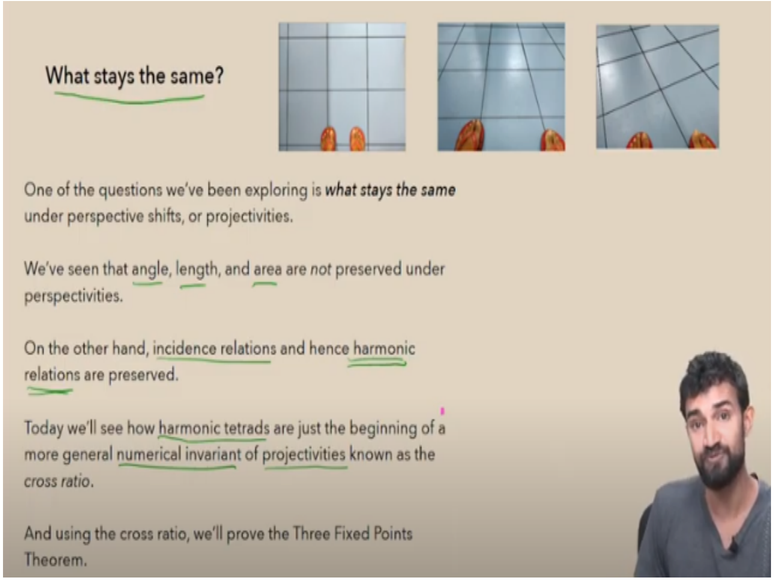
Our Mathematical Senses
The Geometry Vision
Prof. Vijay Ravikumar
Department of Mathematics
Indian Institute of Technology- Madras

Lecture - 13
The Cross Ratio

Hi, welcome back to the Geometry of Vision. This is lecture seven, on the cross ratio.

(Refer Slide Time: 00:20)



So one of the first questions that we asked in this course in the very first lecture was when we think about perspective shifts, perspective shifting maps, changes in perspective, what is changing and what is staying the same. So today, I want to focus on that second question of what stays the same. And we will finally arrive at a complete answer to that question. So we have noticed multiple times that angle, length and area are not preserved under perspectivities.

On the other hand, incident's relations, line and the property of being collinear, that is preserved. And hence, we saw that harmonic relations are preserved, harmonic tetrads. When you have four collinear points that are arranged in a harmonic tetrad, meaning that there exists a quadrilateral for which the two of those four points are vanishing points of the sides of the quadrilateral, and the other two are the vanishing points of the diagonals of the quadrilateral.

That is a harmonic relation that we have learned about, in several lectures back. So we see that that harmonic relation is preserved. But that is a bit complicated. And today, we will see that harmonic tetrads, are really just the beginning of a more general numerical invariant of perspectivities and hence of projectivities.

And that numerical invariant is called the cross ratio. So using the cross ratio, we are going to finally prove the three fixed points theorem, as well as a few other results.

(Refer Slide Time: 02:06)

So I want to introduce the cross ratio via a problem, an exercise, which I found in the YouTube channel Numberphile, it is due to the mathematician, Federico Ardila. And the problem is that here is an actual photograph from 1997. It is a football match Brazil vs. France, and Roberto Carlos is the person you can see right here.

Let me get my laser pointer, this person here, who is kicking the ball, and he is about to kick the ball in this photograph. It turns out, this is the winning kick of the game. I mean, this goal is going to allow Brazil to win the game. But this photograph is taken before the kick even happens, just before. And the question here is how far is the ball from the goal just before it is kicked by Roberto Carlos?

So can we figure that out using only this photograph and whatever other basic information we have about the layout of the football field. So in particular, we have this photograph.

(Refer Slide Time: 03:14)

And we also know the schematic diagram of the football field. We know in particular, that from the goal line, to this first line here is 5.5 meters. And I am not a huge football expert, I do not know the technical terms for these lines. So forgive me for that. But this is the goal line where we are trying to get the ball to. This is the line just after that, which is 5.5 meters from the goal.

The next line out is 11 meters. So going back to this picture, the goal line is this line here. And the next line out is over here. It is the line that we are seeing over here in the photo. We are also seeing this circular arc in the photograph. That is this line here, but we are going to ignore that. We are not actually going to use that for our calculation. So the significant lines are these three lines here.

And we know that the football field has these dimensions, 5.5 meters to here, 11 meters to here. And what we want to know is how many meters to the ball. So the ball is right here, the football, and it is about to be kicked, and it is some distance from this line here. So the question is, how many meters is it from this line here?

And can we figure that out from just this information, the photograph, and this top down bird's eye view of the football field. These are the only two numbers that we are kind of given. So what is x? We want to solve for x. That is our task here.

(Refer Slide Time: 04:50)

So there is one more piece of information. I actually took this photograph.

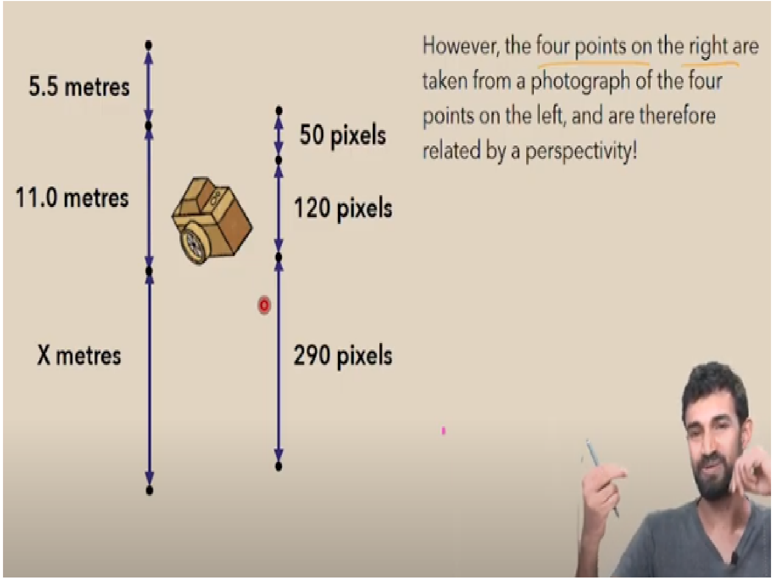
So I am doing this, in case you want to, there is a slightly different treatment on The YouTube channel Numberphile that they do. I am actually doing it in a slightly different way with a slightly different definition of cross ratio, and different measurements. So anyway, just want to mention that.

