Our Mathematical Senses
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Lecture – 11
Proving Pappus's Theorem

Hi, welcome back to the geometry of vision. This is Lecture 6 in which we are going to investigate the fundamental theorem of projective geometry.

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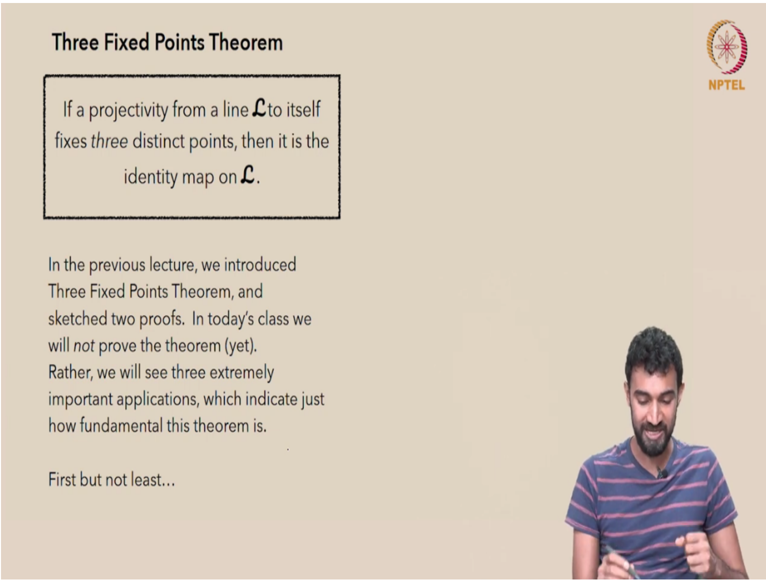

In the last class we introduced something called the three fixed points theorem which said that if a projectivity from a line L to itself fixes three distinct points then it has to be identity map on all of L. And in today's class, we are not going to prove the theorem yet, but we are going to see three extremely important applications of just how fundamental this theorem is for projective geometry.

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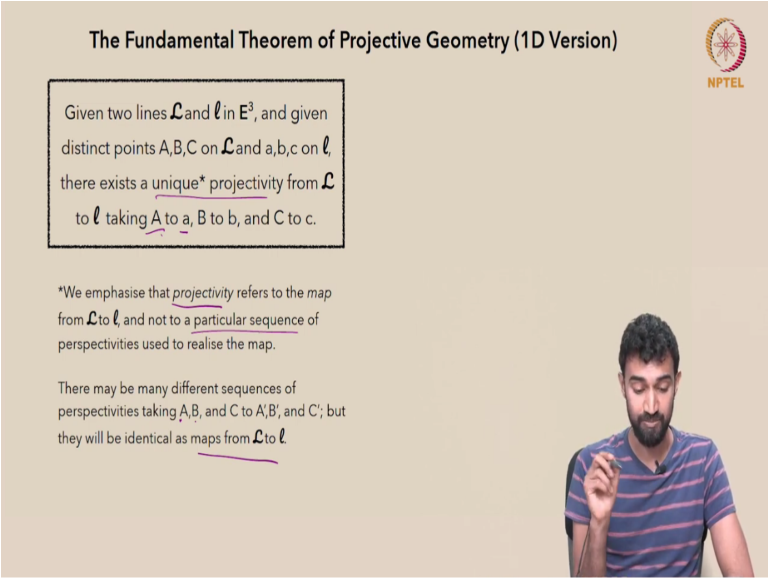

So, the first application, which is certainly not the least important, is the fundamental theorem of projective geometry in one dimension. When I say one dimension, I mean we are looking at projectivities from lines to lines. So, in this setting of projectivities from lines to lines, we have the following statements. Given two lines L and l in the extended Euclidean space E^3.

And given distinct points A, B and C on L and a, b and c on l, there exists a unique projectivity from L to l which takes A to a, B to b and C to c. So, I want to emphasize here something really important, when I say projectivity I am referring to the map from L to l.

I am not referring to a particular sequence of perspectivities that we are using to construct our projectivity. There might be many, many different sequences of different perspectivities that take A to a, B to b and C to c. In other words what this theorem says is that they are going to be identical as maps from L to l. So; even though I might have very different constructions of this projectivity over here and this projectivity over here.

This one is constructed by one set of perspectivities, this is constructed from a completely different set of perspectivities. If they agree on these points A, B and C, then they have to agree everywhere and then have to be identical as maps from L to l. So that is an important thing to keep in mind: the theorem is not claiming that there is a unique sequence of perspectivities. It is just saying, there is a unique map built up from perspectivities.

