**Our Mathematical Senses**
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**Lecture – 12**
**The Fundamental Theorem of Projective Geometry**

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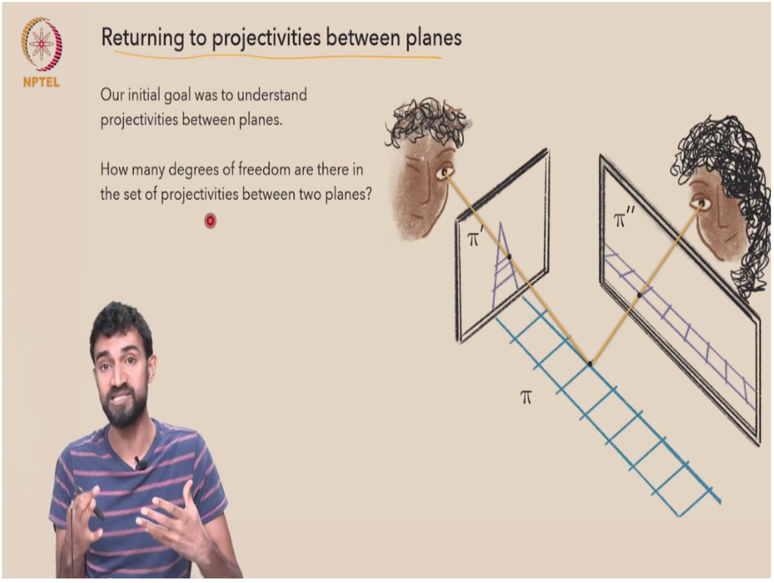


So, finally the third application I want to talk about of the three fixed point theorem is, we first require to return to where we started this discussion which is projectivities between planes. So, our initial goal is to understand projectivities between planes and we moved down to 1D to get a better sense of projectivity. So, I want to move back to projectivities between planes now.

And I want to ask the question of how do we understand this collection of projectivities? And in particular can we say anything about how many maps there are? How many projectivities are there between two given planes? How many degrees of freedom does this set of projectivities have?

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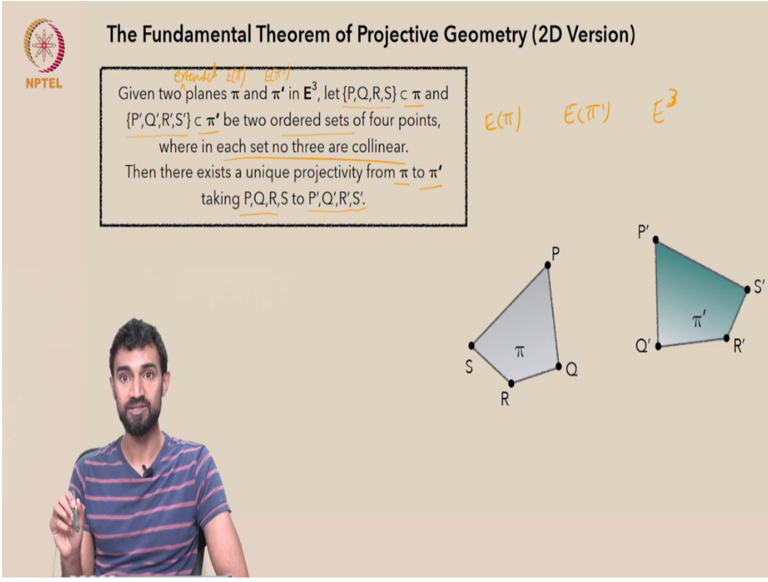


So, that is the content of the fundamental theorem of projective geometry in 2D. So, in this scenario, what is the theorem saying? It saying that given two planes π and π' in $E^3$, technically speaking I should be saying extended planes here and I should write that here two extended planes, it will be bit cumbersome, so I am just going to write π, but really we mean E(π) and E(π'), who want to be little precise so these are two extended planes in $E^3$.

Extended again means they include points in infinity; it is not just the plane we see in space, there is also a whole line at infinity. There are a lot of points in infinity associated with each plane. So, let P, Q, R and S be points in π and let us let P', Q', R' and S' be points in π' and not only distinct points, these are two ordered sets of points. So, there are distinct points given in a certain order with the additional stipulation that in each set no three points are collinear, so it is a nice set of four points.

No three are collinear meaning that it is going to form some kind of quadrilateral. We cannot have something that looks like this triangle. In this case three of them are collinear. No three should be collinear implies it will be something like this. I do not actually have to draw an image because I have already drawn one here. These were examples right here, of two sets of four points.

One in the plane $\pi$ and one in the plane $\pi'$. So, in this case there exists a unique projectivity from $\pi$ to $\pi'$ taking P, Q, R and S to P', Q', R' and S' in that order. In other words P goes to P', Q goes to Q', R goes to R' and S goes to S'.

So there exists a unique projectivity from the plane $\pi$ to the plane $\pi'$ taking these four points to these four points, that is what the fundamental theorem says. So, it is a lot like the fundamental theorem in 1D except there we had three collinear points on a line here. We have four points in a plane such that no three of which are collinear. It is kind of spread out nicely. They are in a general position you can say.

I will just note that we are assuming these are ordinary points P, Q, R and S. I mean for the sake of diagrams, I am drawing all my points. I am not drawing any of them as a line of infinity, but everything in this proof works out if any of these points are at infinity. So, this is truly a theorem about extended planes even though I am writing $\pi$ and $\pi'$ everywhere.

I really should be writing $E(\pi)$ and $E(\pi')$ because this is truly a theorem about the extended space $E^3$ and in my pictures I am drawing ordinary points and ordinary quadrilaterals, but all of our arguments actually work for general points. So the points could be points of infinity and nothing goes wrong. Maybe the first time we look through this we would not confuse ourselves with any of that. We can look through it again and see that indeed everything works out.

