[A',C';B',D'], they are related by a projectivity. How do we know this? Well, we can construct a projectivity relating A, B, and C to A', B' and C'.

So over here, let us say A' B' C' is just some other set of three points. And I can always construct a projectivity relating A, B and C to A', B' and C' because we can always construct a projectivity relating three points to three other points. That was our fundamental theorem of projective geometry in one dimension. In fact, we are not even using the uniqueness, we are just using the existence part.

So we can construct that. And then by the harmonic tetrad invariance theorem that we proved last week, or the week before last week I think, the point the harmonic conjugate of B with respect to A and C, it must go to the harmonic conjugate of B' with respect to A' and C'. So this harmonic tetrads are invariant under perspectivities.

So if we construct a projectivity setting A, B and C to A', B' and C', it will invariably map D to D', wherever that is on this line here. So any two harmonic tetrads are related by a projectivity and we can construct it.

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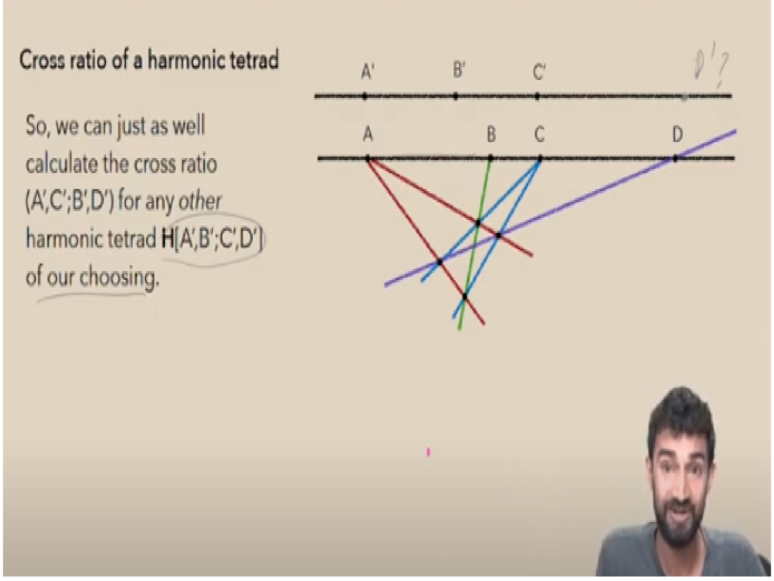

And as a result, we might just calculate the cross ratio of this other harmonic tetrad here, A', B', C', D', I did not draw D', but D' is somewhere you know wherever D' is. Construct or calculate that cross ratio rather than this one. I mean, we might as well calculate it for a harmonic tetrad of our choosing.

And we might as well choose one, then that is easy to calculate. That is the point of this. We do not want something that is difficult to calculate, where I have to get up my ruler, measure the number of pixels and do all of that. Let us choose a better harmonic tetrad that is easier to calculate that we do not need to do measurements and complicated calculations for.

So is there any harmonic tetrad that we would like to calculate the cross ratio of, that will be easier to calculate the cross ratio of. And indeed there is. Maybe you remember. We did look at this once briefly.

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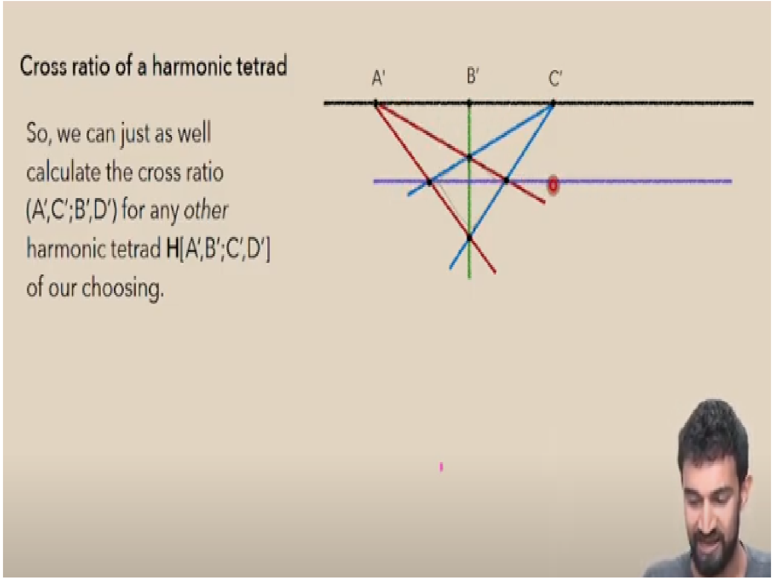

But we can take the harmonic tetrad involving three evenly spaced points A', B' and C'. In that case, the quadrilateral will look like this. A' is the vanishing point of these two sides. C' is the vanishing point of these two sides. B' is the diagonal. And the other diagonal, well since they are evenly spaced, everything is symmetric.

