Our Mathematical Senses
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Lecture – 10
Projectivities as Functions on the Real Numbers

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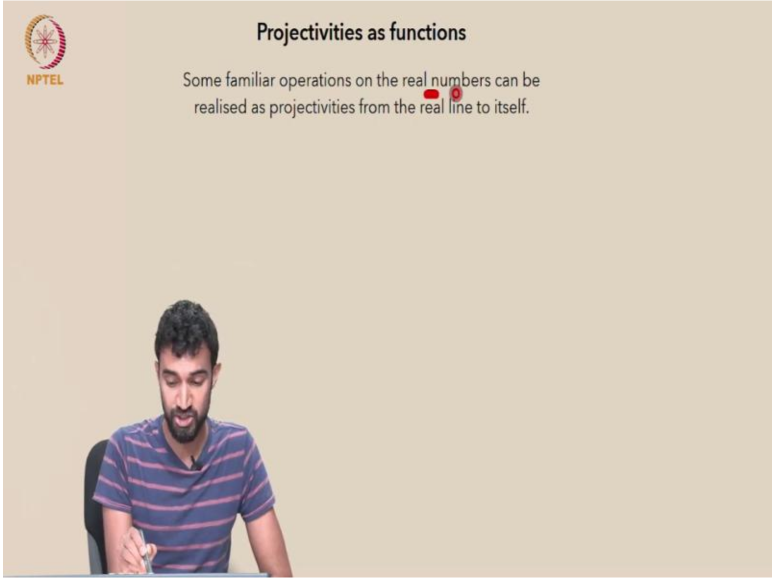

For now I want to talk a little bit about projectivities as functions. In other words, I want to look at projectivities of the real line to itself. Because we can think of that as a function from R to R and in doing so we will see that some familiar operations on the real numbers can be realized or constructed as projectivities from the real line to itself.

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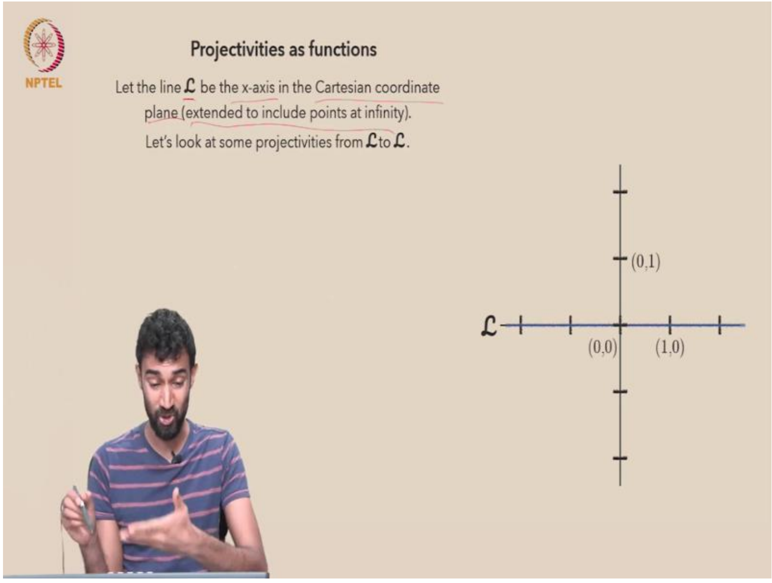

Let L denote the x axis in the Cartesian coordinate plane which we think of as an extended plane so we have points in infinity lying out beyond our sites. So, it is really an extended plane, but we still have Cartesian coordinates on it. So, this is (0,0), this is (1,0), this is (0,1). We have the usual R^2 Cartesian coordinate system and let us look at some plane projectivities from L to itself, from the x axis to itself.

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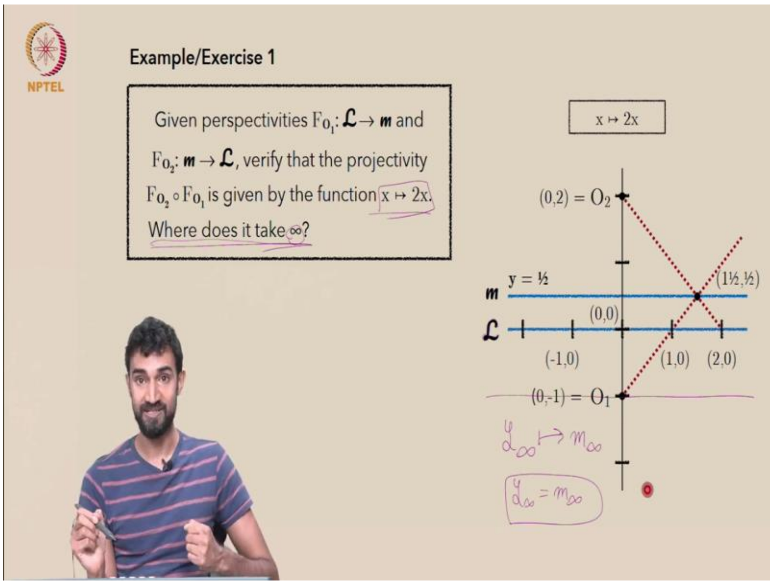

So, the first example that I want to look at, which is an example/exercise because I am not going to fully verify I am leaving it to you to verify that what I am saying here is true. So, the first example is that given this diagram here which we will look into a little more deeply in a second. In this diagram the perspectivity F_{O1} centered at O_1 from L to m followed by the perspectivity F_{O2} centered at O_2 from m back to L, the projectivity F_{O1} composed with F_{O2} gives you the function x goes to 2 x.

So, this is a dilation or a scaling function which doubles everything. I am claiming that this projectivity that I have constructed here, as a map from the x axis to itself, as a map from the real line to itself, takes any real number and doubles it. So, let us see why that is true.

So, let us just look at the number (1,0) first, think that is the number 1 in the real line. So, what does F_{O1} do? Well, it is projecting this point over to this point here. What is this point? Well you can work out that it is (1.5,0.5). By the way, m is the line y=0.5, it is another horizontal line.

So, it is projecting this point (1,0) to this point (1.5,0.5). Then our second perspectivity centered at O_2 is projecting this line m back to L. And it is projecting (1.5,0.5) into the point of (2,0). So, I was taking (1,0) to (2,0) or as a function of the real numbers it is taking 1 to 2. I claim that it is generally taking x to 2x for any x. So, I will leave it for you to verify. But maybe just to see another example, here is the point 1 / 2 let us just see where that appears to go.

It appears to go here from the first perspectivity and it appears to go this way from the second perspectivity and indeed it looks like it goes to 1 more or less, but I will leave it to you to verify that algebraically. Similarly 0, that is a bit easier to verify, goes up to here via the first perspectivity and back down to here via the second perspectivity, so 0 goes to 0.