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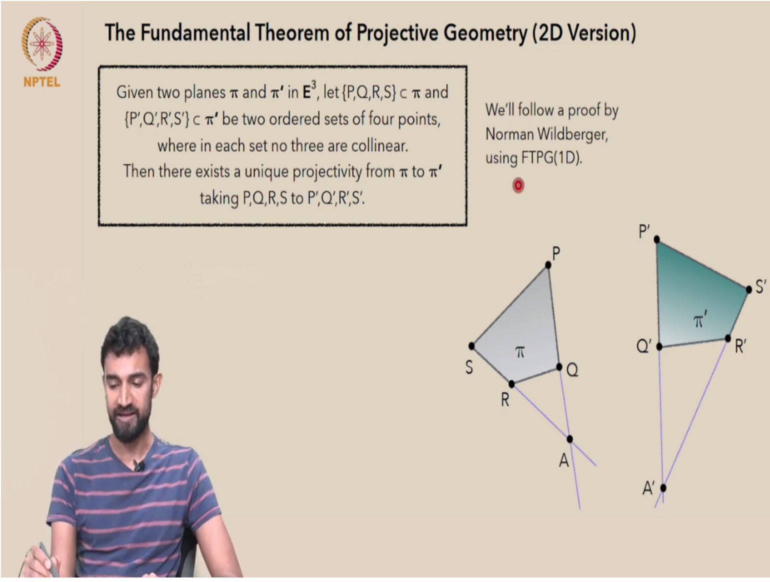


So, let us prove this and I am going to follow a proof by Norman Wildberger which uses the fundamental theorem of projective geometry in 1D. The fundamental theorem in 1D which states that if we take three collinear points and three other collinear points in space there is a projectivity taking one to another. So, I do not have any collinear points yet.

So, in order to use that I need some collinear points. We can easily get them by just extending some opposite sides. So over here on the left, let us extend PQ and SR. Let us call their point of intersection A. Let us extend P'Q' and S'R'. Let us call their point of intersection A'. So, now what I want to do is map P, Q, A to P', Q', A' and by the fundamental theorem in 1D, that can be done by a projectivity.

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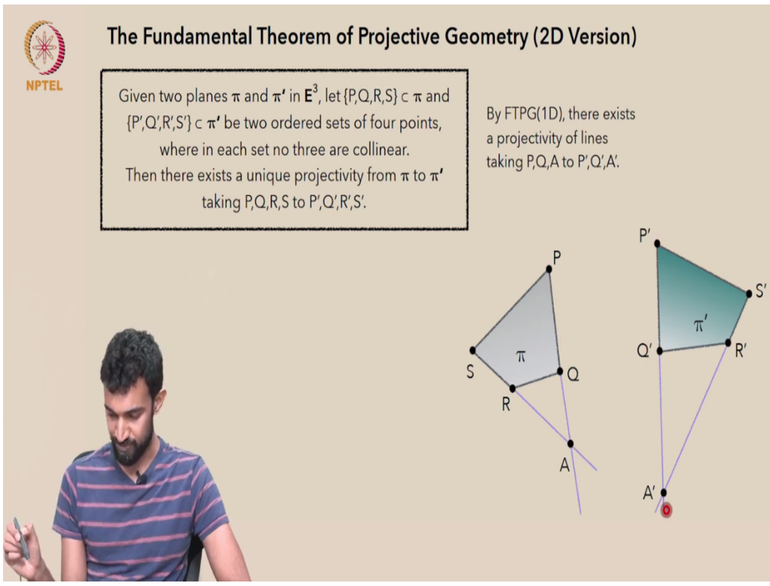

There exists a projectivity of lines taking P, Q, A to P', Q', A'.

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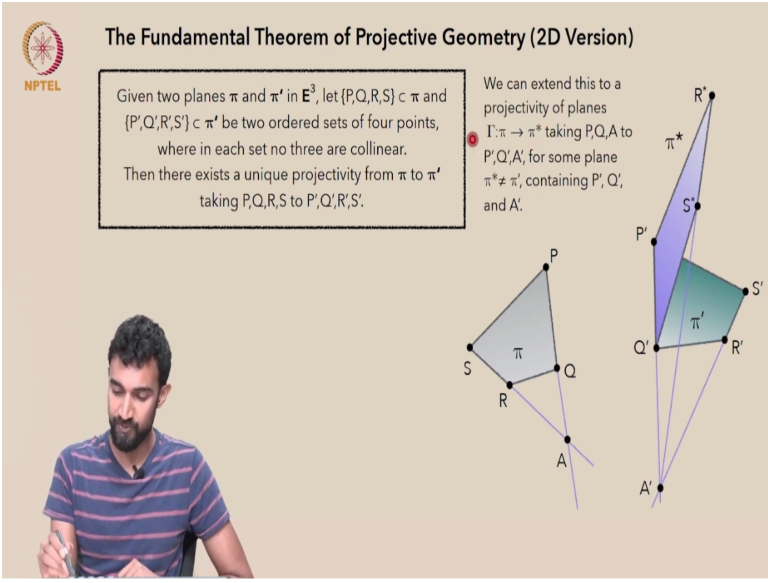


Moreover, we discussed in the last lecture that we can extend this projectivity of lines to a projectivity of planes we call that **Γ** from π to π* and it is taking P, Q and A to P', Q' and A', where π* is some plane containing P', Q' and A' and like we discussed earlier we can actually choose π* to be any plane that we want, as long as it avoids hitting our centers of a perspectivity.

So, we have a lot of freedom there. So, let us just choose a plane π* which is not equal to π' and which contains P', Q' and A'. So maybe it looks something like this π*. This projectivity **Γ** which takes P, Q and A to P', Q' and A' when we extend it to a projectivity of planes from the plane π to the plane π*, it is going to map S and R to some other points S* and R* in that plane.

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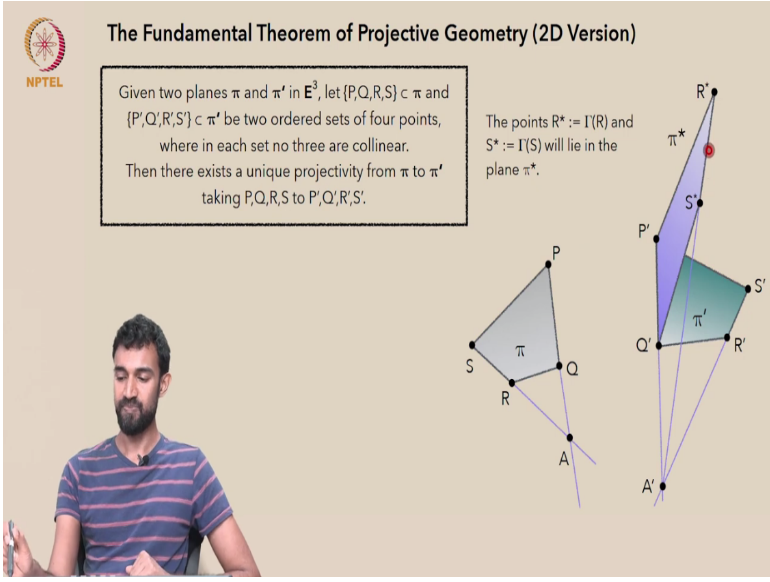


So, let R* be the image of R and let S* be the image of S under the projectivity **Γ**. So, basically **Γ** this is projectivity which has partly completed our mission. We have mapped P to P' and Q to Q', but we have not mapped R to R' and S to S' rather we mapped R to R* and S to S* and we have mapped this quadrilateral to this quadrilateral which is lying in this plane π*.