And the other diagonal is going to be horizontal. It is going to be parallel to this line. And it is only going to intersect this line at infinity.

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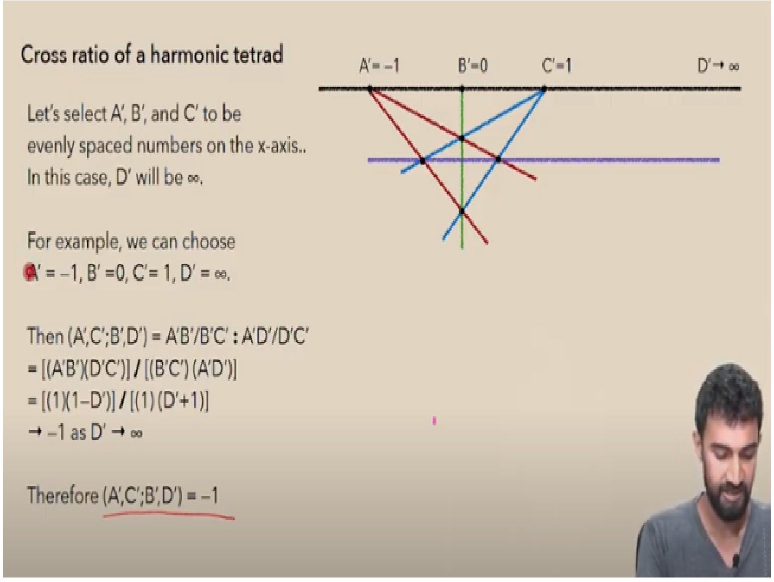

So if we select A', B' and C' to be evenly spaced numbers on the x axis, then D' will be infinity, it will just go off somewhere. And let us make it even simpler and nicer.

Let us say A' is -1, B' is 0, and C' is 1 and D' will be infinity in that case. What is the cross ratio in this case of this harmonic tetrad?

Well, let us calculate it. Now our numbers are really nice and easy.The cross ratio (A', C'; B',D'), by definition, is equal to (A'B'/B'C')/(A'D'/D'C'). Oh, we have two infinite lines here. But let us see, maybe something nice will happen and it will cancel out.

Let us not despair. Let us just try and simplify this. So first of all, I can rewrite this algebraic expression in a slightly nicer way. So I have (A'B'/B'C')/(A'D'/D'C'). But might as well flip this and bring it to the top. So that becomes D'C'/A'D', which is what I have written here.

Okay, well A'B' is just 0-(-1). So A'B' is 1. B'C' is 1-0. So that is also equal to 1. D'C', what is that? D'C'? Well, let us leave D' as a variable, let us not plug in infinity, because that will make things go a bit haywire. But D'C' is let us just consider that as 1-D'.

And similarly A'D', let us just consider that as D'-(-1), right? A'D' is just D'-A'=D'-(-1)=D'+1. Okay fine, so what do we get? We get (1-D')/(D'+1). What happens to this quantity as D' goes to infinity?

Well, (1-D')/(D'+1) as D' goes to infinity. Well, the 1 on the top and the1 on the bottom do not matter much, but we have -D' on top and a positive D' on the bottom. So it is going to -1 as D' goes to infinity. So the cross ratio in this case, (A', C'; B', D') is equal to -1.

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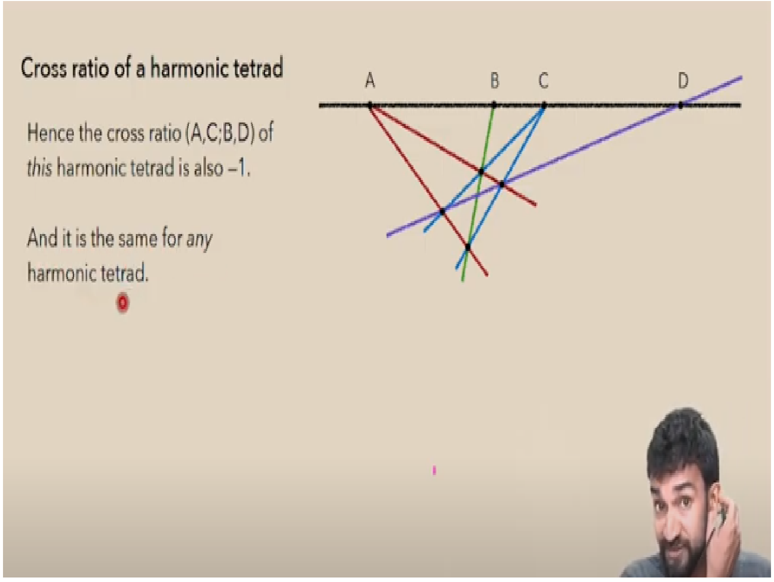

And in fact, the cross ratio of any harmonic tetrad, this one for example, must also be -1, because they are related by a perspectivity, any two harmonic tetrads are related by perspectivity.