So, let us assume this is indeed x goes to 2x. So, I have to do a little work, but maybe for a general point, say (x,0), check algebraically where it gets sent to through this sequence of two perspectivities. That is an exercise. I am assuming that is true. Where does it take infinity? Actually, I do not even need to assume this function. I mean just from the construction itself, let us see where this projectivity takes the point at infinity.

Well, just to refresh your memory, how do we figure out where perspectivity sends a point at infinity? Well the key is to draw a line through the center of perspectivity which is in that class of that point at infinity, but that was in the 2D case, now we are in the 1D case where there is a single point at infinity which I am denoting by the infinity symbol.

So, the class corresponding to that point at infinity is just the class of horizontal lines. So, to find out where the point of infinity is, we have to draw a line through O_1 that hits the point of infinity of L. So, to do that we have to draw a horizontal line through O_1. And we have to see where that line hits m, to see where the line of infinity in O_1 is going to. But this horizontal line does not actually hit m in the plane; it intersects m at infinity.

So in other words, it is taking the point at infinity in L to the point at infinity in m because they share the same point of infinity because they are parallel to each other. L_∞ equals m_∞ and this horizontal line through O_1 is hitting this point m_∞ so it is fixing it. So, infinity goes to infinity.

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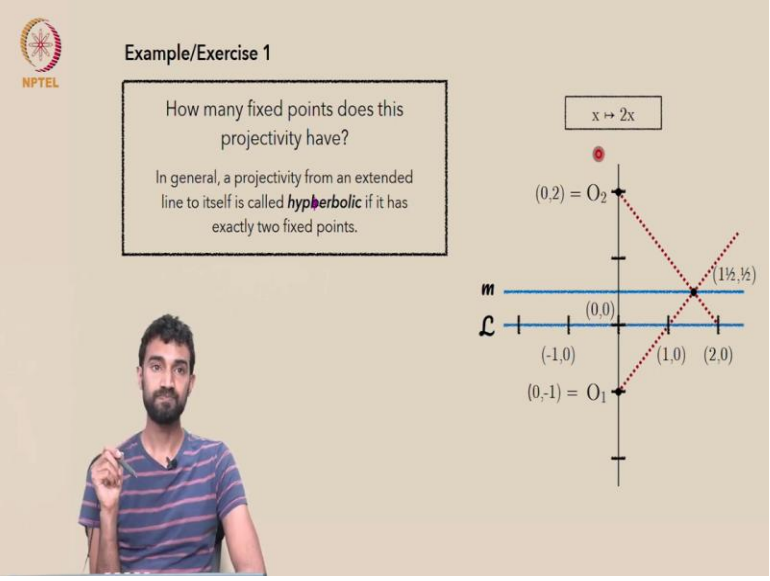

And infinity is a fixed point of this projectivity x goes to 2x. Now is it the only fixed point or other fixed points? Maybe you remember that we just mentioned another fixed point namely 0, it goes up to here and then back down to here, so 0 goes to 0, so there are actually two fixed points. Are there any other fixed points? Well, some familiarity with this function will tell you that there are no other real numbers that are fixed by the doubling function besides 0 and in this case infinity is also fixed.

So, we have two fixed points of this extended line under this projectivity and in general a projectivity from an extended line to itself is called hyperbolic if it has exactly two fixed points. So, this is an example of a hyperbolic projectivity from the line L to itself.

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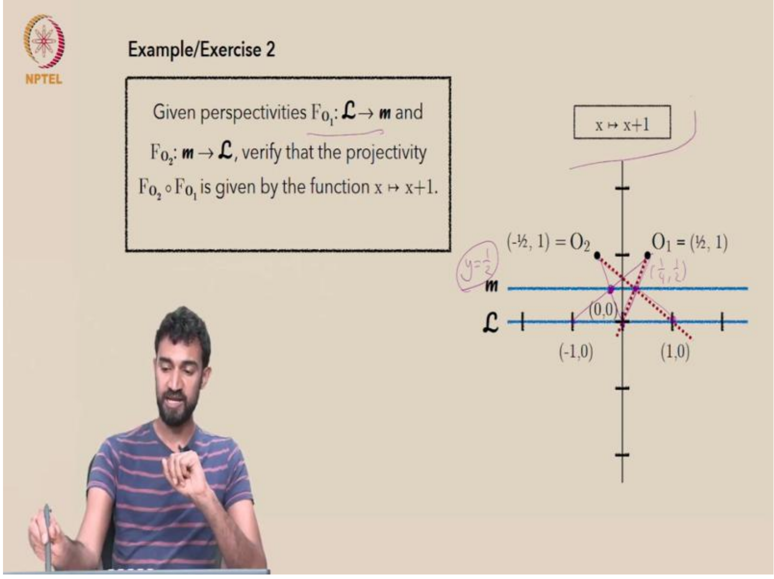

So, let us look at a second example this time of the function x goes to x + 1. Let us first look at this construction here, which I claim, represents this function x goes to x + 1. So, how to see that? Well where is (0,0) going? O_1 is pulling (0,0) up to here. So by the way m is once again the line y=0.5. So, what is O_1 doing?

Well it is pulling (0,0) up to this point. What is that point? Well it is actually (0.25,0.5) you can work that out, because O_1 is at (0.5,1). This is the line with slope 2. So, the first projectivity F_{O1} maps (0,0) up to (0.25,0.5). The second projectivity is going to push this point down to (1,0). So we are mapping 0 to 1. Similarly, what is happening with -1?

So, (-1,0) is pulled up to this point via F_{O1} and then via F_{O2} is pushed down to (0,0). so -1 maps to 0. So, you can verify for yourself algebraically that this construction here actually represents the function x goes to x + 1. Part of that will be on the assignment as well. So, do take a look at that. For now we will accept it as a fact.

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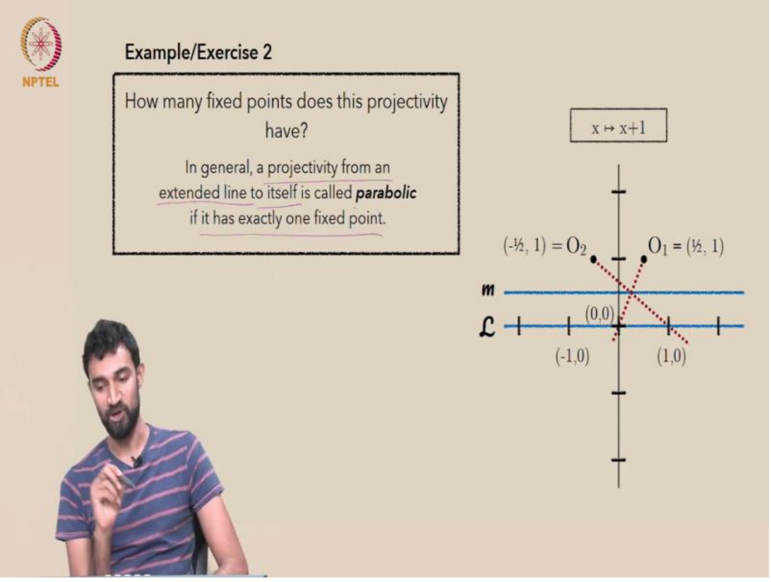

And ask the question. Okay we accept this, this is the construction we can realize this as a projectivity, how many fixed points does this projectivity have? Well, as a function of the real numbers, x goes to x + 1, the standard translation by a unit, has how many fixed points? Well none. You are translating everything along, nothing can possibly be fixed. So, there are no fixed points for the real number version of this, but now we are on the extended line we have a point of infinity.