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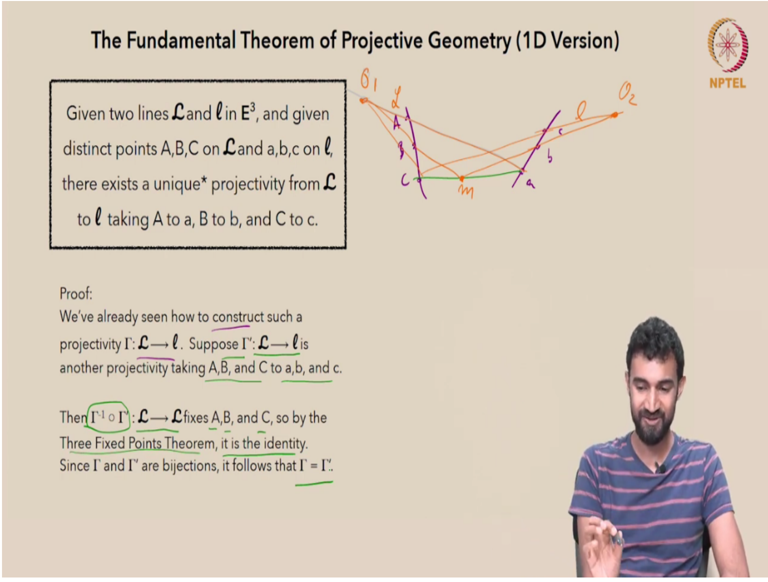

So, how do we prove this? Well, the proof is actually kind of simple, given if we assume the three fixed point theorem. Because, first of all we have already seen a construction. So, we already know the existence, we just have to prove the uniqueness. So, we have already seen how to construct this projectivity Γ from L to l. So, just to refresh your memory, if we have A, B and C here and we have a, b and c here, remember the construction I am talking about was that we linked up C and a with another intermediate line m and then we did a perspectivity from L to m and another perspectivity from m to l. So, the way I have drawn it here it is going to look a little bit different, but basically this is O_1.

And you can imagine, you are sending A to a, you are fixing C and you are sending B to some point here on m. Now we can take a second perspectivity that takes this point to b, C to c and fixes a. It is centered over here O_2 where these lines intersect. That is my construction that I am talking about. That is just one construction, there are other constructions as well.

This is the construction we already saw in the last lecture, so existence is not a problem. We know that we can construct a projectivity taking three points to three other points. The thing we have not proved yet is uniqueness. So, how do we prove uniqueness? Well, suppose there is another Γ' from L to l which is also a projectivity taking A, B and C to a, b and c.

So; just agreeing with Γ on those three points. Well in that case let us look at this composition in which we do Γ' followed by Γ^{-1}. That is going to be a projectivity from the line L to itself which fixes A, B and C. So, by the three fixed point theorem it is the identity. Since Γ forms a bijection, it follows that Γ is equal to Γ'; they have to be identical because one followed by the inverse of the other gives us the identity.

So, that does it. So the uniqueness actually follows directly from the three fixed point theorem.

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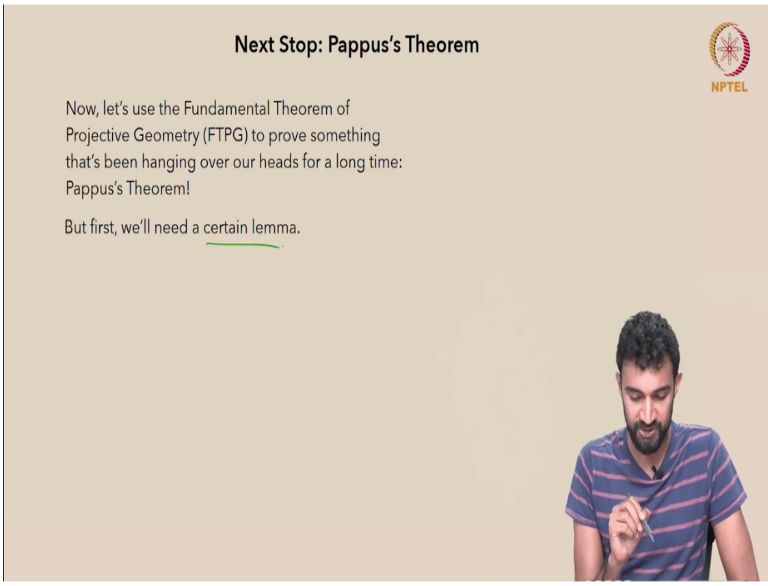

A second application I want to do of the three fixed points theorem is Pappus's theorem. So, let us use the fundamental theorem of projective geometry, which we just proved, to finally prove Pappus's theorem. But for that to work, first we are going to introduce a certain lemma.