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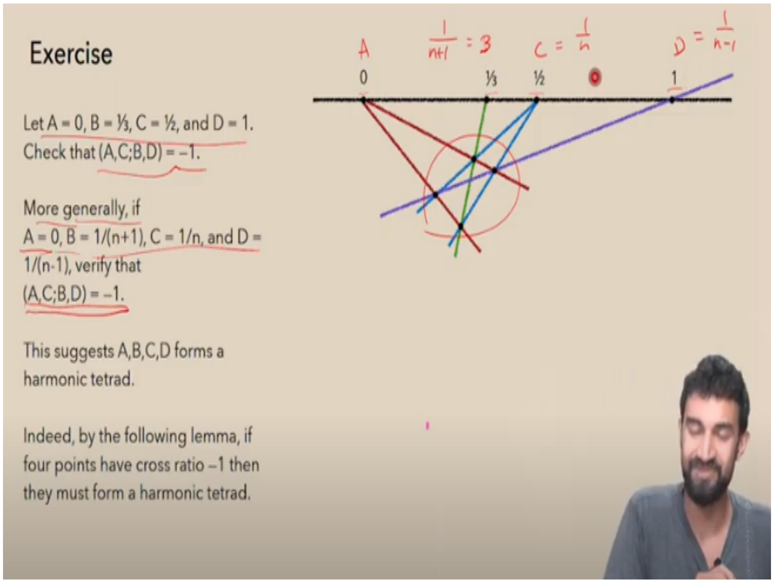

So as an exercise, why do you not verify this for the following harmonic tetrad? So I have given you a harmonic tetrad here whose points A,B, C and D are 0, 1/3, 1/2 and 1. And I claim this is a harmonic tetrad. And rather than verify the harmonic tetrad by doing all these calculations, why do you not first just check the cross ratio and verify that it is -1.

And more generally, as a slightly more advanced, slightly harder exercise, if we take four points A=0, B=1/(n+1), C=1/n, and D=1/(n-1), then I claim that this is also a harmonic tetrad. As an exercise, just verify that the cross ratio is equal to -1 in this case.

So this suggests that this is a harmonic tetrad. And indeed, by the following lemma, we will see that it is. We have seen already that any harmonic tetrad has a cross ratio -1. I claim that any four collinear points with cross ratio -1 has to be a harmonic tetrad. We have not quite proven that yet. But the following lemma will immediately imply it.

And we see evidence for it here in this exercise. If you work out these two examples and verify them, you have two more collinear sets, sets of four collinear points whose cross ratios are -1. And from the next lemma, we will see that that does indeed imply that these are examples of harmonic tetrads.

And this example is particularly nice, because it tells us why we have this word harmonic. You can see that these numbers 1/(n+1), 1/n, 1/(n-1), they are forming a harmonic progression, I guess in the other direction. So that is actually why this name harmonic is there, why we attach this thing harmonic to this set of points.

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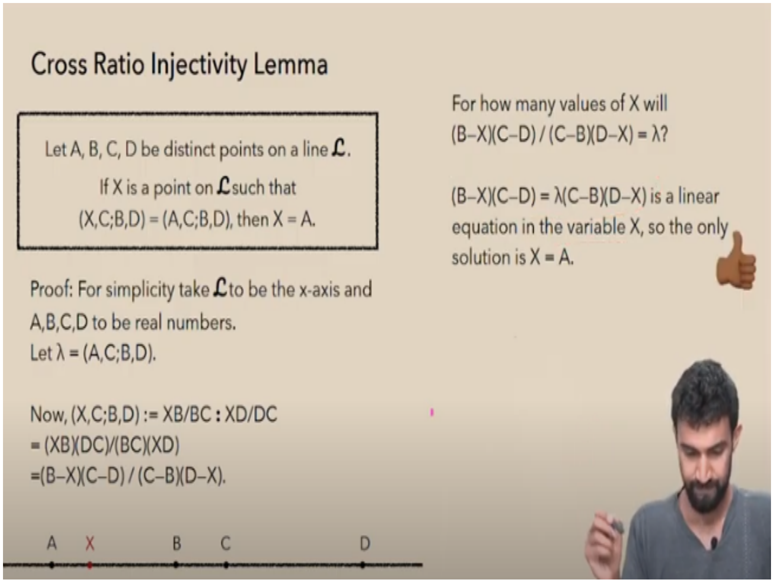

So let us look at the next lemma which is going to finish our activity here. So the cross ratio injectivity lemma states that if A, B, C, and D are distinct points on a line L, and if X is a point on L, such that (X,C;B,D)=(A,C;B,D). So in other words, if the cross ratio of X with C, B and D is the same as the cross ratio of A with C, B and D. Then in that case, X must be equal to A.

So the proof is as follows. So for simplicity's sake, let us take L to be the x axis. And let A, B, C and D be real numbers. And let λ denote the cross ratio of these four points. So we have A, B, C, and D on the real line, and their cross ratio is λ. And the claim is that if X is another point, whose cross ratio with B, C and D is also equal to λ, then X must equal A.