So there is another piece of information we have, which is that I actually took this photograph and I just measured it on my computer using a free open source software called GIMP. So that allows you to measure, I mean, it is like a free version of Photoshop. But on it, you can actually take measurements of the number of pixels between two points.

So in this photograph, this digital photograph, digital version of the photograph, I measured that between these first two lines, it is 50 pixels, between these two lines, it is 120 pixels. And between these two lines, it is 290 pixels. So I just did a quick measurement on my computer screen using that software, which told me this.

There are probably lots of other free apps that will let you do these kinds of measurements on photographs. So it is something you can also try on your phone, if you just download the right app.

(Refer Slide Time: 06:01)



So what do we have? So it is kind of tempting, you might be tempted. In fact, most people when they see this problem, they are tempted to try something. We have these measurements here, 5.5 meters, 11 meters, and x meters. And we have these measurements here 50 pixels, 120 pixels, 290 pixels of exactly the same, you know the lines in the photograph here correspond perfectly to the lines in this diagram here.

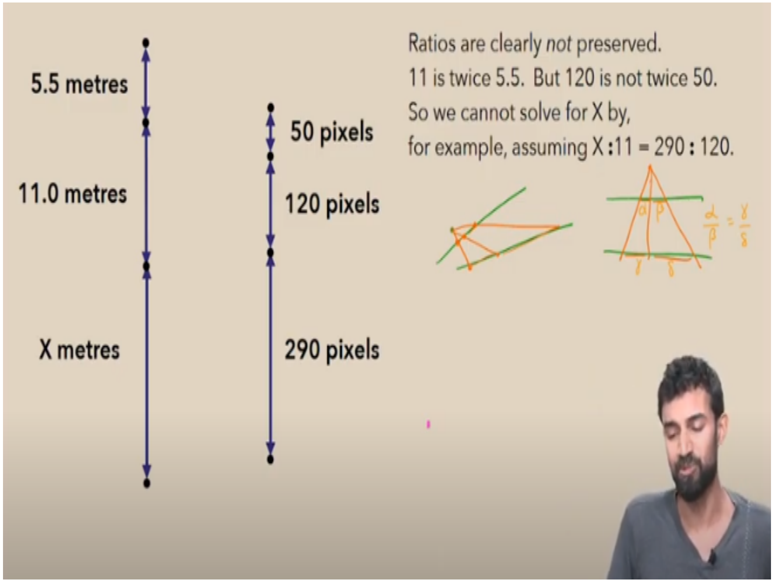
And we know the measurements in pixels here. We know the measurements in meters here, and we have to solve for x. So it does not seem too difficult. And maybe you are tempted to just say okay, well we have, you know this is our setup. And probably it is just proportional. So you know we can just use similar triangles or something or similar lengths, similar ratios.

But quickly when we start examining it we see that that is not the case, we cannot just take a simple ratio, because ratios are not preserved. We have seen that before. But it is good to remind ourselves of it here. And we can see it very clearly, because 11 is equal to two times 5.5, twice 5.5 is equal to 11. However, looking over here, 50 is clearly not half of 120.

So ratios are very evidently not preserved in this scenario here. So we cannot simply solve for x by, for example, assuming that the ratio of x to 11 is the same as the ratio of 290 to 120. x/11 is not equal to 290/120. So we cannot just try that kind of naive approach, although that is kind of the first thought that most people might have.

And maybe just a quick reminder, we have seen this before, but I think it is a good thing to remind ourselves of, because it is a bit counterintuitive that perspectivities do not preserve ratios.

(Refer Slide Time: 07:56)



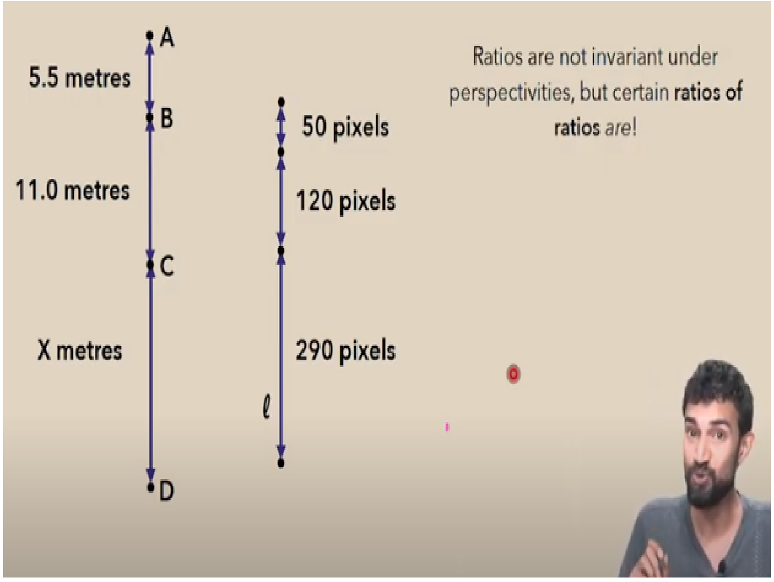
You know even a simple perspectivity of two lines like this, it is not going to preserve ratios, the ratio of this distance to this distance is not the same as the ratio of this distance to this distance, except we have seen one situation where it does preserve ratios. But it is only in this special situation where these two lines are parallel. Then if we have a perspectivity like so then we do end up preserving ratios.

But only in that case. Let us call this α and β, and this γ and δ. So only in this special case, the ratio of α/β is equal to the ratio of γ/δ, where these two green lines are parallel. Otherwise perspectivity in general, will not preserve ratios.

So that is the problem we are facing here. Because we do have a perspectivity here. The four points on the right are taken from a photograph of the four points on the left. These four points here are from a photograph of these four points here. And photography is a perspectivity. We have talked about that and seen that. So these points are indeed related by a perspectivity.

However, it is not a perspectivity between parallel lines in space, so it does not preserve ratios.