Where is the point of infinity going, but once again we can figure that out by looking at the horizontal line through O_1 and the horizontal line through O_2. I just realized that I made a small mistake earlier. I mean I did not finish my explanation for why infinity is a fixed point in this example. I showed that the point of infinity at L is the taking the point infinity at m by $O_1$and they are at the same.

Basically that O_1 fixes that common point at infinity, but you can easily see that O_2 does the same thing. We can draw the horizontal line through O_2 that is also fixing m_∞ which is equal to L_∞. So, the projectivity fixes it as well the composition of those fixes that as well. Okay going back to here, there is a single horizontal line I can draw between O_1 and O_2.

And for similar reasons we can see that the point of infinity is fixed by O_1 and is fixed by O_2. And indeed m_∞ is equal to L_∞. Again they share one point at infinity and it is fixed by both these perspectivities, so the projectivity fixes it. So, we do have a fixed point, the point of infinity, but we only have one fixed point in this case and in general a projectivity from an extended line to itself is called parabolic if it has exactly one fixed point. So, earlier with two fixed points it is hyperbolic with one fixed point it is parabolic.

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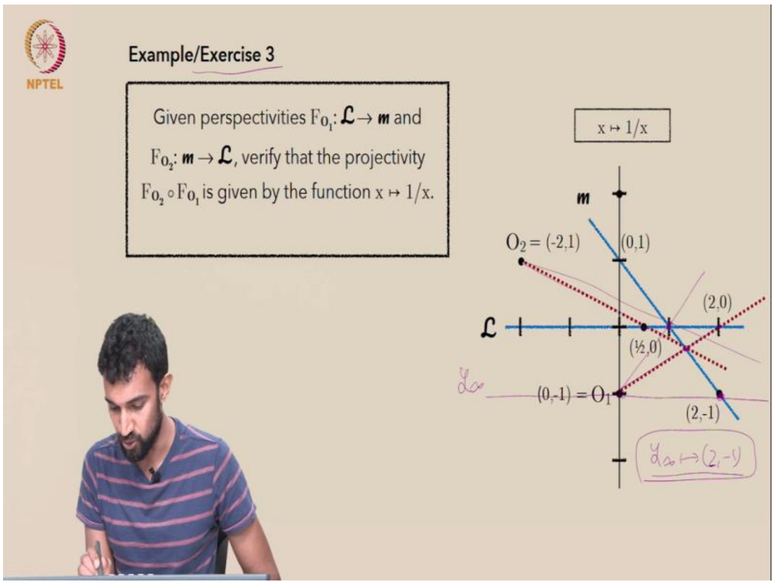

And let us see the third example and this is another function x goes to 1/x. By the way these functions are not fully random. There is a reason that I am choosing these functions to look at as projectivites and we will see why. So, let us look at x goes to 1/x. I know these constructions are getting a little weirder and weirder. This one does take some time to sit with. I do not expect you to look at it now. You need to spend about a few minutes working it out and I am not going to do that right now.

So, that is an exercise. But maybe we will at least see that it works on this point 2. Where do theis 2 go? Well O_1, let us just visually see it, pull it down to here and O_2 pulls it up to here and 2 goes to 1/2. Similarly for 1 here, the projectivity through O_1 does nothing, because it is on both L and m. So this point here is fixed.

So, F_{O1} does nothing to it. What about F_{O2}? Well F_{O2} also does nothing because it is again on the intersection of m and L. So, 1 is fixed. We see that 2 goes to 1/2. Maybe one more thing to quickly check, where does infinity go in this case? This is a little more interesting. So for L, by drawing a horizontal line through O_1, we can see where the point L_∞ gets mapped.

It is mapped to a regular point on the line m, not mapped to the point of infinity for m. It mapped to this regular point (2,-1) of m. So, infinity is finally not fixed in this case. I will leave it to you to verify that this is indeed that map x goes to 1/x. Sorry I did not finish well. I will say it again.

Let me just finish what I was saying: FO1 takes L_∞ to (2,-1). What does F_{O2} do to (2,-1)? We have to see where it ends up. Remember we are looking at the projectivity. We want to see where F o 2 sends (2,-1) in order to see really where infinity goes. Well for that we need to draw a line between (2,-1) and O_2. O_2 is (-2,1). So, the line connecting negative (2,-1) and (-2,1) is just this line which goes through (0,0), which hits L at the point (0,0). So L_∞ is actually maps to (0,0).

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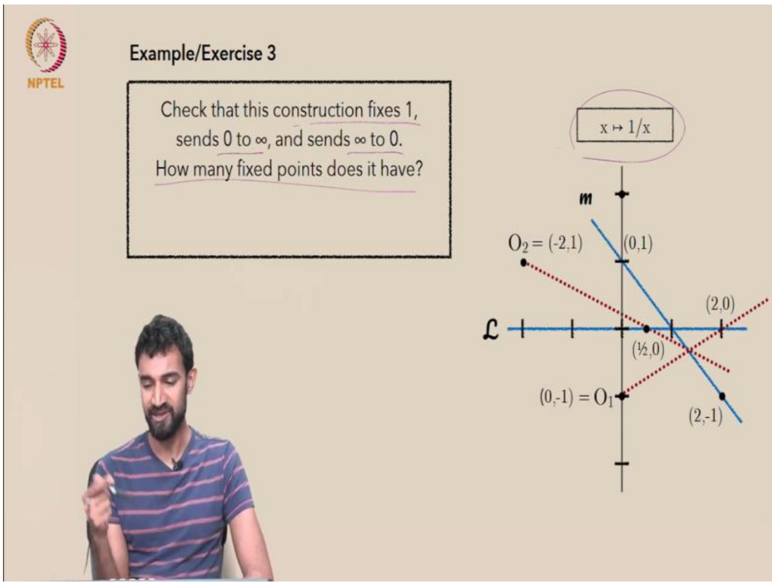

So, infinity is getting sent to 0. I will leave it to you to check that it sends 0 to infinity. Also I will leave for you to check how many fixed points there are. Well we can use the fact that we know a little bit about this function, how many fixed points this function has.