We cannot vary X. When we vary X, we have to change the cross ratio. And as we vary X around, we are never going to repeat values of the cross ratio. Every X is going to have its own cross ratio value. That is what this is saying. So how do we prove that? Well, by definition, this cross ratio with X is equal to (XB/BC)/(XD/DC). And what is that?

Well it is, we can rewrite this as (XB*DC)/(BC*XD). Since these are real numbers, I can write that as (B-X)*(C-D)/((C-B)*(D-X)). So for how many values of X will this quantity here equal λ? Remember, our assumption is that this cross ratio with X is equal to λ.

So as we vary X around on the real line, maybe it is here, here. We can vary X around. Wherever X is, its cross ratio is given by this expression. So when will this expression be equal to λ? For how many values of X will this expression be equal to λ? Well B, C, and D are constants. And we can multiply this out to get (B-X)(C-D)=λ(C-B)D-X).

And that is just a linear equation, right? Everything here is constant except for X. So this is a linear equation in one variable X. So the only solution is X equals A. You can see that the only solution here is when X is equal to A. And maybe, I should not say it in one line like that. Since it is a linear equation in one variable, there is exactly one solution. And what is that solution?

Well, there is at most one solution. And what is that solution? Well, what happens if X=A? We know that is already a solution. Because that was our assumption. And so that has to be the only solution. There can be no other solution. There exists one solution X= A, and that is the only solution. So therefore, X= A. So we are done.

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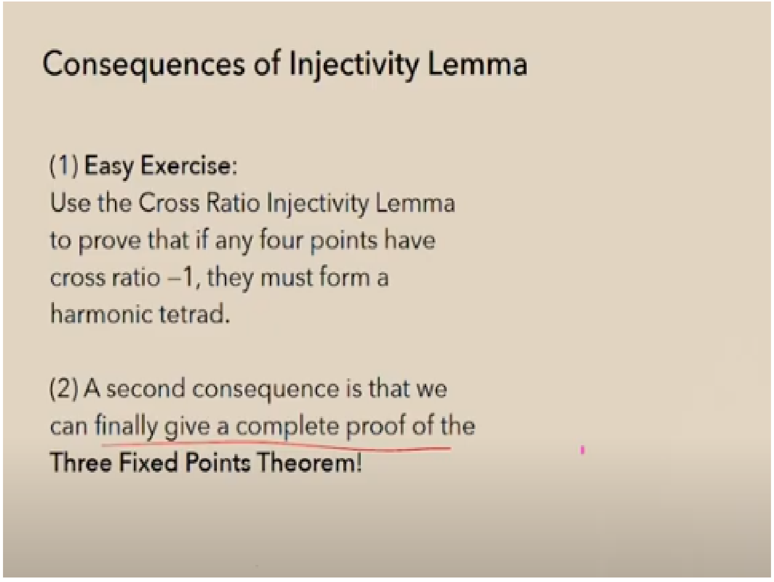

So what are the consequences of this injectivity lemma? Well, an easy exercise is the one we just mentioned. Since it is, by this lemma, if any four points have a cross ratio -1, they must form a harmonic tetrad. Why is that? Well, let us fix A, B and C, and let D vary. There has to be some value of D that forms a harmonic tetrad with A, B and C.

And there definitely the cross ratio will be -1. But that is the only place that will have cross ratio -1 with A, B and C. So therefore, no other point will have cross ratio -1. So if the cross ratio is -1, it has to be that special point, which forms a harmonic tetrad with A, B, and C. So that is easy exercise, I just kind of did it.

The slightly harder consequence, which is not an exercise we are going to see now is that we can finally give a proof of the Three Fixed Points Theorem. We saw two sketches of the proof previously. Now we will finally see a complete proof of it.

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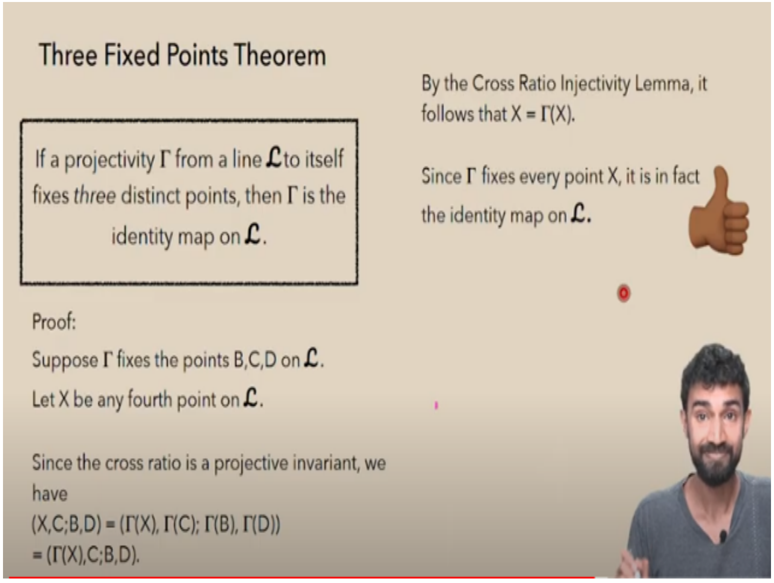

So remember the Three Fixed Points Theorem. What does it say? It says that if a projectivity **Γ**, from a line L to itself, fixes three distinct points, then **Γ** is the identity map. If it fixes three points it has to fix everything, it has to be the identity map. And how do we see that? Well, suppose the **Γ** fixes the points B, C, and D on L. Let X be any fourth point on L.