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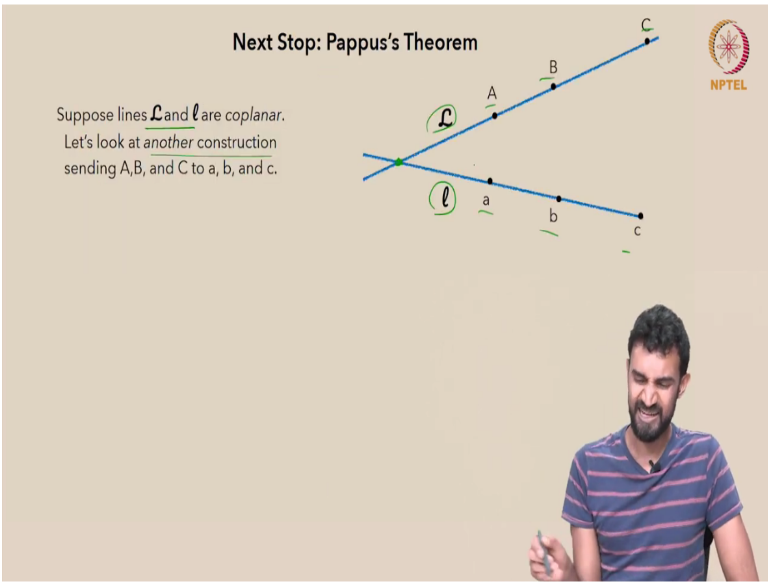

So, let us suppose lines L and l are coplanar, so they intersect somewhere and let us look at another construction that sends A, B, C to a, b and c. So, I am going to do a different construction than the one that I just refreshed our memory right here.

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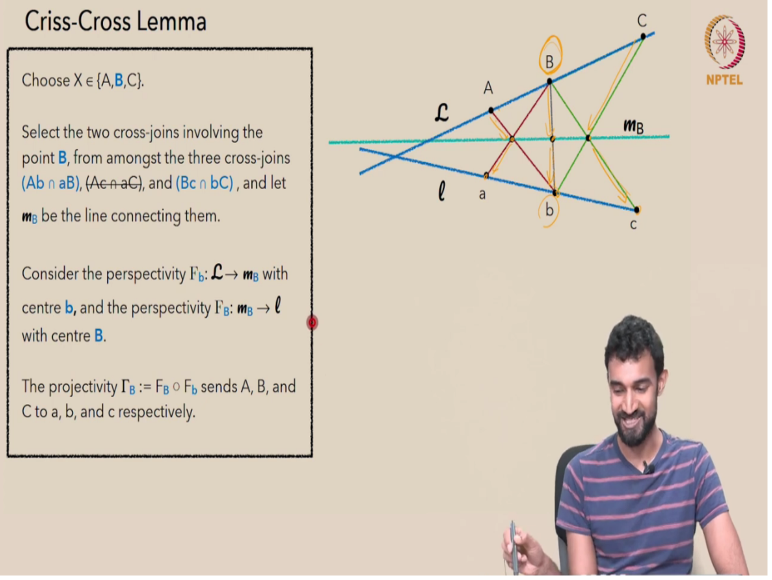

Let us see a new construction like I said there are many constructions that do this. I am going to show another one, but it is one that is quite special. So, I am phrasing it in terms of a lemma which is sometimes called the criss-cross lemma and you will see why. So, this construction is actually three separate constructions which is why the notations are a little weird because there are three constructions corresponding to whether we choose x to be equal to A, B or C.

Basically we have to choose A, B or C to be the center of the projectivity that I am going to construct. The criss-cross projectivity that I am going to construct now, it need a center and at center I can choose it to be A, B or C. Depending on which one I choose I will get a different criss-cross construction. So, there are really three constructions that I am going to describe now and maybe rather than have this weird general notation with x let us do it for B and everything here is fully symmetric.