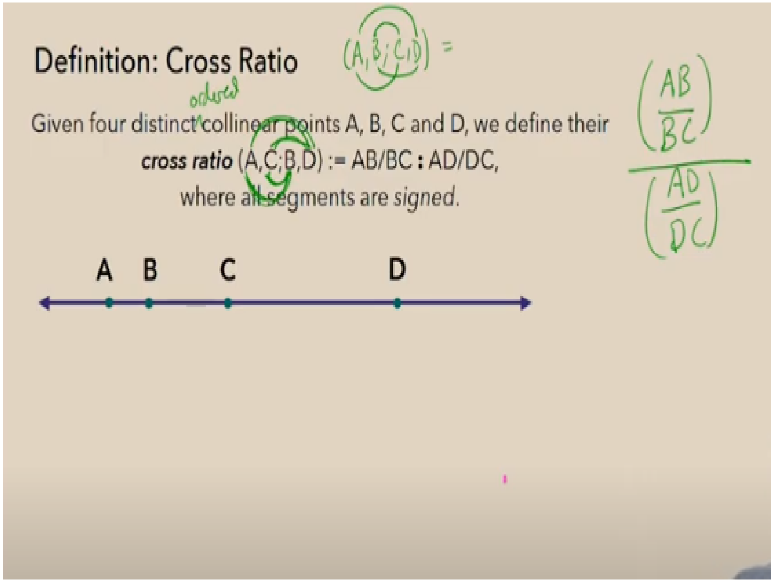
(Refer Slide Time: 09:35)



So that is our problem. Ratios are not invariant under perspectivities. However, and this is the key idea we are going to use today, certain ratios of ratios are invariant under perspectives, which is very surprising and to most mathematicians, and yet at the same time, it was known to Pappus, the mathematician Pappus all the way back in 340 AD.

So it is not a new piece of information. However, it is a fairly revolutionary piece of information in various ways.

(Refer Slide Time: 10:08)



So let us now define the cross ratio and use it to solve our football photo problem. So the definition of the cross ratio is the following. And I am going to repeat it a few times, because it takes a while to wrap your head around it and get it to stick, at least for me. It is a very algebraic definition. And it is hard to kind of see it geometrically.

So given four distinct collinear points, A, B, C and D, the cross ratio is always taken of four collinear points. It only makes sense to take the ratio of four points when they are collinear, all on one line. So given four collinear points, A, B, C, and D, we define their cross ratio, (A,C;B,D) to be the following ratio of ratios. It is (AB/BC)/(AD/DC). Let me just write that in a slightly less clunky notation.

So what I am saying is it is AB/BC. I kind of like my device for remembering it, we are looking at AB/BC. We go from A to B, and then back to C. We are always crossing over this middle semicolon. So AB/BC. That is the first ratio. And that is divided by, all divided by AD/DC.

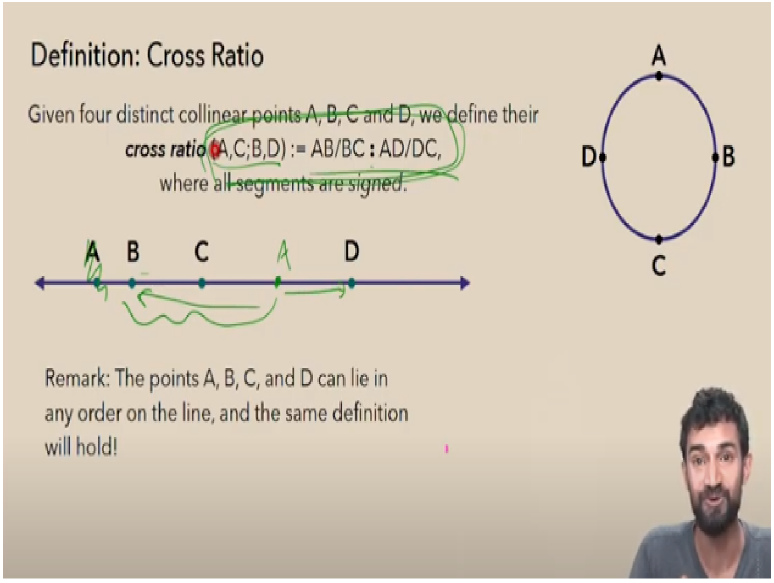
So it is a ratio of ratios. That is the cross ratio. (AB/BC)/(AD/DC). So it takes a while to remember. By the way, I have written it with this strange order A, C; B, D. My points are arranged in alphabetical order A, B, C, D, but I am writing the cross ratio, (A,C;B,D). Not everyone does that.

You might look this up on Wikipedia or somewhere else, you might see a different definition. Some people do use this, it is not that I am randomly using it. But people are kind of divided. Some people use this, some people will just write A, B, C, D. I am using this because it has slightly nicer geometric properties. And I like to kind of think of things geometrically and visually.

So even though it is a little bit weird, it looks a little weird here, we will see later that it actually has a payoff to write it this way. The cross ratio is always of four ordered points. Maybe I should have written that, given four distinct collinear points, A, B, C, D and an ordering on them. So let us say four distinct, ordered collinear points.

Because this cross ratio I am taking here, is not the same as other definitions. Some people will use this other definition for the cross ratio (A,B;C,D), and it is(AC/CB)/(AD/DB). So it is the same thing. So today, and in this class, we are always going to use this ordering, (A,C;B,D).

(Refer Slide Time: 13:41)



So another mnemonic device, a kind of another way to remember this is the following, I kind of use this sometimes. I just arrange the points in a circle A, B, C, D, and the cross ratio (A,C;B,D), it is just, first of all, I am putting in my notation here, A and C are in one partition, B and D are in the other partition. So in the cross ratio, you are always partitioning your four points into two sets.

So A,C is one of my sets, and B,D is my other set. And I have arranged them in the circle in such a way that A and C are on two sides, and B and D are kind of other antipodal points. So they are crossing in that way. So I am looking at A and C and I am looking at B and D. And my cross ratio is (AB/BC)/(AD/DC).

So sorry, I am hammering this in because otherwise it is a little hard to remember. And once we start doing calculations, it is easy to get lost. So I just like to use this as a device to remember the cross ratio of my four points A, B, C and D. I write them in a circle, A, B, C and D, and it is (AB/BC)/(AD/DC). And the other important thing to note is that these segments are all signed.