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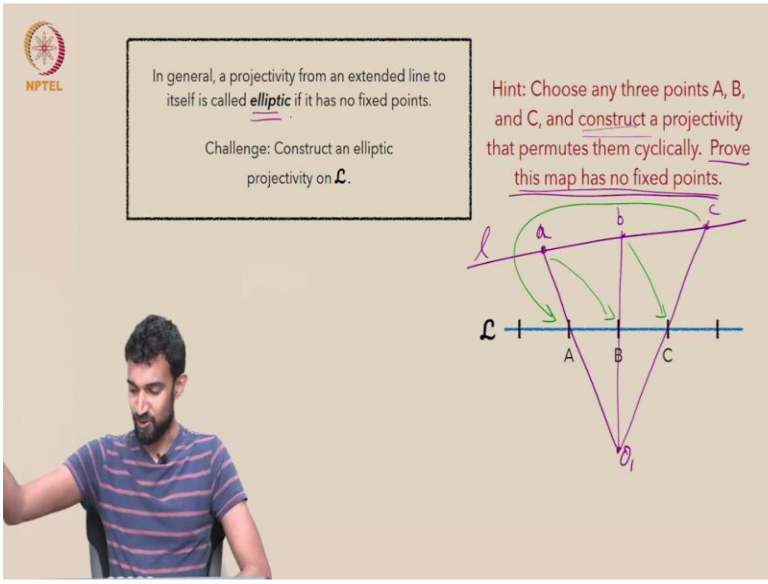

Now, in general a projectivity of an extended line to itself is called elliptic if there are no fixed points. So this is a challenging exercise: can you construct an elliptic projectivity on this L? Construct a projectivity that honestly has no fixed points. Remember translation does not work because translation fixes the point infinity. We want a projectivity of the extended line which has no fixed points including the point of infinity.

So, this is a bit tough. It is not the easiest thing to do, but it is an interesting exercise and it is a pretty hard exercise except let me give you a hint which will make it hopefully quite a bit easier. So, my hint is the following. Choose any three points A, B and C. And construct a projectivity that permutes them cyclically.

For example you could send A to B, B to C and C to A. That is an example of the cyclic permutation of these three points and how do you do that construction? Well we saw that you can always construct a projectivity that takes any three points to any other three points here we only have one line though. So might be helpful to first project these three points on to another line, let us call it l.

And let us just project, we can just choose any center of perspectivity if you want and we can project over here and that will give us all this a, b and c and now using the construction we have already seen you can now map a, b and c via another projectivity to A, B and C, but permuting the order. You can now find a projectivity that sends a to B, b to C and c to A.

So, I will leave it as an exercise to construct that projectivity that taken together with this triple projectivity here, let us call this O_1. So, F_{O1} followed by this green projectivity together is a projectivity that will permute these A, B and C cyclically. So that is relatively easy we know how to do it, it just a matter of working it out, but the harder part now is to prove that this map which permutes A, B and C actually has no fixed points anywhere clearly A, B and C are permuted cyclically none of those are fixed.

But I claim that this map will have no fixed points at all anywhere including infinity, including 0. So, in other words it is an example of an elliptic projectivity. So, I am leaving that as a challenge for you to see if you can construct that and prove that it is indeed an elliptic.

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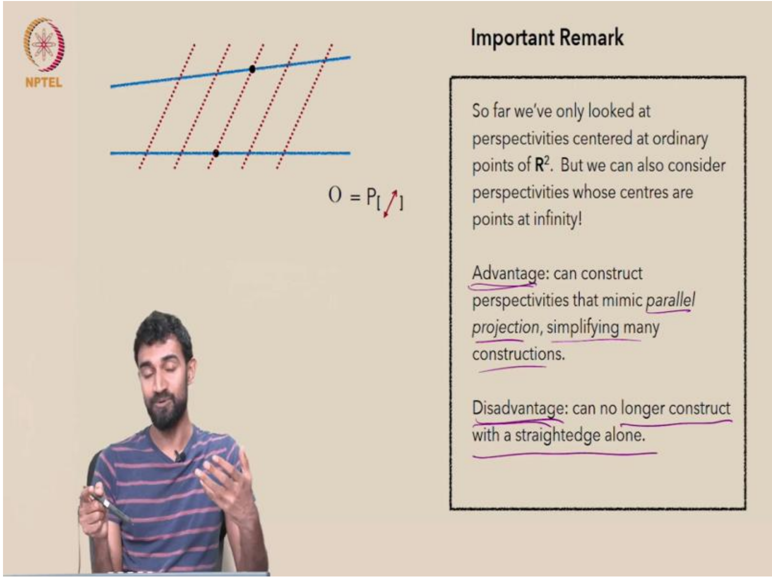

So we looked at a bunch of projectivities that captures function of R^2 and so far we have only looked at perspectivities centered at ordinary points of R^2, but I just want to mention that there is nothing to stop us from considering perspectivities whose centers lie at infinity, at different points at infinity and when we do that here is an example of that. We end up mimicking parallel projection.

Because our center of projectivity is infinitely far away, the lines from that are all going to appear parallel to each other. Remember any points on this ordinary plane when we connect it to a fixed point in infinity is going to give us a line in that parallel class corresponding to that point in infinity. So, they are all going parallel to this line here which is defining the point of infinity that I have chosen.

So, the advantage to doing this is that we can construct perspectivities that mimic parallel projection. This is parallel projection, what you are seeing here is projecting by parallel rays of light and this simplifies many constructions. So this is a useful thing to be able to do parallel projection as opposed to only doing central projections which always does some distortion, parallel projection is a lot more friendly when it comes to certain Euclidian notions.

So it does not distort ratios, it is the right way to say I guess, where central projection distorts ratios and creates a lot of overall distortion to deal with. So, it is nice to be able to do parallel projection, but for that we need points of infinity to be our centers of perspectivity. On the other hand the disadvantage is that if we are using points at infinity for our perspectivities then we can no longer do these constructions with a straightedge alone.

Up until now all of these constructions that we have been doing, all these functions that we have been constructing, I can do these constructions just using a straightedge, no need for a compass, no need for a ruler or a measuring device. I am just using a straightedge so that is kind of cool, that these are constructions that you can do with a straightedge alone we are not taking any measurements except for we are given a coordinate plane.

We are starting with a kind of a God given real line with measurements on it, but we are not using a measurement device beyond that, we are just using a straightedge. So, it is not a major disadvantage, but it is something to keep in mind that if we are doing parallel projection we cannot do that with a straightedge alone.