Now since the cross ratio is a projective invariant, we know that the cross ratio (X,C;B,D) is equal to the cross ratio (**Γ**(X), **Γ**(C); gamm(B), **Γ**(D)). Cross ratio is preserved under projectivities. But we also know that C, B and D are all fixed by **Γ**, they are fixed points.

So this cross ratio is just equal to the cross ratio of **Γ**(X) with C, B, D. But we just saw from the injectivity lemma that if X and **Γ**(X) have the same cross ratio value with the fixed C, B D, then X has to equal **Γ**(X). So by the cross ratio injectivity lemma, it follows that X=**Γ**(X). In other words that X is also fixed.

But now X is arbitrary. So Γ fixes every point X. It fixes everything. And it has to be the identity map on L. So that is done.

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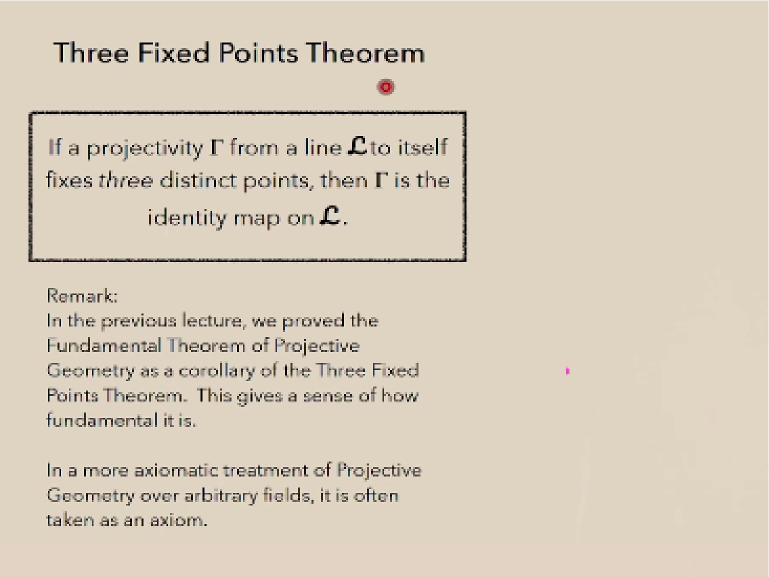

Finally, we have proved it using the cross ratio. And just as a remark, you know, we have seen three different proofs, I mean sketches. We have also seen many important consequences. We saw that the fundamental theorem of projective geometry is a corollary of the Three Fixed Points Theorem. And that gives us a sense of how fundamental it is to this field.

I just want to point out to people who are interested in more axiomatic treatments of projective geometry, over arbitrary fields, you can do projective geometry over complex numbers, or over finite fields. You can do it over in other settings. And in those settings sometimes this Three Fixed Points Theorem is just taken as an axiom.

It is over the real numbers however, where we assume some basic properties of real numbers that we can actually prove it directly in different ways, as we have seen. So that is the Three Fixed Points Theorem, which is sometimes even just an axiom of projective geometry. It is so fundamental.

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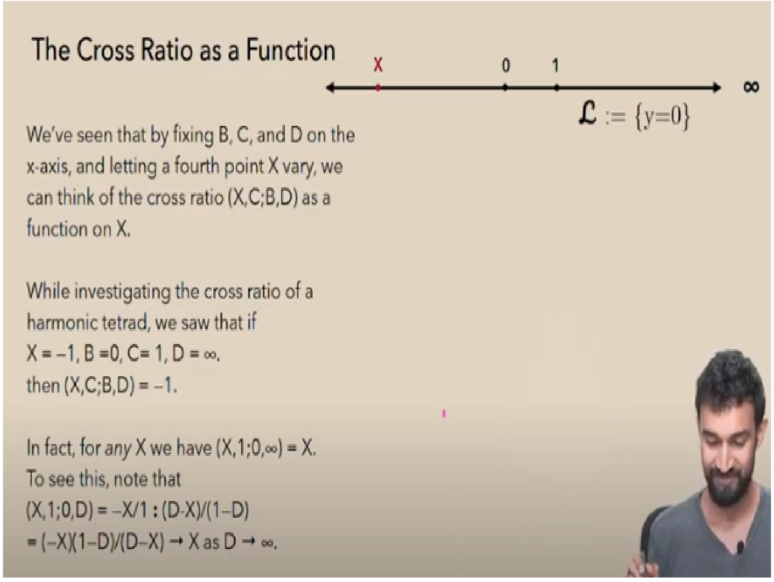

And I want to conclude the lecture today with one last discussion on the cross ratio. Thinking of the cross ratio as a function, which we have just seen, but I want to clarify what we have done with that. So we have seen that by fixing B, C and D on the x axis, and letting the fourth point X vary, we can think of the cross ratio as a function of X.