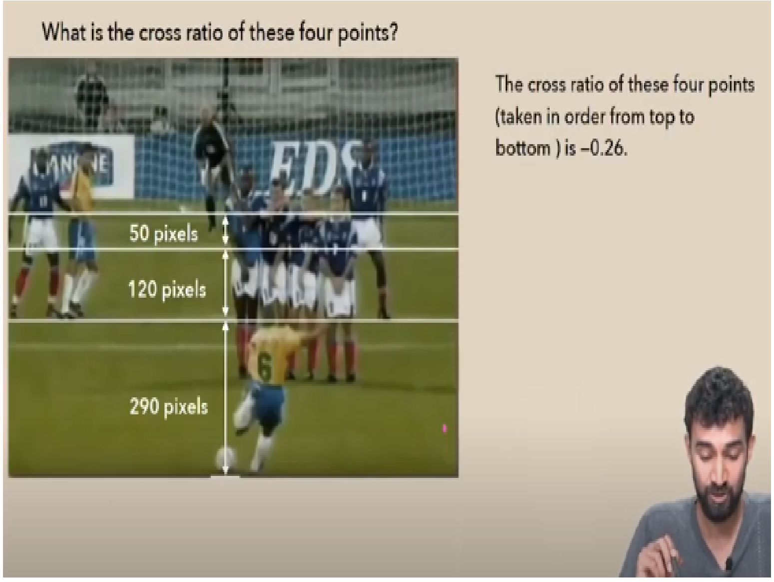
So AB is positive, AB is positive, BC is positive. AD is also positive. But DC is going to be a negative length, a negative value, because we are going from a point here D on the right to a point C on the left. So we are thinking of all of these as signed lengths. So that is the other important thing to remember. So I think it will become clearer now when we start using it.

So let us start using it. And just the other remark is that these A, B, C, and D can lie in any order on the line, and the same definition will hold. So I do not have to have my points A, B, C and D lie in this order. I could have had, A might be here. And I could still take the cross ratio of A, B, C and D, and it would still be calculated this way.

Only some of my signs would change. So now AB is going from here to here. And that is going to be a negative length. BC is still positive, AD is still positive, and DC is negative. So the signs change depending on the placement. But the formula is still the same, the formula never changes. So this is our formula for the cross ratio. It does not matter how the points are arranged in the line.

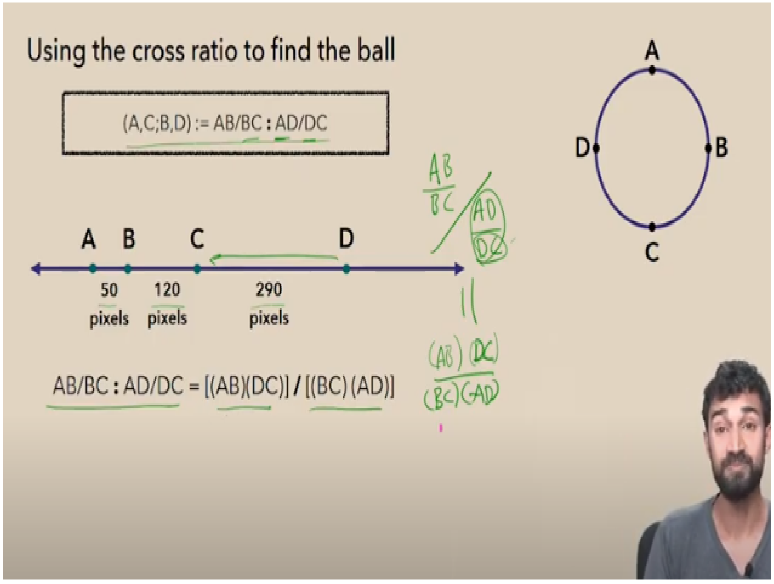
So just remember this formula here and you will be in good shape. It is just that it is a little tricky to remember sometimes. That is why I brought in this circle here. Let us get back to our football photo and try and figure out how far the ball is from the goal.

(Refer Slide Time: 16:31)



In other words, how far is the ball from the goal? So let us do that by calculating the cross ratio of these four points. What are the four points? We have a point here, a point here, a point here, and a point here. And I want to know the cross ratio of these four points.

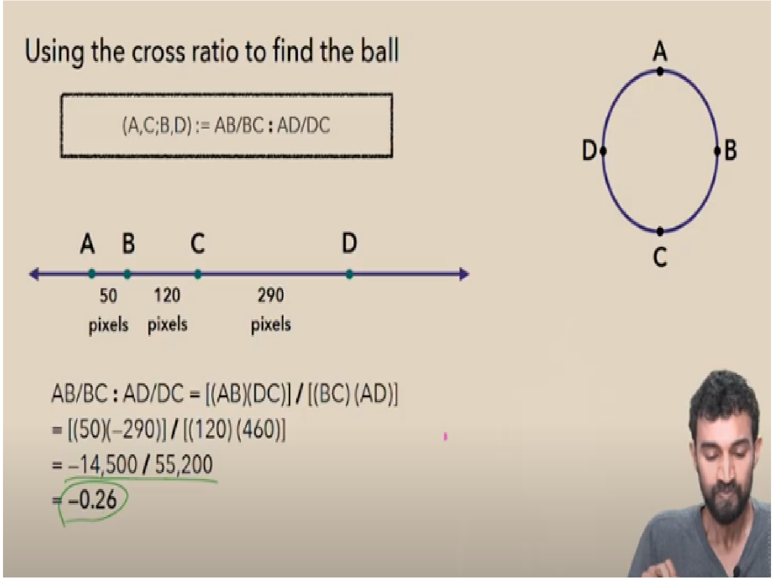
(Refer Slide Time: 17:00)



So let us use the cross ratio to find where the ball is. So remember, our formula is here. And let us calculate the cross ratio. So we have AB is 50. BC is 120. AD is 50+120+290. And finally, we have DC, which is going in this direction. So that is -290. So we have a negative number there. Okay, so let us solve that. Let us calculate the cross ratio.

So first of all, just notice that I can write it like this (AB/BC)/(AD/DC). But then I can actually take the reciprocal of the denominator and combine these two fractions. So this is actually equal to (AB*DC)/(BC*AD).