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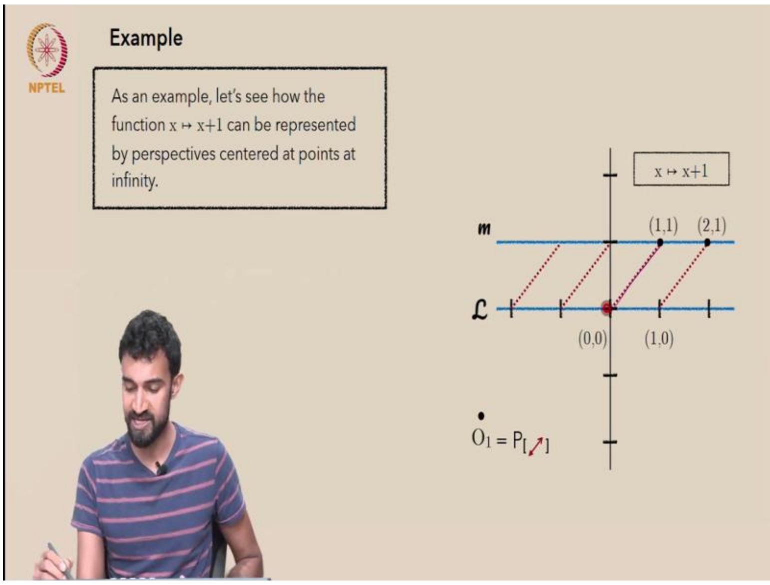

But let us take a look at the advantages. Let us see how this function x goes to x + 1 can be represented by perspectivities centered at points at infinity. So, we will use the following picture: let us take this point in infinity here and two parallel projections, this is just a slope 1 line here and via parallel projection, by the slope 1 line, I am sending (0,0) to (1,1) and (1,0) to (2,1).

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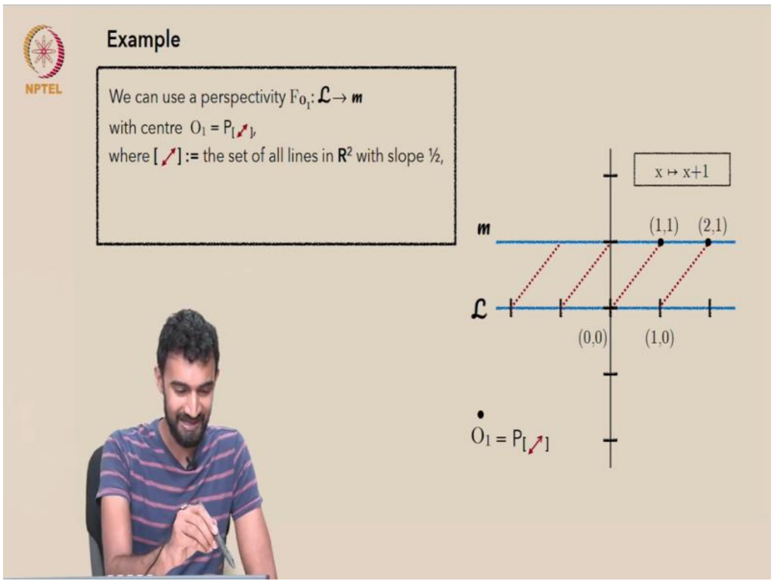

Let me just say it a little more clearly, as a perspectivity this is F_{O1} from L to m with center O_1 equal to this point of infinity. Sorry why I am saying slope 1, I should be saying slope 1/2. No sorry, it is a typo, it is just slope 1.

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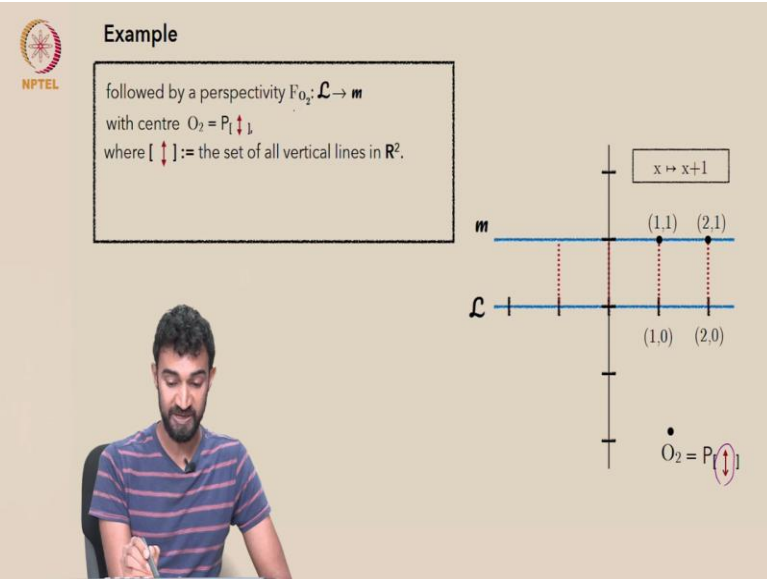

And next we just parallel project everything down via lines like this. So, we follow with a perspectivity F_{O2} from m back down to L with center O_2 which is the point of infinity corresponding to the set of vertical lines. That will just take (1,1) to (1,0) and (2,1) to (2,0).

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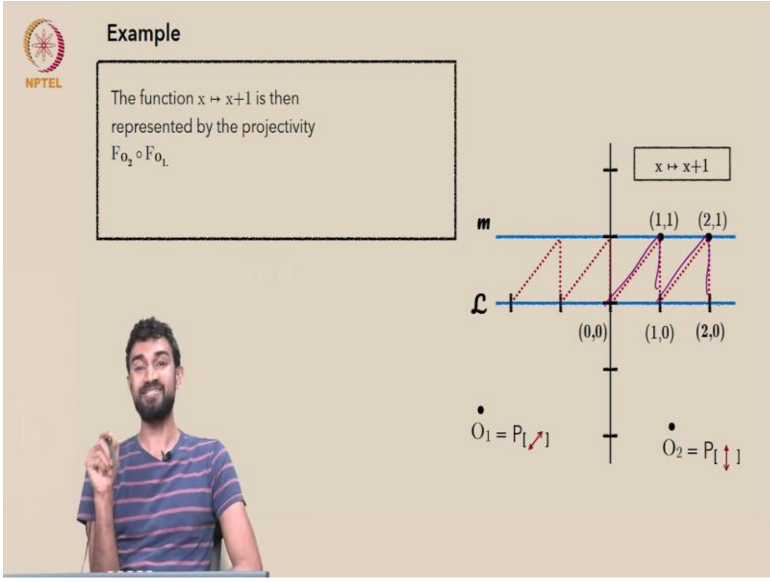

I have taken together we are mapping (0,0) to (1,0) and (1,0) to (2,0). So, this is a little easier to see why this is a translation, it is actually much easier to see. So, that is the advantage to thinking of translation as given by this projectivity here.