Actually it is a good thing that this plane is not equal to our plane π' because now we can easily create another perspectivity which will map it down.

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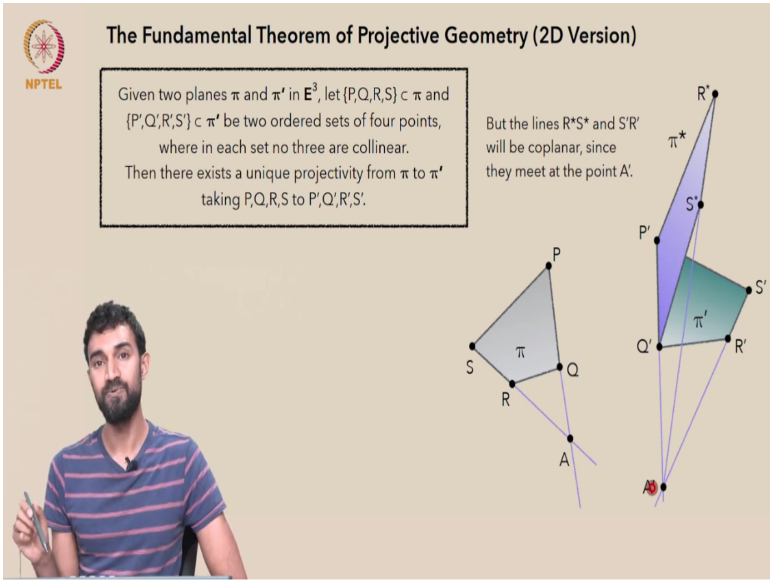


Note that R*S*, this line here, actually meets A', because **Γ** is a projectivity and projectivities take lines to lines and preserve incidence relations. So, **Γ** took P, Q, A to P', Q', A', it has to take S, R, A to this line containing R*, S* and A'. If you are confused by the order being rearranged do not worry even though **Γ** takes lines to lines, it does not necessarily preserve the order of points on those lines.

So, this can happen where the order gets changed around, we have seen already. But we do know that the line formed by R* and S* does meet the point A'. Similarly, A' was defined to be the intersection of P'Q' and S'R'. So S'R' also meets the point A', in other words R*S* and S'R' are coplanar, since they meet at the point A'.

So, why does it matter that they are coplanar? Well we can do a very simple perspectivity that will bring us to where we want to go.

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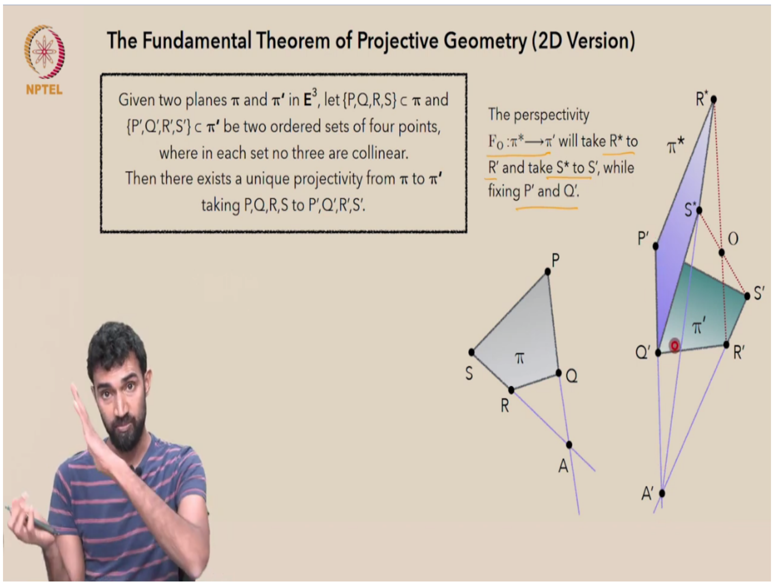


Namely this one. We can connect R* to R' and S* to S' and since R*S* and S'R' are coplanar, their intersection will exist. These lines will not be skew because they lie in the same plane. Let us call their intersection O. The perspectivity centered at O is going to interchange from π* to π', is simply going to take as S* to S'.

And it is going to take R* to R'. So, the perspectivity $F_O$ from π* to π' will take R* to R' and S* to S'. Most importantly it is going to fix P' and Q'. Why is it going to fix P' and Q'? Well P' and Q', they lie in the intersection of π* and π'. So, they are fixed by any perspectivity from π* to π' including this one.

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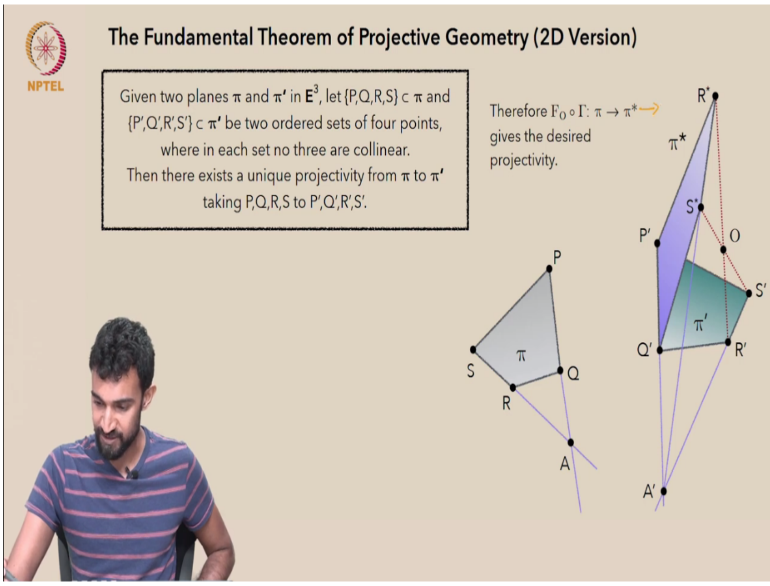


So, composing $F_O$, this perspectivity with **Γ** which is the perspectivity that took this quadrilateral to this one, we get a projectivity of planes from π to π'. The composition is from π to π' and it takes P, Q, R and S to P', Q', S' and R' as desired.