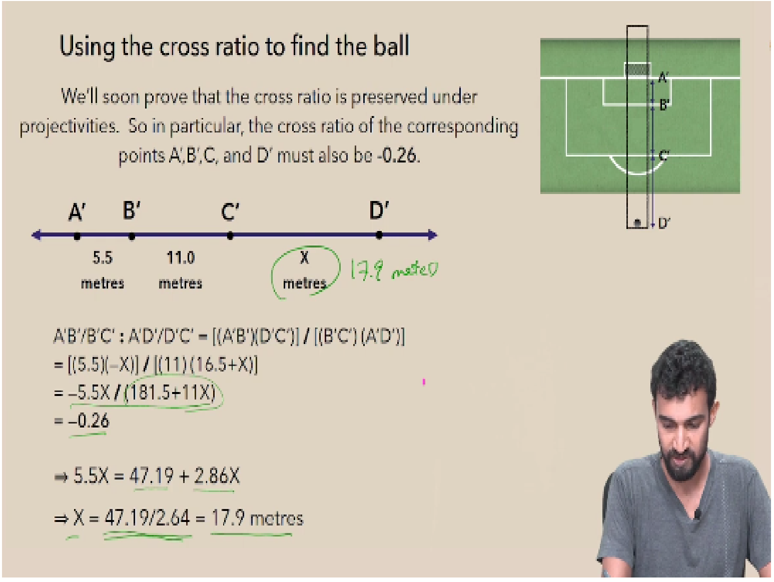
(Refer Slide Time: 18:24)



Now let us substitute in the values of these lengths. AB is just 50. DC, which is our second term here, is -290. BC is 120. And finally, AD is the sum of 50+120+290. So those of you who can quickly add up numbers will know that that is equal to 460. Just putting them into your calculator, you will get that this is -14,500/55,200, which is just equal to -0.26, close to -0.26.

So that is the cross ratio of these four points taken in order from top to bottom. It is -0.26.

(Refer Slide Time: 19:13)



Okay, how does that help us find the ball? Well, I alluded to it, but I have not, we have not proved it yet. But we are going to use it anyway. So the cross ratio, the big important thing about the cross ratio is that it is preserved under projectivities. In this particular case, the cross ratio of the corresponding points in this diagram here, the points A', B', C', and D' must also be -0.26.

In this line here we have these four collinear points coming from my schematic diagram of the football ground. And we know that A'B' is 5.5 meters. B'C' is 11 meters, and C'D', we do not know. If we knew that we know how far the ball is from the goal. So let C'D' be x meters.

And the cross ratio, since it is preserved under perspectivities, in particular photographs, it must also be -0.26. So let us calculate the cross ratio of these four collinear points. And using that we will solve for x. So what do we have for the cross ratio? Well, going back to the definition, it is (A'B'/B'C')/(A'D'/D'C').

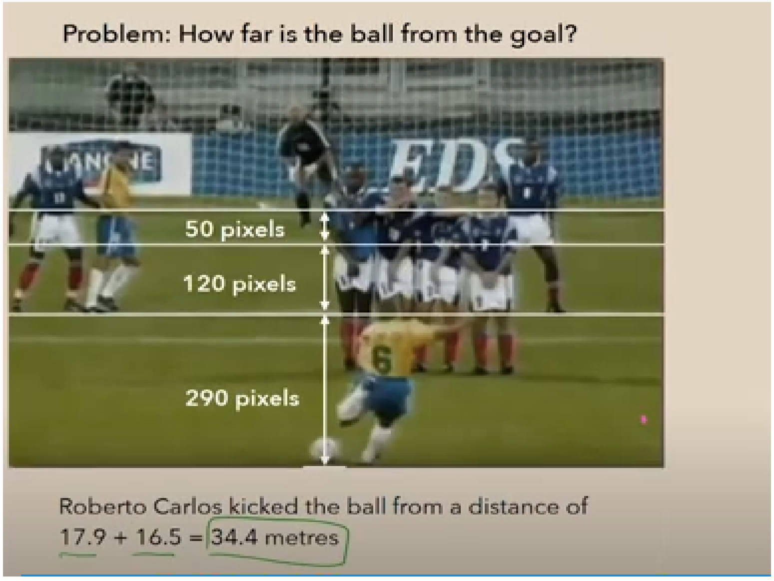
And like I did last time, that works out to (A'B'*D'C')/(B'C'*A'D'). So this is my quantity.

We know that A’B’ is 5.5 meters. So this is 5.5. We know that D’C’ is -x meters. We know that B’C’ is 11 meters. And we know that A’D' is 5.5+11. That is 16.5+x meters. So we have 5.5(-x)/(11(16+x))= -5.5x/(181.5+11x).

Okay, so we know that since the cross ratio is preserved, we have not proved it yet. But assuming the cross ratio is actually preserved under photography, under perspectivities, this had better be equal to -0.26, because that was the cross ratio from the pixels on the photograph. And let us solve for x.

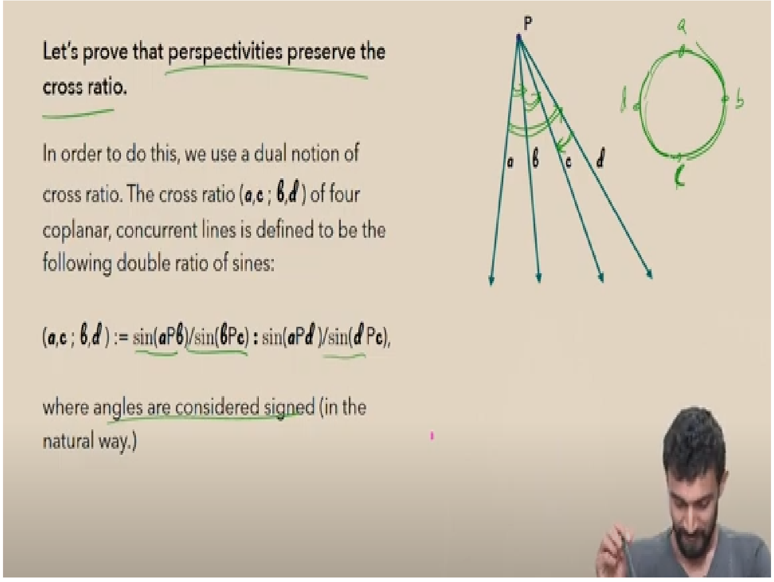
I am just, let us just multiply both sides of this equation by the denominator here. So we get that 5.5x is equal to 47.19+2.87x. And now solving for x, we get that x is equal to 47.19/2.64, which so obviously I am not doing this in my head, I did this earlier on a calculator. So you can check this for yourself, just check that I am not making a mistake here. But if you can verify it, we get that this is equal to 17.9 meters. X is equal to 17.9 meters.

(Refer Slide Time: 22:53)



In other words, Roberto Carlos kicks the ball from a distance of 17.9+16.5 meters. So that is 34.4 meters. So that is pretty cool. So from this photograph, and from this diagram here, we can actually recover the distance the ball is from the goal, just before it has been kicked. So that is an application of the cross ratio. And you can see that it is quite powerful.