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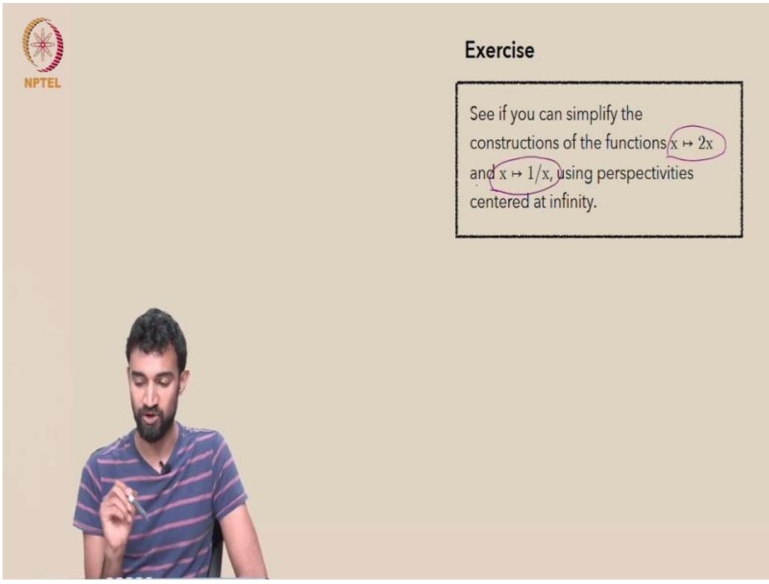

So this is an exercise: can you simplify the constructions of the function x goes to 2x and x goes to 1/x using perspectivities centered at infinity. In other words, it is using parallel projection.

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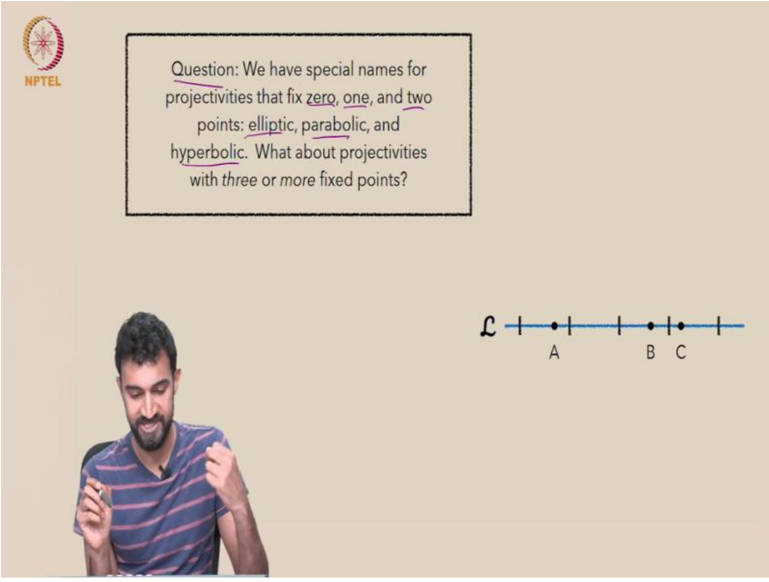

Now I have a question. We have special names for projectivities that fix zero points, one points and two points. Zero points is elliptic, one point is parabolic, two points is hyperbolic. What about projectivities that fix three or more points. So for example a projectivity that fixes A, B and C does it also get a special name.

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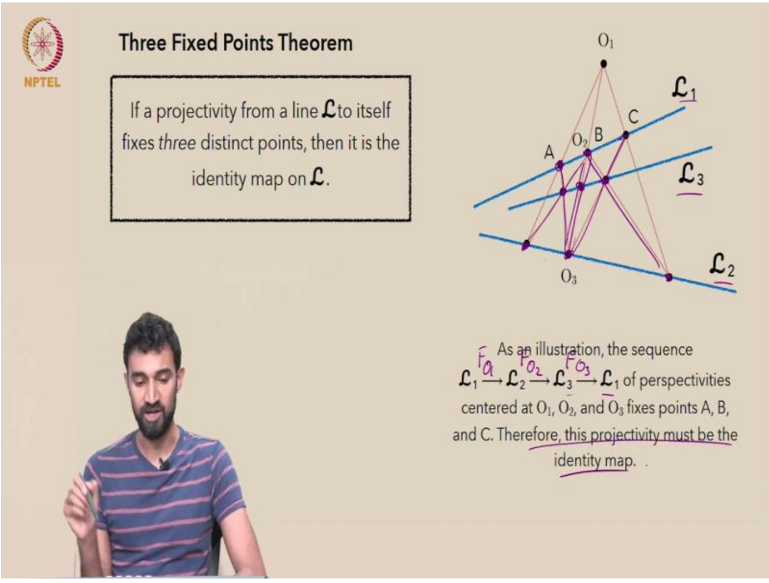

And the answer is, if a projectivity from a line L to itself fixes three or more distinct points then it is the identity map on L. So, it is kind of a special name it is the identity. If it fixes three or more points it fixes every point which is why we are calling this the three fixed points theorem. If a projectivity from a line to itself fixes three distinct points, then it fixes every point on that line. When I say line here I really mean an extended line.

So, the line includes a point infinity that is important to keep in mind. Just a quick illustration of this here is a projectivity that fixes three points. So, first I am looking at F_{O1} from L_1 to L_2 that taking A, B and C down to these points here. Next F_{O2} from L_2 to L_3 that is taking these points to these three points here and finally F_{O3} from L_3 to L_1 that takes these three points to these original points here.

So, A, B and C are going back to A, B and C in that order no permutation of them, A is going back to A, B is going back to B, C is going back to C, so these are fixed points. So, according to the theorem this projectivity must be the identity map that is the content of the three fixed points theorem.

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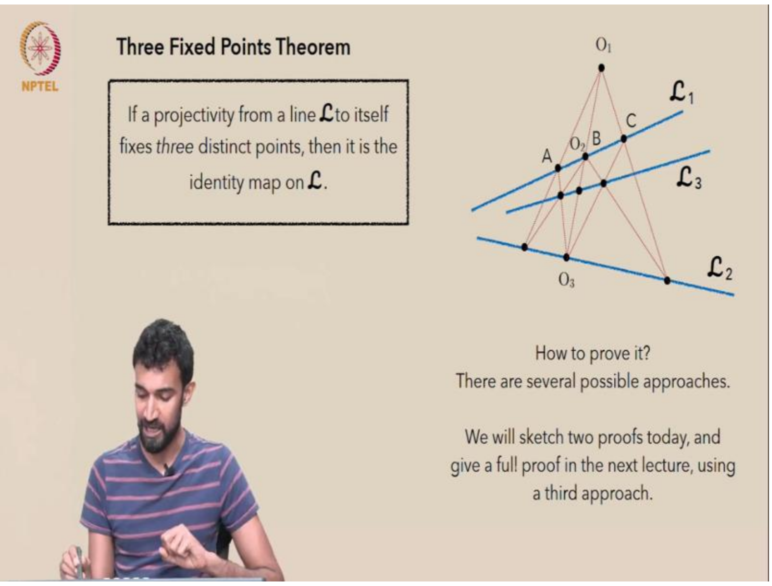

So, how do we prove this? There are actually several possible approaches. There are many different proofs and this turns out to be very crucial results. So there are many ways of approaching it and we are going to sketch two proofs of it right now and in the upcoming lecture or actually maybe the next lecture we will prove it fully using a third approach.