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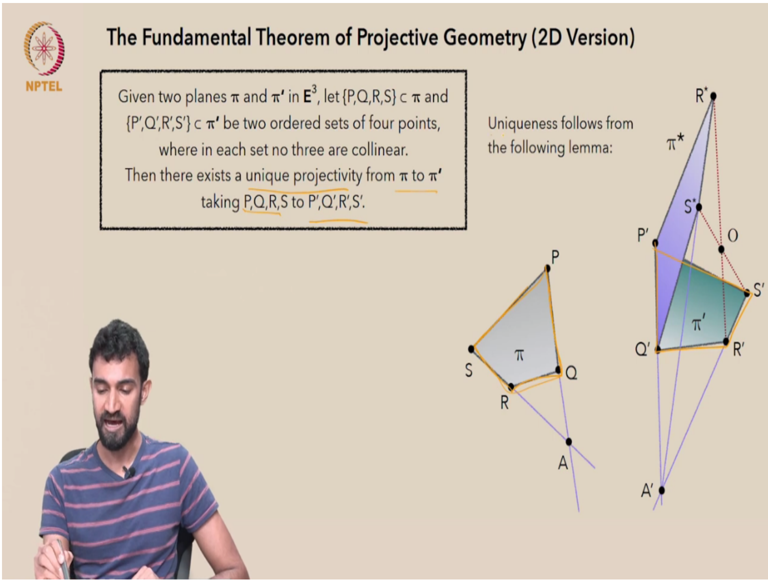


So, we have constructed a projectivity taking P, Q, R and S to P', Q', R' and S', but the theorem says more than that. The theorem says there is unique projectivity. Once again uniqueness means as a map that is unique, there is a unique projectivity map from π to π' taking these four points to these four points in that order. It does not say that the sequence of perspectivity is unique.

So how do we see the uniqueness well it is going to follow from a lemma which will prove now.

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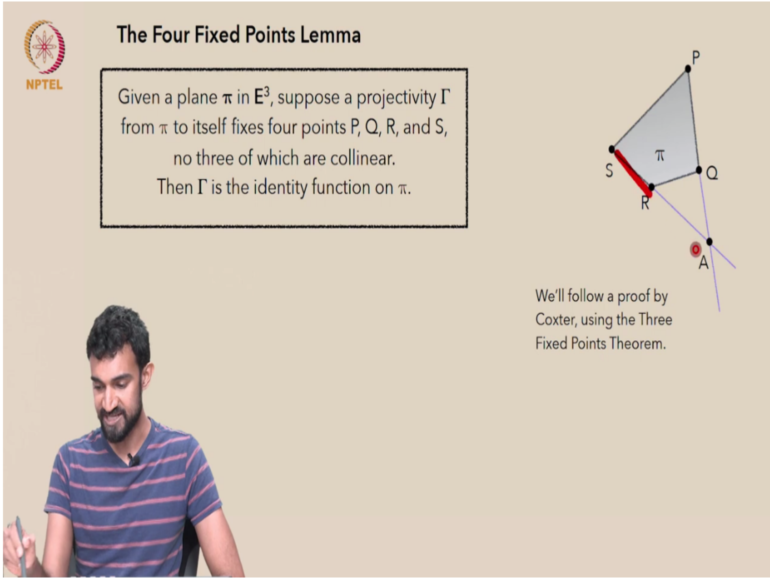


That lemma is called the four fixed points lemma and what does it say? It says that given a plane π in the extended Euclidean space $E^3$, suppose a projectivity **Γ** from π to itself fixes four points P, Q, R and S such that no three of which are collinear. So, it fixes a quadrilateral like this PQRS, but in this case **Γ** must be the identity function on π. If it fixes four distinct points with no three of which are collinear, it has to be the identity function that is what the four fixed points lemma says.

So, why is this true? So to prove this, we are going to follow a proof by Coxeter and the proof uses the three fixed point theorem. So, to use the three fixed point theorem, we need three collinear fixed points. Right now we do not have that, but if we extend this line PQ and this line SR, we get a point A.

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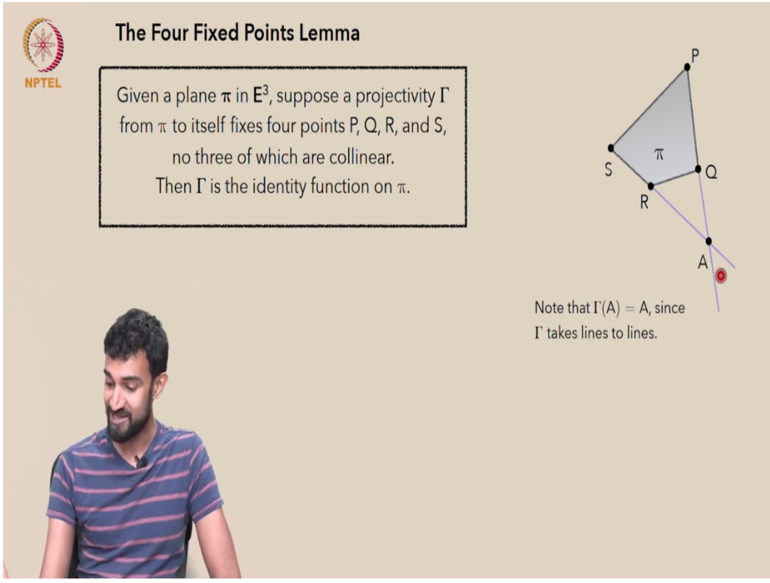


Well, what happens then? Notice that **Γ**(A) is equal to A, since **Γ** takes lines to lines. So, the lemma we need to prove is known as the four fixed points lemma and what it states is that given a plane π in the extended Euclidean space $E^3$ if we have a projectivity **Γ** from π to itself which fixes four distinct points P, Q, R and S such that no three of which are collinear then our map **Γ** is in fact the identity map on π.

So, in other words if our map **Γ** from π to itself fixes a quadrilateral PQRS fixes four non collinear points then, it has to be the identity. So, it is a lot like the three fixed points theorem except in three fixed point theorem we had three collinear points on a line that were distinct from each other. So, they were in general position on a line, here we have four points on a plane which are in general position.

So, there are four coplanar points, but they have to be distinct and they also have to satisfy this property that no three are collinear and so if they are in general position like that and if our map **Γ** is fixed on those four points, then it is fixed everywhere it is the identity. So that is the content of the four fixed points lemma.