(Refer Slide Time: 23:21)



So we obviously used a significant fact, which we have not proved, which is that perspectivities preserve the cross ratio. So let us prove this fact, let us see how we can actually prove that perspectivities preserve the cross ratio. In order to do this, we need to introduce another dual notion of the cross ratio, another cross ratio. The second cross ratio is not a cross ratio of points, it is a cross ratio of lines.

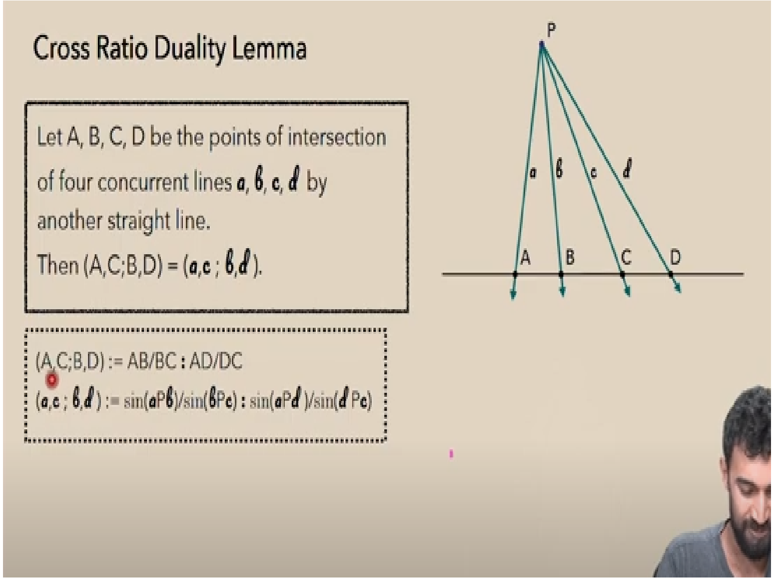
So we are going to just define a cross ratio (a,c;b,d) for given four coplanar, concurrent lines, a, b, c and d. We define their cross ratio (a,c;b,d) to be the following double ratio of sines of angles. Three or more lines are concurrent means that they all share a common point of intersection.

So if these four lines a, b, c and d are concurrent, that means they all meet at a point P. So given four coplanar concurrent lines, a, b, c and d, their cross ratio (a,c;b,d) is defined as follows. It is (sin(aPb)/sin(bPc))/(sin(aPd)/sin(dPc)). And let me actually draw these arrows.

I should have drawn all these as arrows because they are signed angles in this natural way, just like we used signed lengths earlier, we are using signed angles now. And yeah it is really up to you, how you want to remember this. Again, you can use that mnemonic device I mentioned earlier. We are looking at (sin(aPb)/sin(bPc))/(sin(aPd)/sin(dPc)).

So it is the exact same ordering that you have been using for the other cross ratio. So nothing new here. It is just that the only difference in the definition is that instead of measuring signed distances between points, we are measuring signed angles between lines and taking their sine.

(Refer Slide Time: 26:05)

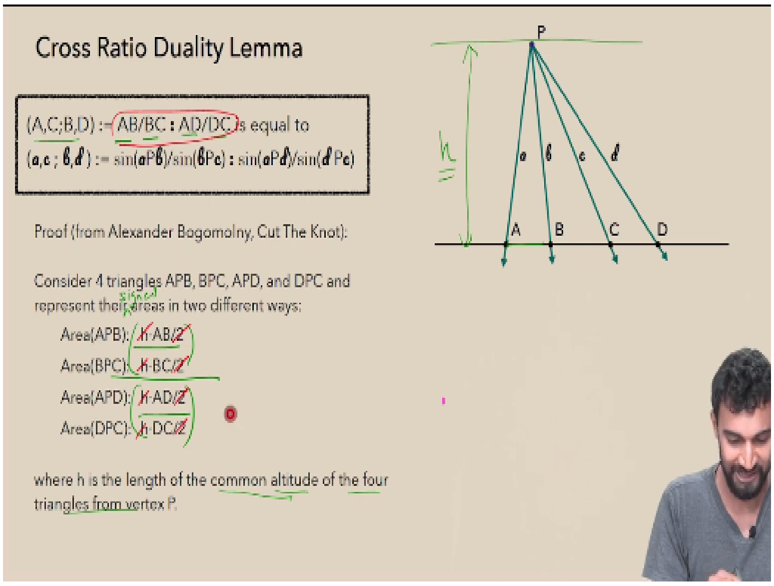


So hopefully it is not too much more to remember. And now we have the cross ratio duality lemma, which is essentially saying that these two definitions are strongly related to each other. So let A, B, C and D be the points of intersection of four concurrent lines a, b, c, and d with another straight line. I have not even given it a name, it is just some other straight line. I guess if we want, we can call it l.

But it does not really matter. So we have another line l here. And A, B, C and D are the points of intersection of the lines a, b, c and d with l. So that is our setup. In this scenario, the cross ratio (A,C;B,D) is equal to the cross ratio (a,c;b,d) of lines. The cross ratio of points is equal to the cross ratio of lines. That is what the cross ratio duality lemma states.

Let us just keep our formulas here for handy reference. I would not repeat them again. But we will keep them on the screen up here.

(Refer Slide Time: 27:15)



And how do we prove this? Well, there are multiple proofs of this. There are actually many different proofs. I am using a proof, again from the internet from the website, Cut the Knot. This is compiled by Alexander Bogomolny. And the proof is quite neat. It goes as follows. We are going to relate these two cross ratios by comparing them to a third cross ratio actually.

This is a cross ratio of points. This is a cross ratio of lines. And we can think of this third thing as a cross ratio of triangles anyway. Let me not put it that way. I want to just rephrase this. Let us not introduce a third cross ratio. That is way too confusing. And it is not actually what we are doing.

A better way to say this, we are going to take four triangles. And we are going to think about their area. And we are going to see that their areas relate to both of these cross ratios in very nice ways. So what are our four triangles? Well, I want to consider the triangles that correspond to these angles that we just used in our cross ratio. So the first triangle APB, next triangle BPC, next triangle APD, the big one.

And finally the triangle DPC. Those are the four triangles I want to think about. I want to think about their areas. Well what are their areas? We actually have two common ways of computing the area of a triangle. One comes, you probably learned one of them in grade school, early grade school, which is that area of a triangle is one half its base times its height.

And maybe later in high school, you learned another formula for the area of a triangle using trigonometry. Using the sine, maybe you learned it from the law of sines, but basically relating the area of a triangle to the sine of any one of its angles. So let us first look at the first way we talked about, that you have probably learned which is to relate the area to one half base times height.