And while investigating the cross ratio of the harmonic tetrad, we saw that if X= -1, B=0, C=1 and D is equal to infinity, kind of like we are seeing over here, then the cross ratio (X,C;B,D) is equal to -1.

The cross ratio of X with 0, 1 and infinity as we are seeing in this picture here, will always give us back X. And to see this, we can just calculate it. Note that this cross ratio (X,1;0,D) is equal to ((0-X)/1)/((D-X)/(1-D)).

We can simplify this to -X(1-D)/(D-X). Letting D go to infinity what does that give us? I will just write it so it is a little easier to see what this is saying, -X(1-D)/(D-X).

But remember X is fixed right now in this discussion. D is going to infinity. So as D goes to infinity, (1-D)/(D-X) just goes to -1. So we end up with (-X)(-1), as D goes to infinity. So that just goes to X as D goes to infinity.

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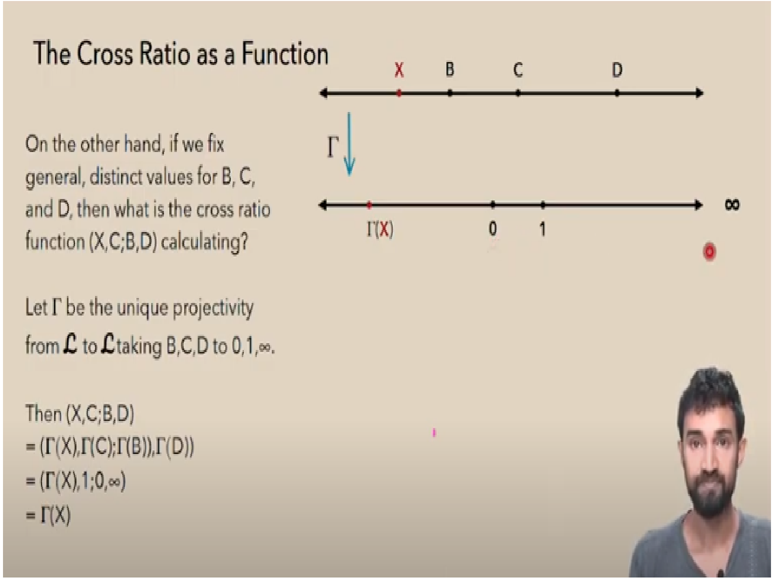

So if B, C, D are 0, 1 and infinity, then taking the cross ratio of X with these guys just recovers X. What if B, C and D are just B, C and D? We do not know what they are, they may or may not be 0, 1 and infinity. Then what is the cross ratio as a function actually calculating? B, C and D are fixed. X is varying.

What is this function calculating? Is there a geometric content to that? And they kind of are. So let **Γ** be the unique projectivity from L to itself that takes B, C and D to 0, 1 and infinity. Well X is not going to go to some number **Γ**(X). But we know that the cross ratio (X,C;B,D) is preserved under projectivity.

So that is equal to the cross ratio over here, (gamm(X), gamm(C); gamm(B),gamm(D)). And that, well here the **Γ**(C) is 1. **Γ**(B) is 0 and **Γ**(D) is infinity. So that is just the cross ratio of gamm(X) with 0, 1 and infinity, which we have just seen is just recovering gamm(X). So in some sense, what is the cross ratio, measuring here (X,C;B,D)?

It measures the image of X under the unique projectivity that takes B, C and D to 0, 1 and infinity.

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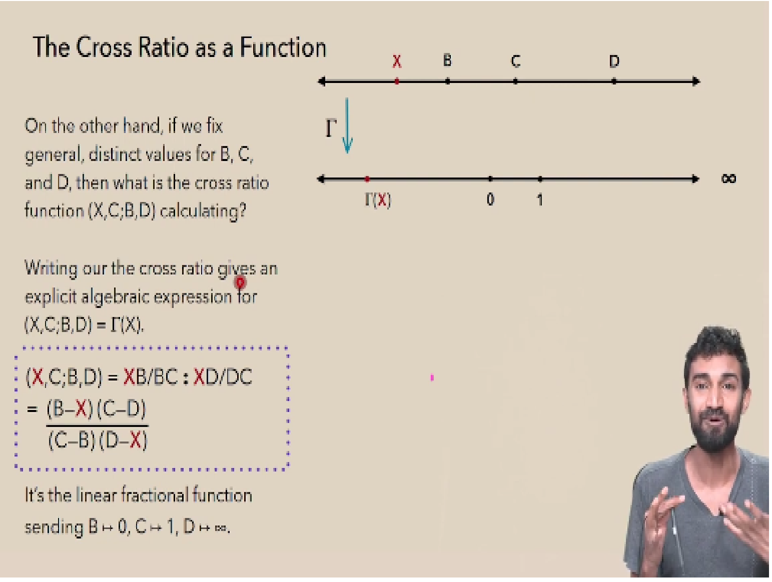