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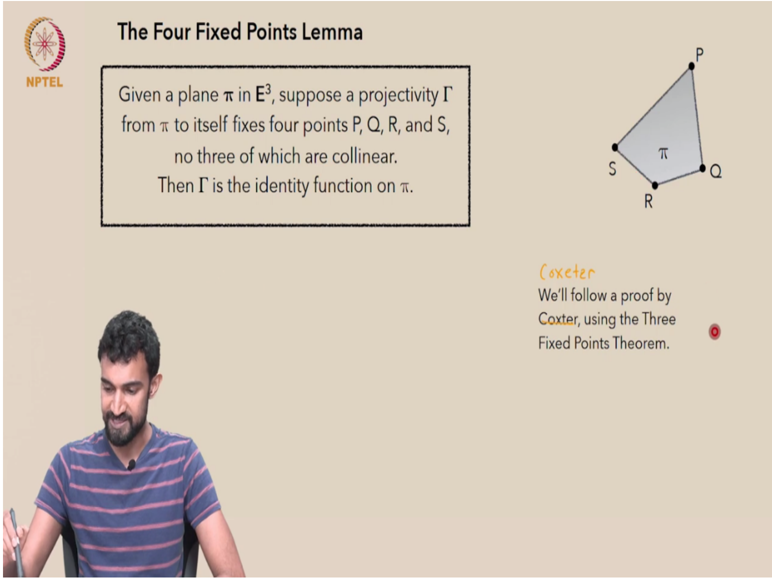


And to prove it, we are going to follow a proof by Coxeter and which uses the three fixed point theorem in order to prove the four fixed point theorem. So, in order to use the three fixed point theorem we need three collinear points that are fixed. Right now we have four fixed points, but no three points of them are collinear.

However, we can connect these two sides to get another point A which I claim will have to be fixed.

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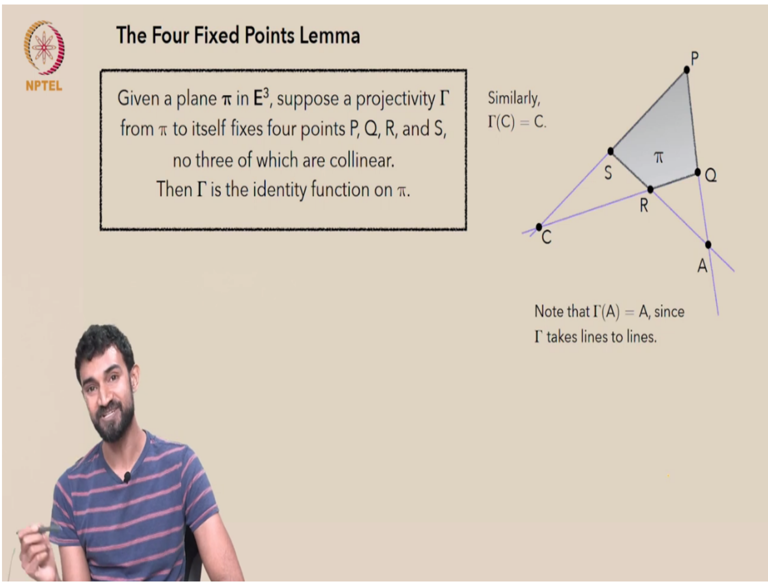


Remember **Γ** is a projectivity, so it takes lines to lines. We know that it is taking P to P and Q to Q. So it has to take the line PQ to PQ. Similarly it is taking S to S and R to R. So, therefore it has to take the line SR to SR.

Now that does not mean it is fixing every point in that line necessarily, but it is taking the line PQ to PQ, it might be jumbling it up. It is taking the line SR to SR, but it might be jumbling it up. However this point A is in the intersection of PQ and SR. So, **Γ**(A) must lie in the intersection of **Γ**(PQ) and **Γ**(SR) because it is a projectivity.

And it preserves these incidence relations of course. So, **Γ**(A) is actually equal to the intersection of **Γ**(PQ) and **Γ**(SR). But **Γ**(PQ) is just PQ and **Γ**(SR) is just SR. So their intersection is just A. So, **Γ**(A) is equal to A. So, we come from this fact that projectivity is a linear map. It takes lines to lines which do not sound like much sometimes, but it can be quite powerful.

Similarly we can extend QR and PS to get a point C and **Γ**(C) will equal to C for the exact same reason that **Γ**(A) is equal to A. So, now we have six fixed points not just four. And some of these subsets of fixed points are collinear now which means we can invoke the three fixed point theorem.

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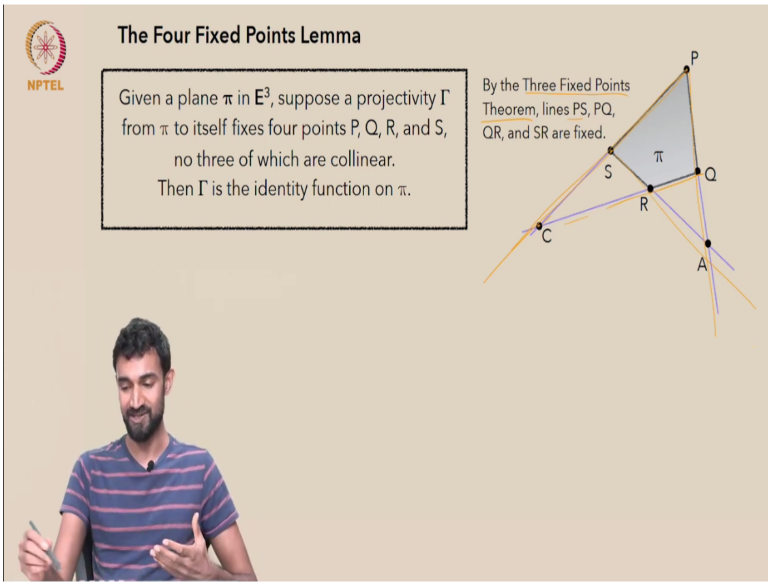


We already knew that line PS is taken to PS by **Γ**, but now we know that, since the line PS contains three points that are fixed, the line PS is actually fixed by **Γ**. So, it was earlier as fixed set wise, now it is fixed point wise by the three fixed point theorem. So, the line PS is fully fixed.

The line PQ is fully fixed point wise, every point on PQ is fixed. Similarly every point on QR is fixed and every point on SR is fixed. So, we have infinitely many fixed points, though it is still not every point to the plane, it is still just these four lines that are fixed.