So rather than B, I could have picked A, I could have picked C to get a criss-cross construction centered at A or criss-cross construction centered at C. So, there are three different criss-cross of construction you are getting for the place of one. Let us see how it works with the one centered at B. Remember there is a construction, there is a projectivity that takes A, B and C to a, b and c on the l. But how does it work?

Well, the first step is to select two cross joints that involve point B, since we are choosing B as our center. Well there are two cross joints that involve B. By the way, what I mean by cross joint is the intersection of aB and Ab that is one cross joint and the second cross joint involving B is the intersection of bC and Bc, which is right here.

So, those are the two cross joints involving the point B. The third cross joint which I did not pick is Ac intersect aC which does not involve B. By the way we have our two cross joints, let us connect them with a line and let us call that line m_B. Now let us construct our projectivity, it is going to be a sequence of two perspectivities. The first perspectivity from L to m_B , is centered at b.

So, it is a perspectivity centered at b from L to m_B. Where is it taking A, B and C? Well, it is taking A to here. It is pulling them down to taking A, B and C to these three points here. That is what the perspectivity centered at b is, if we then follow with a perspectivity centered at B going from the line m_B to the line l.

Well you can see that it is pushing them down to a, b and c. So, taken together this composition is sending A, B and C to a, b and c. If you want to see it again, the first perspectivity is centered at b so we end up taking A, B and C to these three points.

The second perspectivity centered at B, pushing these points out down to a, b and c. So, basically A travels on a path like this to get to a, B travels on a path like this to get a b, C travels on a path like this to get a c and it accomplishes what we wanted.

So maybe from the picture you can see why it is called a criss-cross construction and you can also probably imagine how it would be if we have picked A or C as our center? If you cannot imagine it, picture it very well. That is fine. We are going to see it now because now we are going to use the lemma twice and those times are centered at A and C. So we will actually get to see all three constructions.

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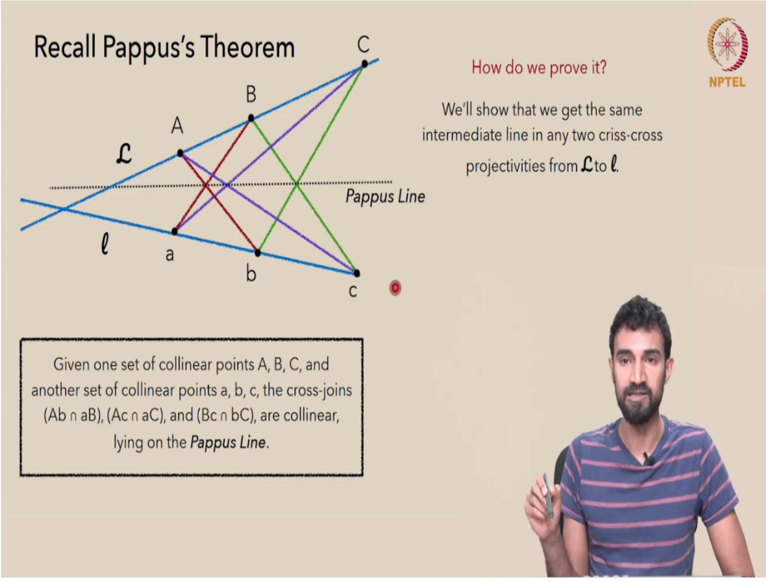

Let us recall Pappus's theorem and let us prove it. So, what does Pappus's theorem say? It says that given a set of a collinear points A, B and C and another set of collinear points a, b and c the cross joints Ab intersect aB, Ac intersect aC and Bc intersect bC are collinear and they lie on the Pappus's line. Pappus's line is defined to be the line containing them.

So these three cross joints are collinear and we will call the line that they all lie on, the line that they all share, we will call that as the Pappus's line. So, that is Pappus's theorem. How do we prove it? We will show that we get the same intermediate line in any two criss-cross constructions, any two criss-cross projectivities from L to l.