And there is just another way to see this. We can write our cross ratio function explicitly and get an algebraic expression for this function, **Γ**(X). So to do that, just write it out (X,C;B,D). It is by definition equal to(XB/BC)/(XD/DC).

This is (XB*DC)/(BC*XD). It is equal to (B-X)*(C-D)/((C-B)*(D-X)). What is that? Well, it is a linear fractional function. B, C, and D are all constant. X is a variable.

But it is a special one. It is the one that sends B to 0, because if we plug in B for X, we get zero, correct? Imagine that you plug in B for X over here. And we get B - B on the top, so that goes to 0. What if we plug in D? Well, if we plug in D for X, we are getting a D down here for X. D - D is 0. So we get a 0 in the denominator. So that goes to infinity.

And if we plug in C, then we get B - C here, which cancels out with this C - B here, and we get D - C here, which cancels out with a C - D here, kind of cancels out. So we get (-1)(-1) which is 1. So C goes to 1. So it is a very special linear fractional function. It is the unique one that maps B to 0, C to 1, and D to infinity. So as a function, that is what the cross ratio is doing.

It is kind of bringing us to this common frame, this common projective frame consisting of the points 0, 1 and infinity and seeing where our fourth point sits with respect to that frame. That is kind of how I like to think about it.

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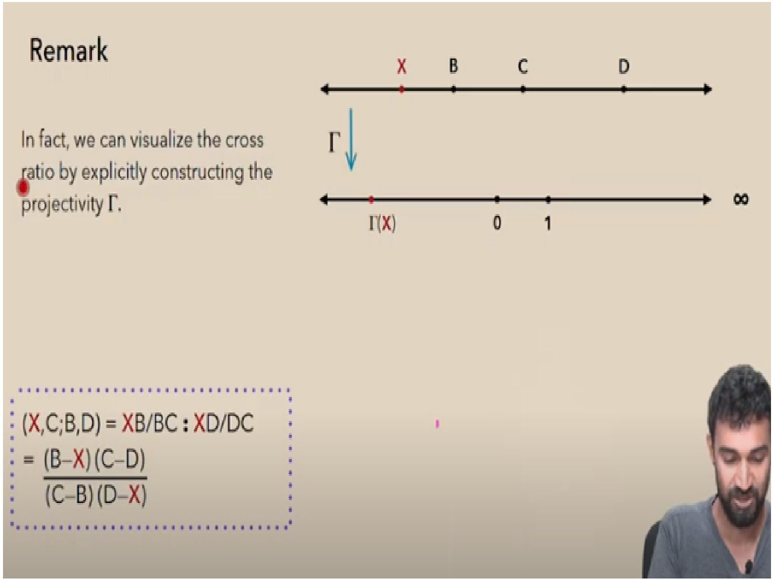

And we can visualize this cross ratio function explicitly by constructing the projectivity **Γ**.

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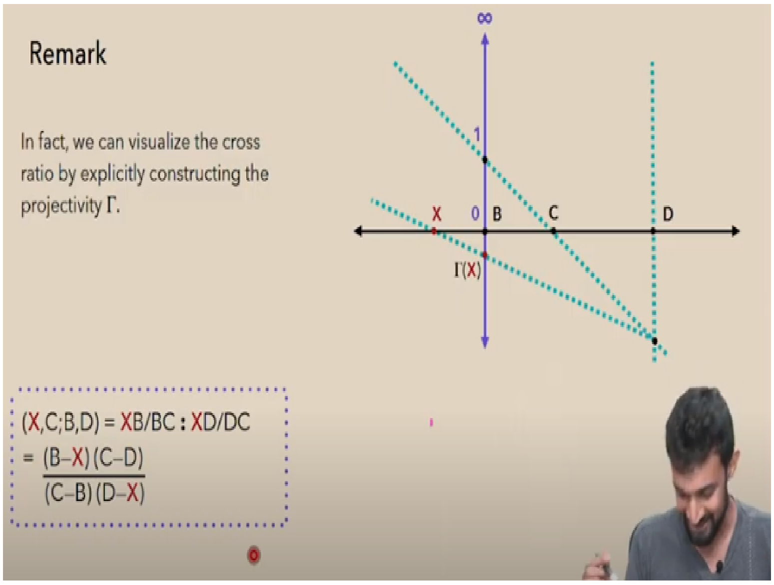

And it looks something like this. Here is X, B, C, and D. And I am taking another line, a vertical line with 0, 1 and infinity. And I am just taking my line here, and positioning it so that B lines up with 0. And I am drawing a projectivity in fact a perspectivity, that center here, which sends C to 1, and sends D to infinity. In other words, it is mapping B, C, and D to 0, 1 and infinity, this perspectivity.