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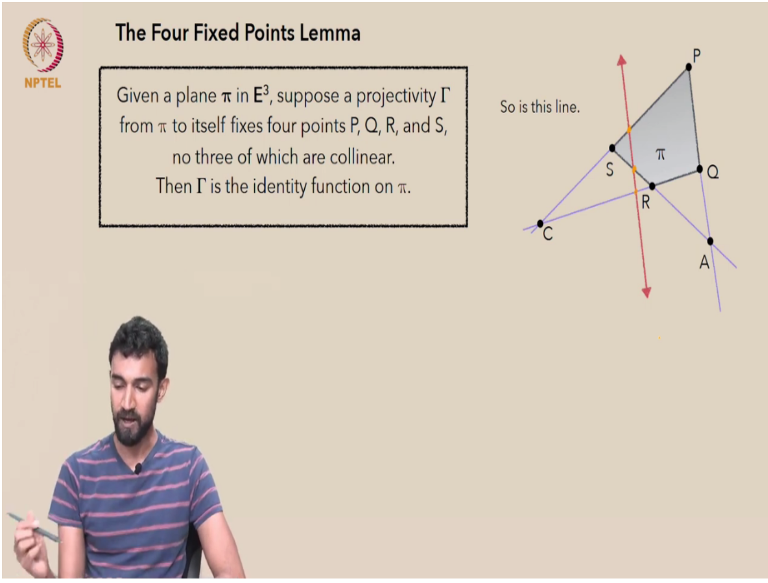


But we can keep going, we can draw this line here. This line has a point of intersection here with PS, a point of intersection here with SR and a point of intersection here with QR. All three of those points must be fixed by **Γ**. So, again by the three fixed point theorem this entire line is fixed by **Γ** that is pretty good.

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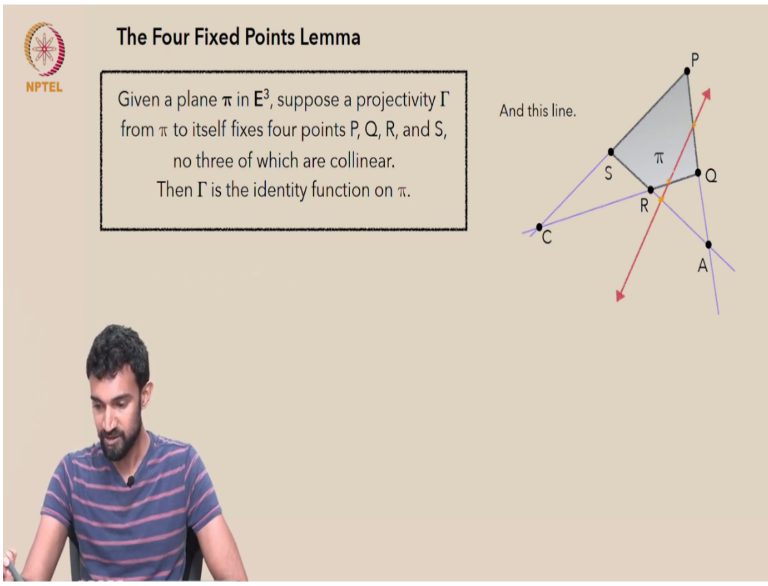

Similarly, we can draw this line here. This point is fixed by **Γ**, this point is fixed by **Γ**, this point is fixed by **Γ**, therefore the entire line is fixed by **Γ**.

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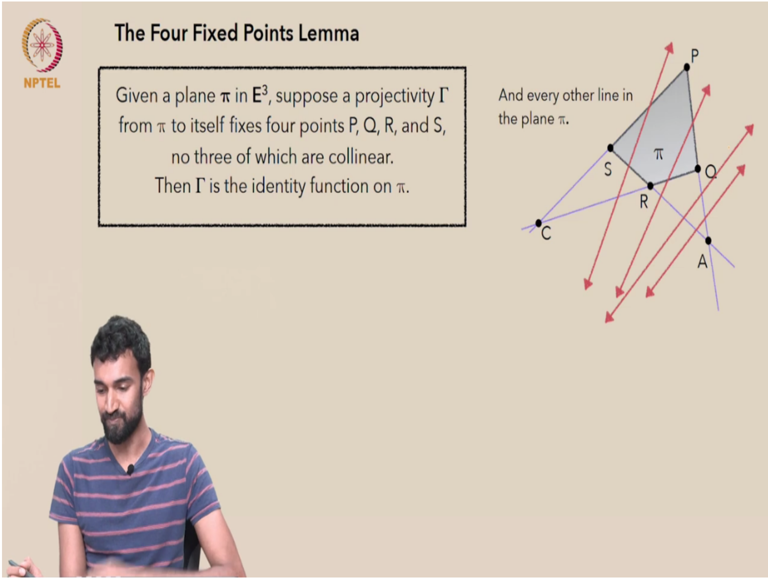


And we can keep doing that. Every line we draw, it intersects at least three of these lines and in fact it will intersect four of those lines and as we get more and more lines there, and more lines to intersect. So anyway we are getting more and more fixed points and every single line in the plane π is going to have at least three fixed points and therefore be fixed.

So, as a result **Γ** is just the identity, it fixes every point in the plane π. So, that is the four fixed points lemma.

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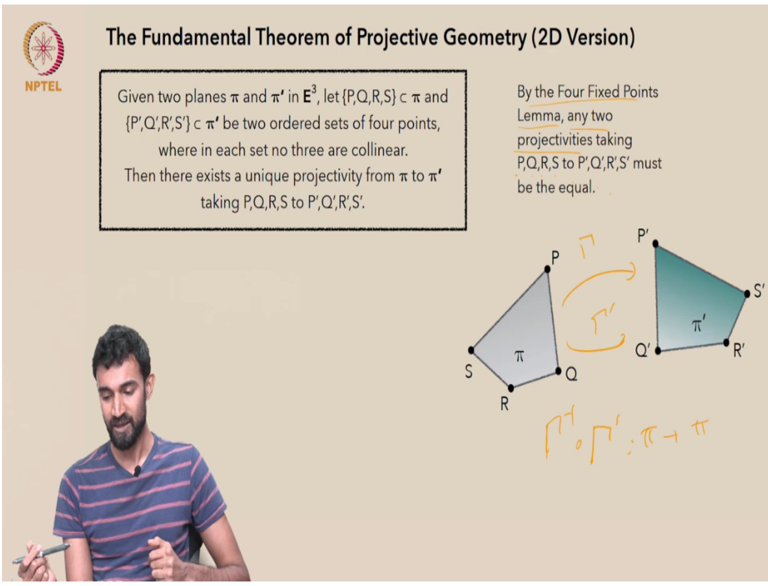

And going back to the fundamental theorem we started, fully proved the uniqueness that we can mostly have now. So by the four fixed points lemma, any projectivity which fixes four points must be the identity. So, how does that help us? Now imagine that you have two different projectivities taking P, Q, R and S to P', Q', R' and S'.