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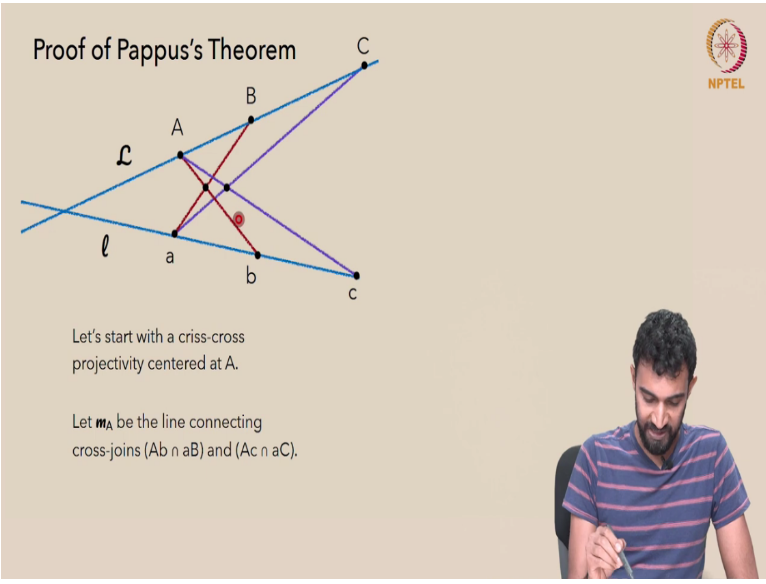

So, let us start with the criss-cross projectivity centered at A. So let us let m_A be the line connecting the cross joints Ab intersect aB and Ac intersect aC. So, we have m_A as the line connecting this.

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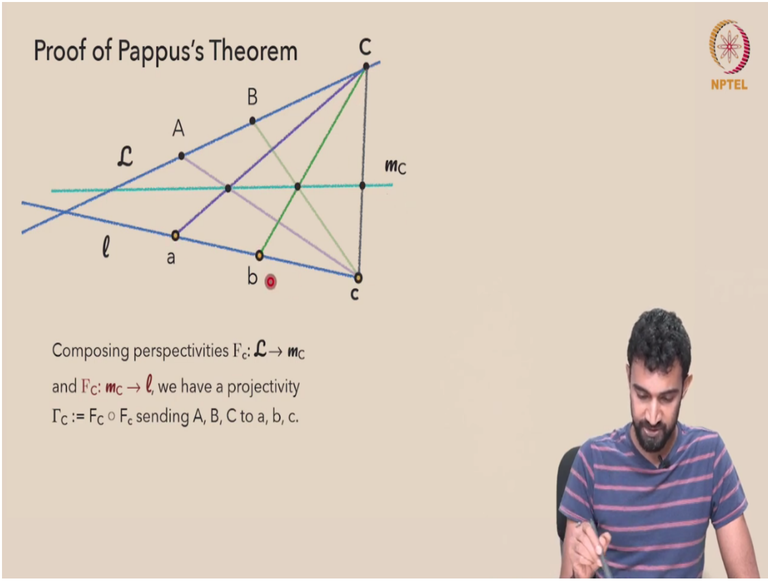

Now how does the criss-cross construction work? We compose the perspectivities F_a from L to m_A and F_A from m_A to l. So, when we do that we first look at F_a from L to m_A. It is centered at a. So it is pulling these golden points down to these three points here. Now we follow that by F_A and what does that do? It pushes these points along these lines down to a, b and c.

So taken together we end up getting to a, b and c like we want. So that is how the criss-cross construction centered at A looks. Now let us compare that to the criss-cross projectivity centered at C. So here we are taking the cross joints Ac intersect aC and Bc intersect bC and we are connecting them with a line m_C.

And now we compose perspectivity centered at c and C. So, the perspectivity centered at c, what is it doing to A, B and C? Well, it is pulling them down to these three points here. Now the perspectivity centered at C. What is it doing to these three points? It is pushing them down to a, b and c, so that is again we are doing and that is all fine.

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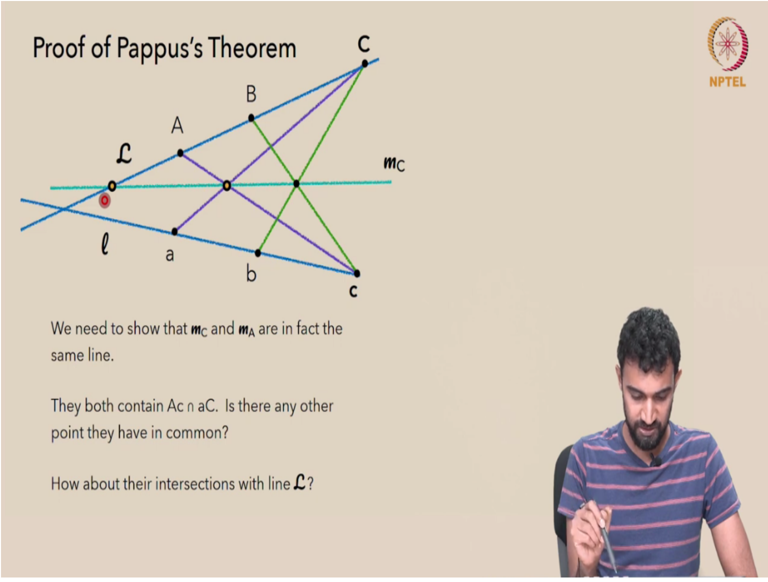