In that case, the area of APB is going to be one half of AB times the height. And when I say height, I mean this height h. All these triangles share a common height. So let h be the common altitude of the four triangles from the vertex P. So this is h. So the area of APB is (h*AB)/2, the area of BPC is (h*BC)/2 the area of APD is (h*AD)/2.

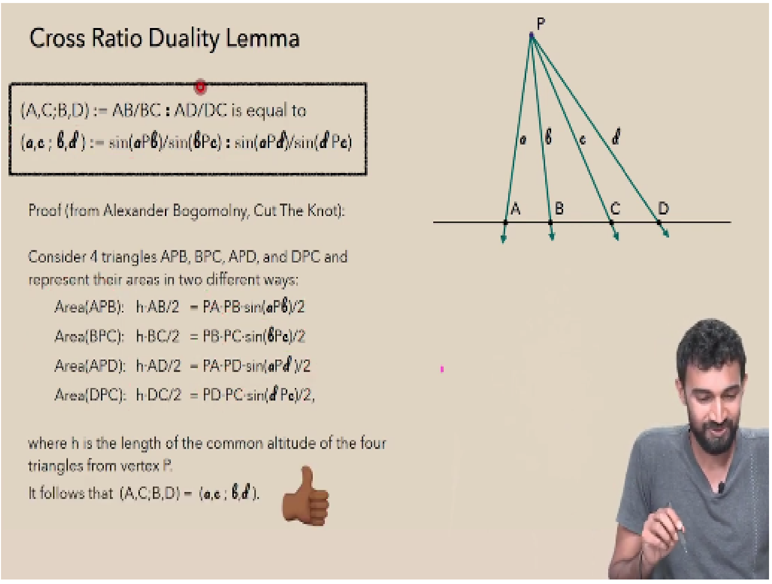
And finally the area of DPC is (h*DC)/2. I do want to think about these as signed areas. So just, we do want DC to be negative, so this is going to be a negative area we get here. So these are actually signed areas. They are signed areas in two different ways. Okay, but how is this helping us? Well notice, we can actually do something kind of nifty here.

We have written exactly the ratios, we see exactly the ratios that we need for our cross ratio, exactly the lengths that we need. Our cross ratio of points is (AB/BC)/(AD/DC). And that is kind of what we have here. We have (AB/BC)/(AD/DC). We actually have the cross ratio written right here, literally, because these h's cancel on the numerator.

In the denominator also these h's cancel. Similarly, all these one halves cancel. And we are left with exactly the cross ratio here. (AB/BC)/(AD/DC). That is exactly the cross ratio. So we can see the cross ratio. So in fact that cross ratio is equal to that is why I mentioned there is a third cross ratio. We can actually think of the cross ratio equivalently as being ((area of APB)/ (area of BPC))/ ((area of APD)/(area of DPC)).

So that is also a kind of a third way of thinking about the cross ratio. But we do not want to, I do not want to define a third cross ratio. We already have two cross ratios, that is more than enough. So this is one way of writing these areas. But now we have our trigonometry way or slightly more advanced High School trigonometry way of writing areas of triangles.

(Refer Slide Time: 32:35)



We can also write the area of APB as PA*PB*sin(aPb)/2. That is the area of this triangle. Similarly the area of the next triangle is just these two lengths times this angle over 2. The big triangle, this length times this length times the sine of this angle over 2.

And finally the last triangle, it is this length times this length times the sine of this angle all over 2. So we have these other ways of writing it. And now what can we do with these quantities here? We will look once again, we actually have our cross ratio of lines right here under our noses. We have (sin(aPb)/sin(bPc))/(sin(aPd)/sin(dPc)).

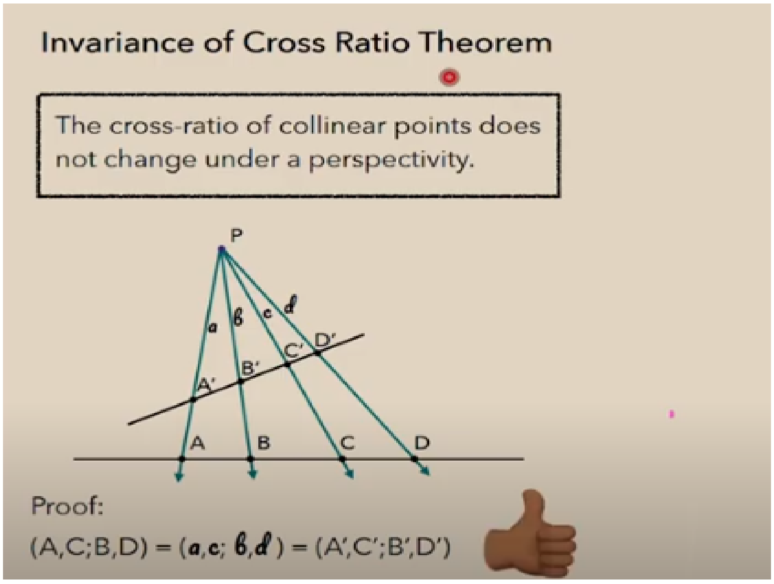
I have this all over this second ratio here. So we have the same ratio of ratios of signs of lines, except we have a bunch of other junk here. But just like before, things very nicely cancel out. First of all, obviously, we can get rid of these one halves. We can say goodbye to all of these, they all cancel out. What else cancels out? Well, here we have to be a little careful, we have to look a little bit more closely.

But if you see we have a PA over here on top in the numerator, and we have a PA over here in the numerator on the bottom. So those PAs cancel out. We have a PB over here in the numerator of the numerator and in the denominator of the numerator we have another PB. So those two cancel out. Similarly, these two PDs cancel out.

And finally we have a PC in the top denominator and a PC in the bottom denominator, those also cancel out. So everything cancels out. And we are left with our cross ratio of lines. So by looking at these, these areas, we actually equate these two cross ratios of points with the cross ratio of lines. So I really like this proof. It is kind of my favorite proof of the cross ratio.

There are many other proofs out there if you are curious. But what is the upshot of this? This shows that we have not actually shown what we want to show. We have shown the cross ratio duality lemma. We have proven this. These two cross ratios are equal, the cross ratio points and the cross ratio of lines are equal in this set up here.