Well we can play the same game we did in the last lecture. Just compose the inverse of one of those maps with the other map. In other words, you have a $\Gamma$ here and you have a $\Gamma'$ here, both take P, Q, R and S to P', Q', R' and S'. If you just compose $\Gamma'$ by $\Gamma^{-1}$ you will have a map which goes from $\pi$ to $\pi$ which fixes P, Q, R and S.

So, it is equal to the identity by the four fixed points lemma and as a result these have to be the same map because they are both bijections.

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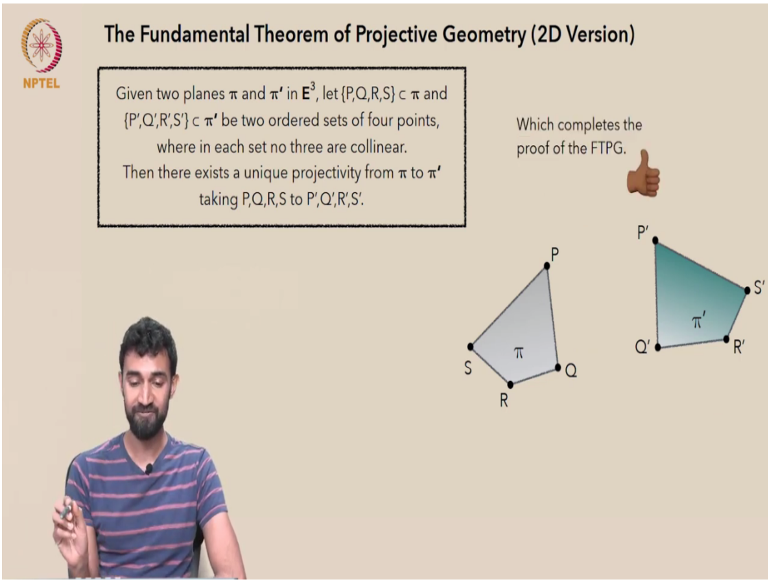


So that completes the proof of the fundamental theorem of projective geometry.

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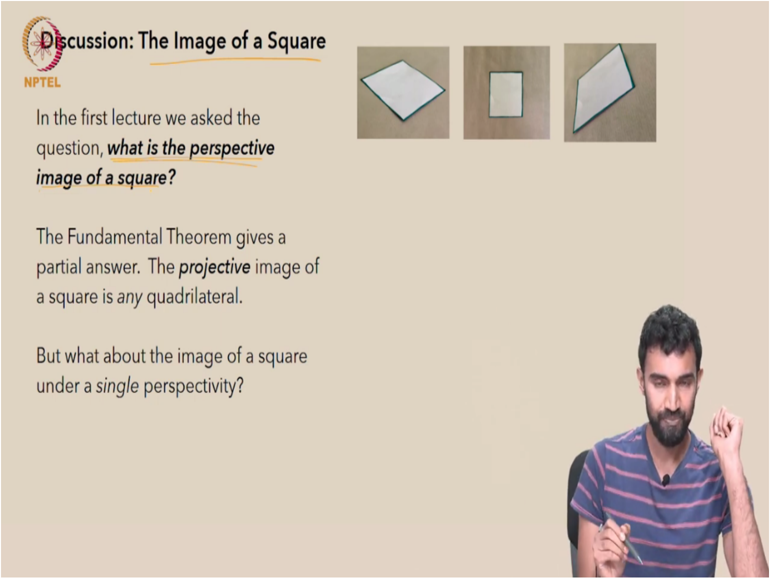


So, with that in mind I just want to go back to a question that we asked a long time ago. What is the perspective image of a square? In fact, I think the very first lecture or maybe the second lecture we ask what is the perspective image of a square? Now the fundamental theorem of projective geometry that we have just proven gives us a partial answer because the projective image of a square is any quadrilateral.

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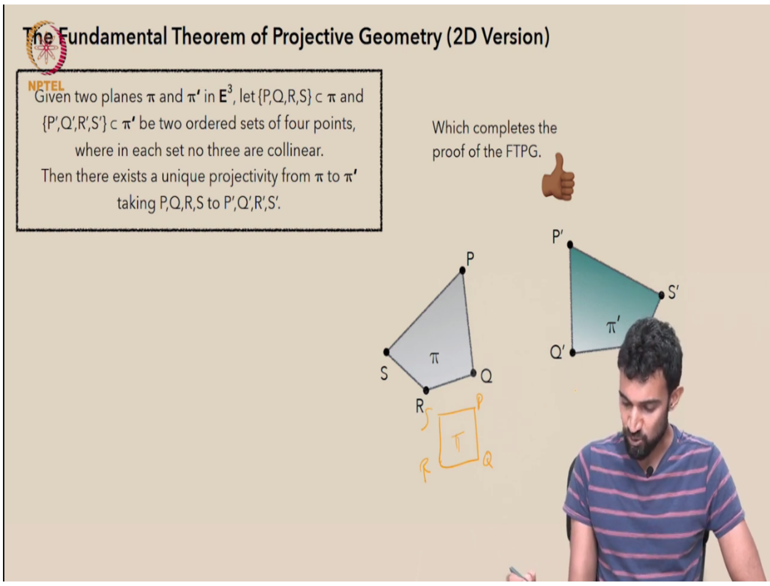


We can take a square as PQRS in π and we can choose any quadrilateral in any other plane π' and we can find the map projectivity mapping the square to that quadrilateral. So, the projective image of a square is any quadrilateral, in fact we can even do that in staying in a single plane π, we can even choose any other quadrilateral in π and find a projectivity that takes a square to this quadrilateral by the fundamental theorem.

But that does not fully satisfy answering the question in the spirit we originally asked. We are interestingly wondering what are all the ways that a square can appear to us. In other words we are asking about the perspective image of a square, not the projective image of a square. So, what is the image of a square under a single perspectivity?