But what we really want to show is that m_C which is the intermediately line we just choose here and m_A which is this intermediate line defined by these two cross joints. We want to show that those are the same line. We need to show that m_C and m_A are in fact the same line. So, both m_C and m_A contain the point Ac intersect aC because that cross joint is used in both criss-cross constructions, the one centered at a and the one centered c, they both involve this cross joint here.

Is there any other point that they have in common? Is there any other point that is contained in both m_C and m_A? The answer is yes, right here, their intersections with the line L. From the pictures, it looks like the intersections should be the same place. So can we prove that they intersect the line L in the same place?

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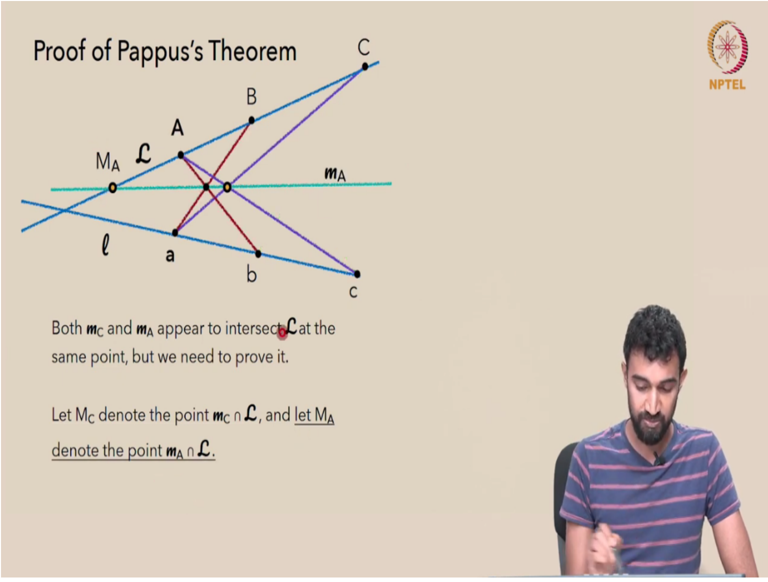

So, let M_C denote the point m_C intersect L, this is the point of intersection of m_C and L. Let M_A denote the point of intersection between m_A and L. So, now our task is to prove that $M_A = M_C$. The point M_A was defined from the first criss-cross construction, M_C was defined from the second. If we can prove that they are the same then we can show that these two intermediate lines m_A and m_C are the same and that will kind of solve our problem.

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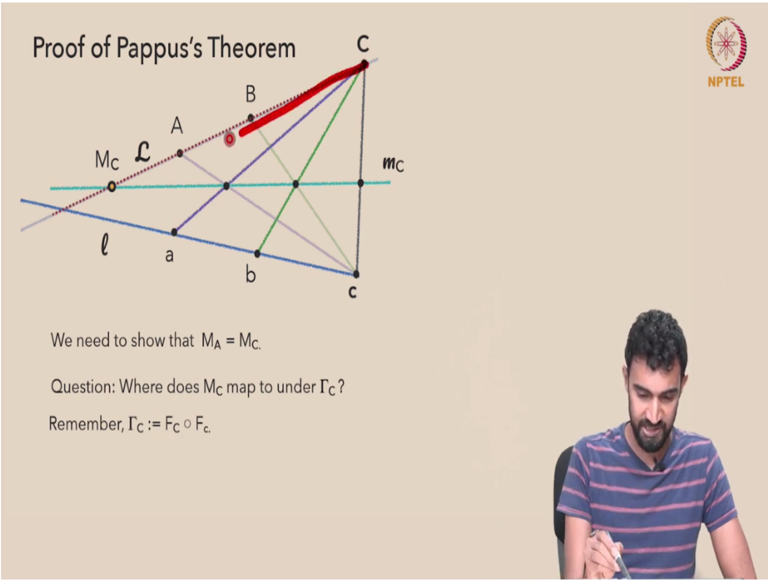

How do we show that they are the same? Well, here is a question. Where is M_C map to under criss-cross projectivity Γ_C? Remember Γ_C is F_c composed with F_C. So, F_c is pulling these points down and what is it doing from L to m_C? What is it doing to the point M_C? M_C lies in the intersection of the two lines that F_c is mapping between. So M_C is fixed. On the other hand, what does F_C do? It is centered at C, so it is pushing points down from m_C to l.