(Refer Slide Time: 35:48)



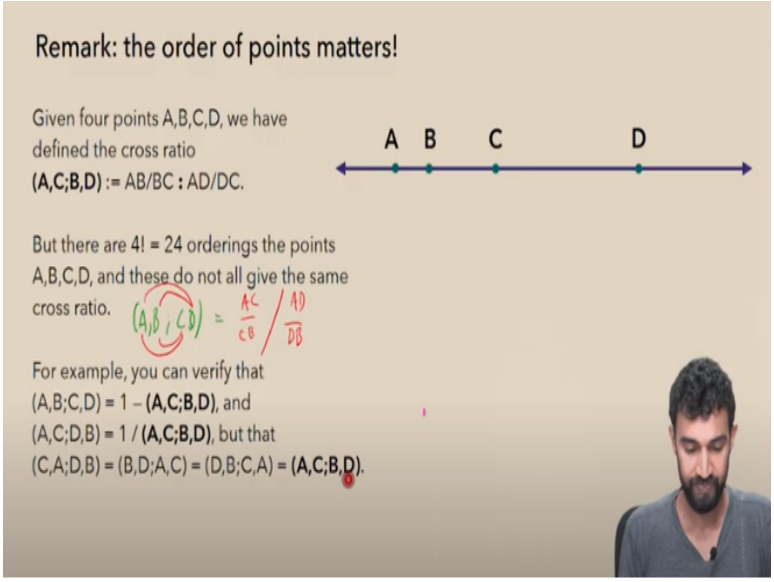
But how does that prove the invariance of the cross ratio? Meaning how does that prove that under a perspectivity, the cross ratio is invariant, it is preserved. So we want to show that the cross ratio never changes, when we project four collinear points to four other collinear points. Well, it actually follows directly because we can just imagine a setup like this.

Here is a perspectivity taking the four collinear points, relating the four collinear points, A', B', C', D', with the four collinear points A, B, C and D. And by our previous cross ratio duality theorem, we know that the cross ratio of these points A, C, B, D is equal to the cross ratio of lines a, c, b, d.

But then we also know from the same duality theorem that the cross ratio lines a, c, b and d is equal to the cross ratio of points, A’, C’, B’, D’. So these two cross ratios of points are equal if they are related under a perspectivity. So that proves the invariance of the cross ratio. And it justifies our method for finding the distance of the ball in that photograph.

We use the fact that the cross ratio is invariant in order to calculate x, the distance of the ball from the goal. And now we can see that that was justified, the cross ratio is indeed invariant under a perspectivity. In particular, it is invariant under a photograph. So that is very cool.

(Refer Slide Time: 37:15)



And I just want to make a quick remark here, the order of the points matters. I mentioned this earlier briefly, but I just want to emphasize it. Let us fix our points A, B, C and D on a line. We can take the cross ratio (A,C;B,D) which is defined to be (AB/BC)/(AD/DC). But there are actually many other orderings of the points.

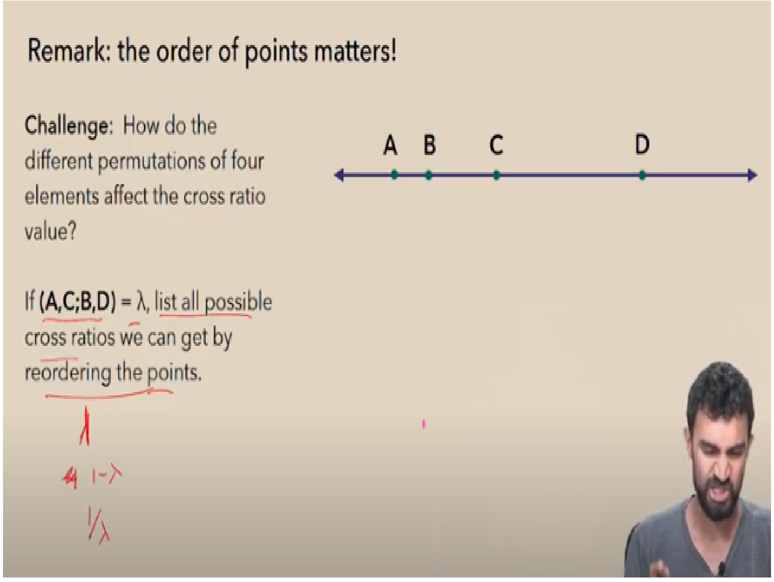
There are 24 other orderings, 4 factorial, and they do not all give the same cross ratio. For example, there is the obvious other ordering that you may see or that you might want to look at, which is what if we just calculate the cross ratio (A,B;C,D). Well, by definition, that is equal to (AC/CB)/(AD/DB). Is that the same thing? It turns out it is not. If you work it out. It is not the same, but it is related.

If you just work out the algebra, you can verify this, I will leave that as a homework exercise for you to do to verify this, but it is actually equal to one minus our cross ratio we are working with which is (A,C;B,D). So it is related, but it is not the same. Similarly, if you were to write (A,C;D,B), you get 1/ (A,C;B,D).

On the other hand, certain permutations do not change the value. So (C,A;D,B) in other words, instead of (A,C;B,D), if you flip the A and C and flip the B and D, that does not change anything, you get the same cross ratio value. Similarly, if you switch these two partition sets, so switch A, C with B, D. So you get (B,D;A,C). If you take that cross ratio that ordering and take the cross ratio, you will get the same value.

Or you could do both of those permutations together and get the same value. So the order of the points matters, but not every permutation changes the cross ratio. So we do not necessarily get 24 different cross ratio values.

(Refer Slide Time: 39:34)



So as a challenge, how do the different permutations of four elements affect the cross ratio value? See if you can figure that out by working out some examples. In particular, if (A,C;B,D) is equal to **λ**, can you list all the possible cross ratios we get by reordering the points? We have already listed two. We have seen that, if **λ** is a possible cross ratio, then 1-**λ** is possible, 1/**λ** is also possible.

Are there any other values. In case you are curious and you want to work out all the details, you will also check the Wikipedia page on cross ratios. That also explains this nicely and succinctly. So I do not want to spend too much more time on this right now. It is just a remark that I think you should know about. But in this class today, and in the next lecture, we are only going to use this value of the cross ratio.

So we do not really have to worry about this. But if you ever encounter the cross ratio in another place in another source where it is written differently, this is why. We actually have to choose a convention when we are working with cross ratios. And this is the convention I have chosen for certain geometric reasons. But there are various other choices which are equally valid.