And it is relating this point X to this point **Γ**(X). So that is how we can visualize the cross ratio as a function. So that is kind of the best geometric visualization I can give. I know it is not crystal clear, it is still a bit confusing, the cross ratio. There is something kind of inevitable about that. There is something very algebraic about it. And it is hard to see it geometrically, even though our brains do see it geometrically, which is something that is kind of surprising about it.

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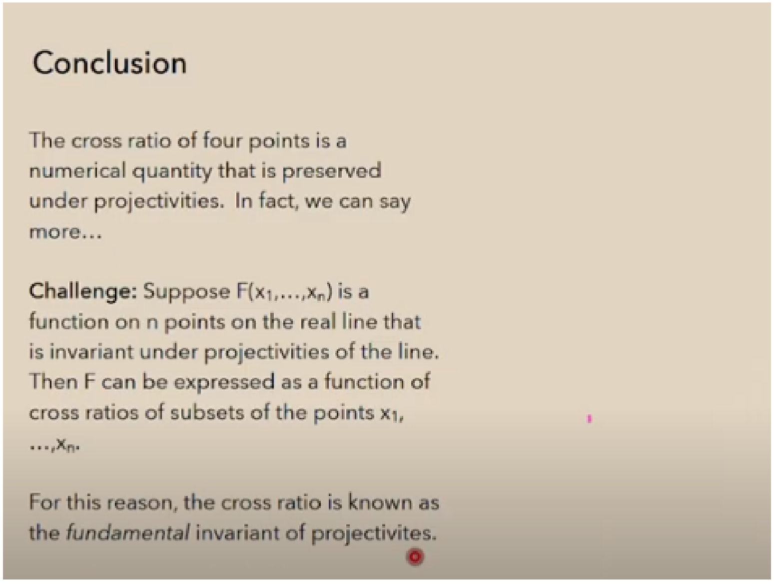

So in conclusion, the cross ratio of four points is a numerical quantity that is preserved under projectivities. And in fact, we can say even more.

It is really kind of the most fundamental invariant in the sense that if we have any other function F of n points on the real line, which is invariant under projectivities, meaning that that function when we have taken on $x_1, x_2, ..., x_n$ will be the same value that we get if we apply the function to $\mathbf{\Gamma}(x_1), \mathbf{\Gamma}(x_2), ..., \mathbf{\Gamma}(x_n)$ for some projectivity **Γ**. In that case, F can be expressed as a function of cross ratios of subsets of the points $x_1, x_2, ..., x_n$.

So we can rewrite this invariant F as a function of cross ratios. So it is the most fundamental invariant of projectivities.

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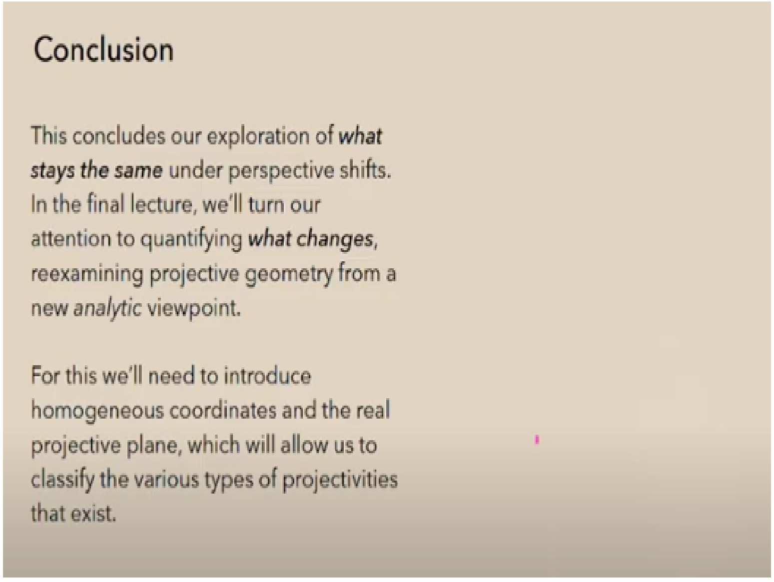

And that really concludes our exploration of what stays the same under perspective shifts. And in the final lecture, we will turn our attention to quantifying what changes when we shift perspective. And we will do that by reexamining projective geometry from a new analytic point of view. So in order to do that, we are going to have to introduce homogeneous coordinates, and something called the real projective plane, which is an example of a manifold.

And doing so will allow us to actually classify and get a much more concrete sense of various types of projectivities that exist. So I will see you then. And that is all for now.