So, what is it doing to this point M_C? It seems to be mapping M_C to a point over here which we have not named yet. What is this point over here? Well, it is the intersection of l with L, that is where M_C is mapping to under the perspectivity F_C centered at C.

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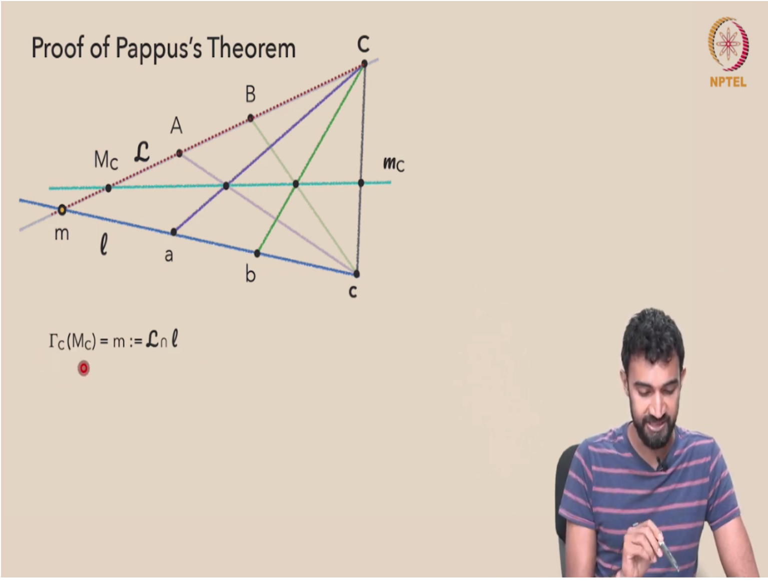

So let us give that point in name. Let m denote the point L intersect l. Clearly $\boldsymbol{\Gamma}_C$ maps M_C to m.

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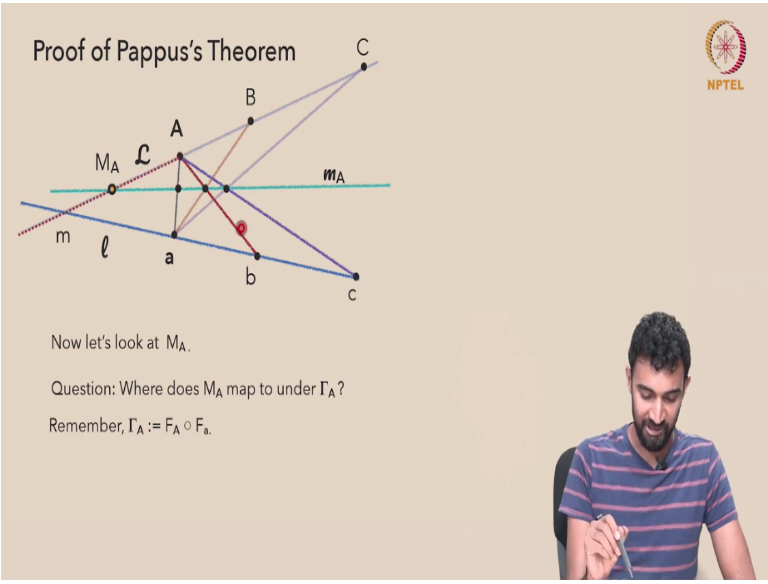

So, now let us look at M_A. Where does M_A map to under $\boldsymbol{\Gamma}_A$? Well, $\boldsymbol{\Gamma}_A$ is F_a composed with F_A. F_a in perspectivity centered here, it is pulling points down from L to m_A. What is it doing to M_A? It is fixing it because that is in the intersection. Now what does F_A doing? Well, it is pushing these points down from m_A to l.

And what is it doing to M_A? It is mapping it to the intersection of l with L, it is mapping it to the same place, it is mapping it to m.