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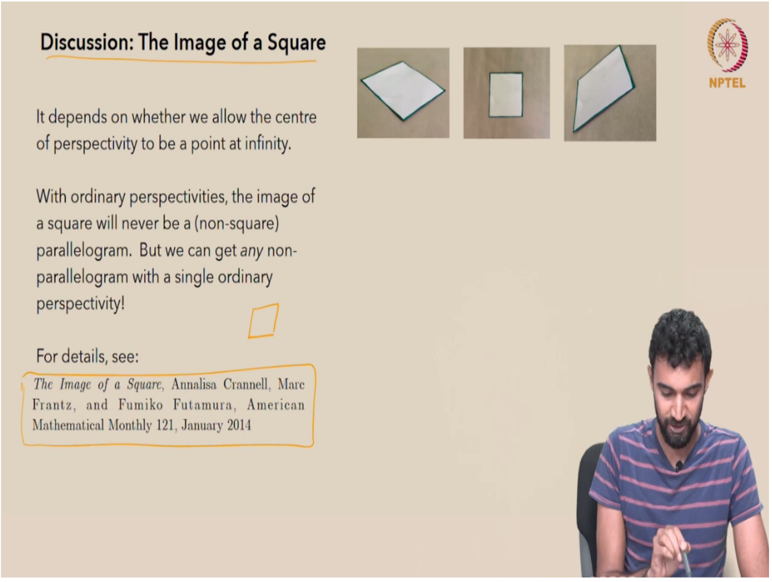


Is it still any quadrilateral for example or are there now some restrictions? The answer is a little tricky; it depends exactly on what we allow. In particular do we allow the center of the perspectivity to be a point at infinity. For ordinary perspectivities, where the center of perspectivity is not a point at infinity but an ordinary point, there are some restrictions.

The image of a square is never going to be a non square parallelogram and that maybe we will leave it as an exercise, it is actually a nice exercise, it is not too difficult. You can never get a non square parallelogram, you will never get a rectangle or a rhombus or any other non square parallelogram as your image of your square. A consequence of that is if you take a photograph of this square here, then you will never get a rectangle or a rhombus or something like that.

Similarly you are never going to use your cell phone torch and make a shadow of it which appears to be a perfect rectangle. You might approximate it, you might get something that kind of looks like a rectangle, but if you were to be super precise it would not really be in your measurements, but maybe I will mention so in the other hand we can get any non square parallelogram with just a single ordinary perspectivity.

So, when I say a non square parallelogram, I misspoke we cannot get any non parallelogram quadrilateral, we can get any quadrilateral that is not a parallelogram through a single ordinary perspectivity. So, another word for that is it is sometimes called a complete quadrilateral though that is not the best term because it has multiple meanings. So a complete quadrilateral, but really there is no amazing term for this. Let me just erase that, let us not call it that, let us call it a non parallelogram quadrilateral.

So, it should not be the case that these opposite sides are parallel to one another and similarly these opposite sides are parallel to one another. It is however okay to be a trapezoid to have one set of opposite edges and the other set converge that is okay. So, this is kind of okay, but this is not okay as the image of a square under a single perspectivity where the center is in ordinary point not a point at infinity.

And for detail proof, the proof is actually fairly elementary you can see this paper here from 2014 where the authors carefully construct that perspectivity which will take a square to a non parallelogram quadrilateral and I will say one more thing what if we do allow the center of the perspectivity to be infinity well then it is pretty easy to see why you can actually get any quadrilateral.

So, let us assume from this paper that we can get any non parallelogram. If we include point of infinity as centers of perspectivity we get parallel projections suddenly which allows us to map a square to a parallelogram pretty easily. In fact, parallel projection will map a square to a parallelogram only, you would not get any non parallelogram quadrilateral that way.

So, you will only get parallelograms when you look at that kind of infinite far away projections and a nice application of this is that you can actually take a square outside and look at its shadow under sunlight. Sunlight basically is parallel projections because the sun is so far away that it might be as well as infinitely far away for any measurements that we can easily take and the shadows of your square under sunlight will all be parallelograms.

They might be rectangles or they might be rhombuses or they might be parallelograms. Whereas if you take the same square indoors in the night time and turn on the cell phone torch and look at the shadow from your cell phone torch, your shadows will never be parallelograms. They will give you all kinds of crazy quadrilaterals, but they would not give you parallelograms.

So, this is a very nice way to illustrate the difference between central projection and parallel projection or in other way saying that is between perspectivities whose center is at infinity versus perspectivities whose center is an ordinary point. If you want to complete the proof, I recommend you to check this paper, The image of a square by Crannell, Frantz and Futamura from 2014.

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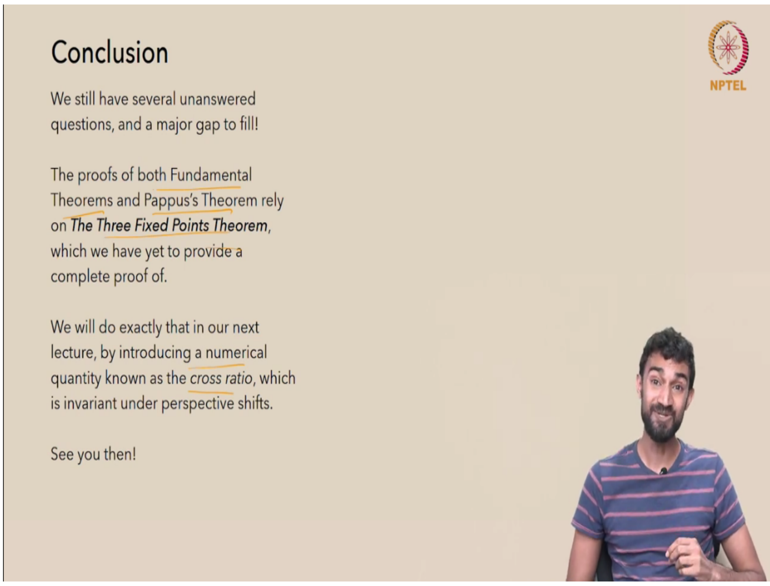


So, in conclusion, we still have several important questions to answer and we still have one very big gap to fill. So, the proofs of both the fundamental theorems and the Pappus's theorem lied on this three fixed point theorem that we keep referring to and we have given two sketches of it in the last lecture, but we have not provided a complete proof yet. So that is exactly what we are going to do in the next lecture.

We are going to complete the proof by introducing a numerical quantity known as the cross ratio and this cross ratio will turn out to be invariant under perspective shits and it will be the key to proving the three fixed point theorem. So, see you next week when we will introduce the cross ratio, prove the three fixed point theorem and answer all the other questions that are remaining from the last few weeks. So see you then.