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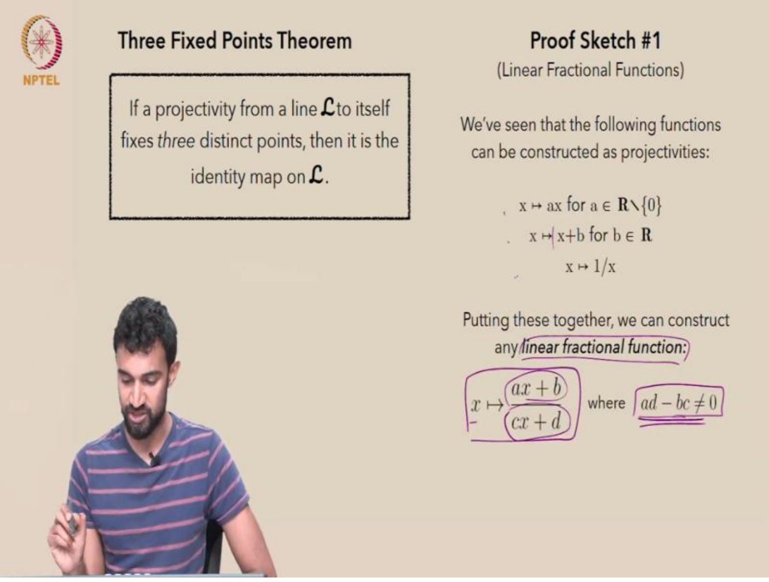

So right now, let us give two sketches. Each sketch is going to have a gap which I will leave for you to fill, but they are a bit non-trivial. So, which is why I do not want to call it a complete proof and we are not going to use these proofs in the course. So, we will do a third proof which we will do fully after this. So, the first proof sketch uses something called the linear fractional function. So, this is my linear fractional function proof.

And how does it work? We have already constructed several functions of the real line using projectivities. We have seen that the following functions can be constructed as projectivities x goes to ax, x goes to x + b and x goes 1/x, which means I am exaggerating a little. We saw x goes to 2x and we saw x goes to x + 1. But it is not much of a stress to imagine you can alter our construction a little bit to get any non zero real number a, as your scaling factor and any real number b, as your translational factor.

So, I will leave that for you to verify, but using these constructions we have already seen it is pretty easy to get any of these functions as projectivities. Putting these together, what other functions can we get? So, taking combinations of these three functions, basically composing these functions, that is what I am allowing.

Taking these three types of functions and composing them as many times as you want and in any order. What other functions can we get? It turns out that there is a nice name for those and nice structure for the entire class of functions that are generated by these three, namely they are linear fractional functions and they look like this x goes (ax+b)/(cx+d), where ad-bc is non zero.

I guess, technically speaking ad-bc being non zero makes it an interesting linear fraction of function. If ad-bc=0, then what happens? Let us check that quickly. If ad-bc=0, then we have a problem, actually because then the numerator is going to be a scalar multiple of the denominator. What will happen then? Well in that case we are just going to get a constant, the x will cancel out and will get that x is going to a constant.

So, that is not a particularly interesting function. I guess we do not want to call that a linear fractional function. So, we will exclude this case where ad-bc=0. But how do we see that composing these functions gives us a function of this form?

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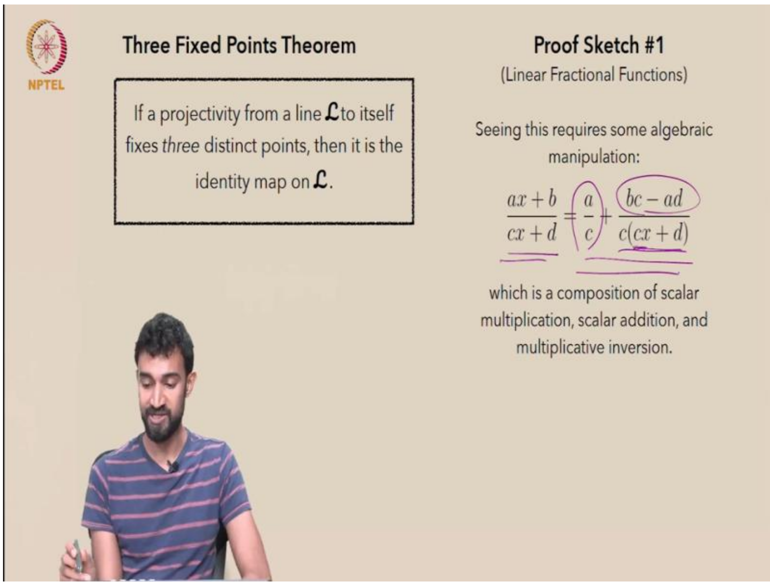

So that requires some algebraic manipulation, namely you can rewrite the standard linear fractional function in this form. Over here which is more clearly a composition of these three types of functions that I mentioned. In particular you can take x, you can multiply by c, so you can do a dilation of it. Then you can add d you can do a translation followed by another dilation by c followed by a reciprocation you put in the denominator followed by scaling by the constant bc-ad after that you can translate by a/c.

So, it is a composition of these three types of functions when it is written this way and you can work out for yourself that these two expressions are indeed equal to each other. So, what this shows is that any linear fractional function can be realized as a projectivity because any linear fractional function can be written as a composition of these three types of functions and these three types of functions are all constructable as projectivities.

Therefore, any linear fractional function can be composed, can be constructed as a sequence of many, many projectivities.

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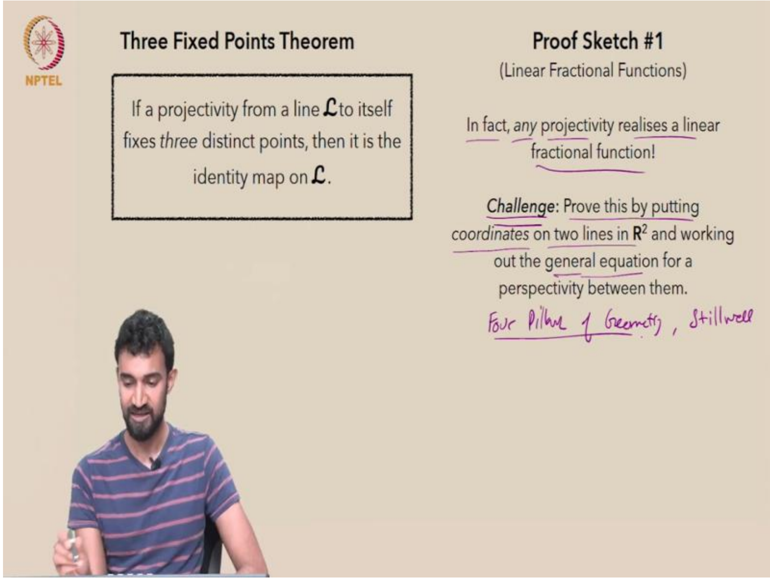

So, the interesting thing which is harder to see is that any projectivity realizes a linear fractional function. We have seen that any linear fractional function can be constructed as a projectivity, but conversely any projectivity you create, no matter how complicated, from the real line to itself is actually capturing some linear fractional function. It can be written down concretely as a linear fractional function.