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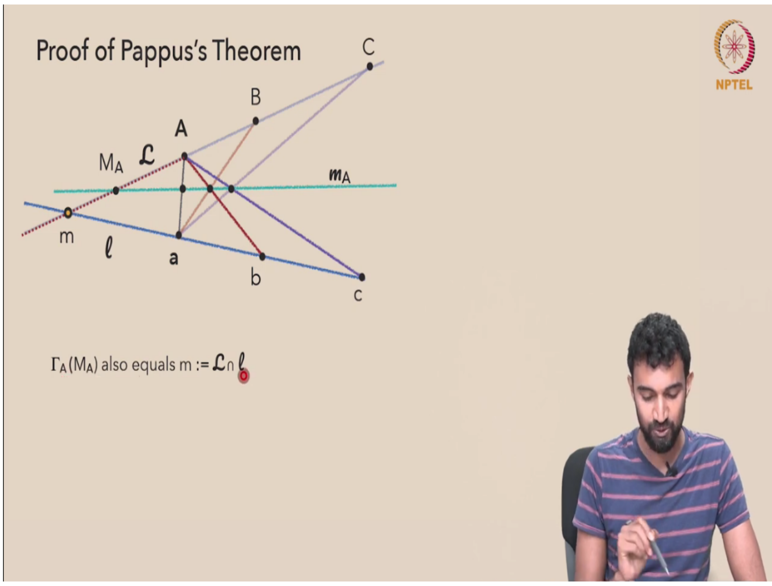

So, $\mathbf{\Gamma}_A$ also maps M_A to m, the intersection of L and l.

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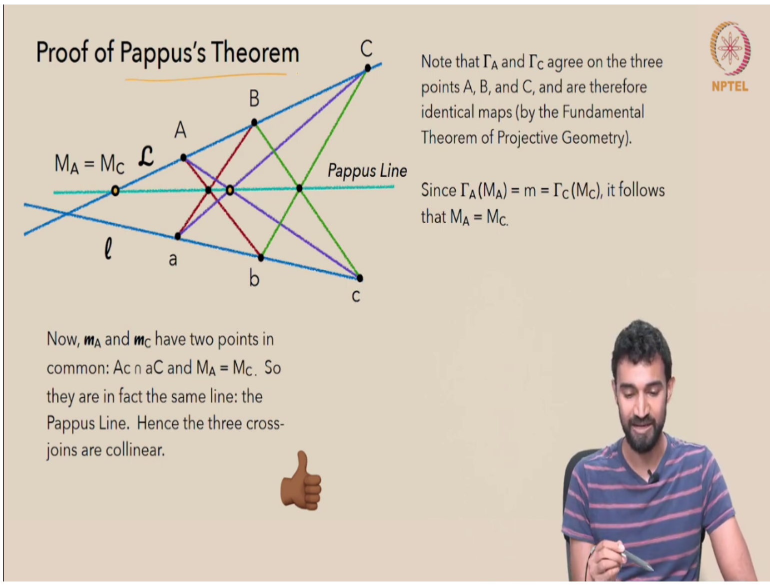

So, note that $\mathbf{\Gamma}_A$ and $\mathbf{\Gamma}_C$ agree on the three points A,B and C already. I mean we have already shown that $\mathbf{\Gamma}_A$ and $\mathbf{\Gamma}_C$, the two criss-cross projectivities, both agree on A, B and C. So, they are actually identical maps as maps from L to l. They are identical by the fundamental theorem of projective geometry. Now, since $\mathbf{\Gamma}_A(M_A) = m$ and $\mathbf{\Gamma}_C(M_C) = m$.

So it means that both of these projectivities which were in fact identical are taking M_A and M_C to m. So, it follows that the point M_A is equal to the point M_C. What does it tell us? It tells us that both m_A and m_C have two points in common: they have this cross joint and they have this point M_A which is the same as M_C. So, they are in fact the same line.

These two m_C and m_A are in fact the same line, they are both the Pappus line and hence the three cross joints are all collinear. So, we can just call it the Pappus line and these are all collinear as we suspected. So, that proves Pappus's theorem. I just mentioned that there are actually many, many more proofs of Pappus's theorem. Pappus's theorem was first stated in 340 that is the first written record of it when Pappus himself wrote it in the year 340.

And in the intervening years there have been many different proofs to it especially as we come towards the present day. So, it is kind of an interesting window to projective geometry because there are proofs involved in just about every concept you can think of in projective geometry, but this proof I like in that it does introduce too much additional machinery, there is nothing extra we have to really introduce beyond the basic principles, basic concepts of projective geometry so it is kind of nice that way.