So, the challenge that I am not going to do which I am leaving as a kind of a challenge exercise is to try and prove this by putting coordinates on two lines in R^2 and working out a general equation for a perspectivity between them. So, if you are interested in trying this and our reference is the book, Four pillars of geometry by Stillwell in which he does not completely finish this proof, but he does it as a series of exercises which might help you out if you are trying to do this.

So, I would recommend that book and why does this matter so let us assume this is true. Projectivities from a line to itself are basically the same thing as linear fractional functions, if that line has coordinates, if it is a real line. Why does this matter? Why does it help us prove the three fixed point theorem? Maybe that is what we are trying to prove.

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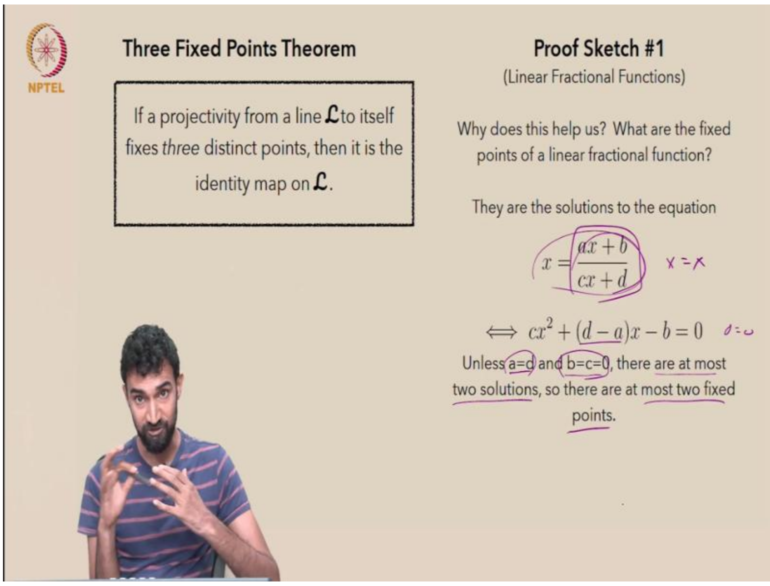

Well what are the fixed points of a linear fractional function. They are just solutions to this equation and we can easily rewrite that, we multiply both sides by cx + d and combine like terms and that gives us $cx^2 + (d-a)x - b = 0$. This nice quadratic equation, how many solutions does it have? The solutions to this are precisely the fixed points of a linear fractional function.

There is a special case to consider, it could be the case that a=d, so d-a=0 and b and c are also 0 in which this just becomes x=x. So, in that case it has infinitely many solutions. Because in that case we really just looking at the identity function. If a=d and b=c=0 then this is just the identity function and then there are infinitely many solutions.

On the other hand if that is not the case, you can solve this equation and there are at most two solutions. So, there are at most two fixed points; there cannot be three or more fixed points without being the full identity function. So, that proves the three fixed point theorem. Modulo, a challenge exercise which I will leave for you.

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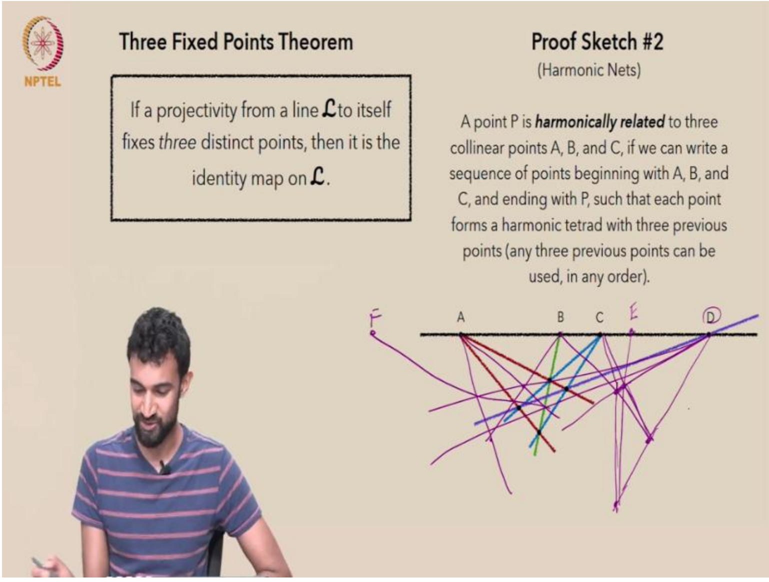

So, let us see another proof sketch. The second proof sketch I want to look at, I am calling the harmonic net sketch. This is going to be a totally different proof than the linear fractional functional proof. A completely different direction and in order to fully finish it, it would be helpful if you had a course in real analysis, but in fact you need that to fully complete the proof.

But it is interesting to think about even if you have not had a course in analysis. I want to just describe it. So, remember that a point P is harmonically related to three collinear points A, B, C if we can write a sequence of points beginning with A, B and C and ending with P such that each point in a sequence forms a harmonic tetrad with three previous points in the sequence.

Some three previous points, any three can be used in any order. So, remember harmonic tetrads means that harmonic tetrads are related by a diagram like this. They are a set of collinear points, there is a quadrilateral whose sides converge to A and C respectively and these diagonals hit that line AC in the points B and D. So from this picture we see that the point D is harmonically related to A, B and C or A is harmonically related to B, C and D or B is harmonically related to A, C and D, actually all of those are true.

Let us try and make another point that is harmonically related to these. Let us draw a point here and let us do my harmonic tetrad constructions, we can have a quadrilateral here. I messed up. I am getting the same point A again because remember that is the entire coincidence of harmonic tetrads. This is again I am looking here at the harmonic conjugate to C with respect to B and D which is just going to be A.

So, I cannot kind of mess up. I did not choose that. Now let us take these two points and let us connect C to both of them and D to both of them so that is going to give us this very small quadrilateral here. Now this is a quadrilateral and when we connect up to these two points we end up with a new point here which I am going to call E.

So E is harmonically related to B, C and D and so on. We can keep adding new points this way that are harmonically related to existing points by doing more and more of these harmonic relations and it is obviously kind of cumbersome to actually do, but the interesting thing is that we can consider the set of all points generated by A, B and C harmonically which is called the harmonic net of A, B and C.

And it consisted of all points harmonically related to A, B and C. As a challenge can we prove that a harmonic net is dense in the real line. Basically the point is that we can keep adding more and more points to this harmonic net.

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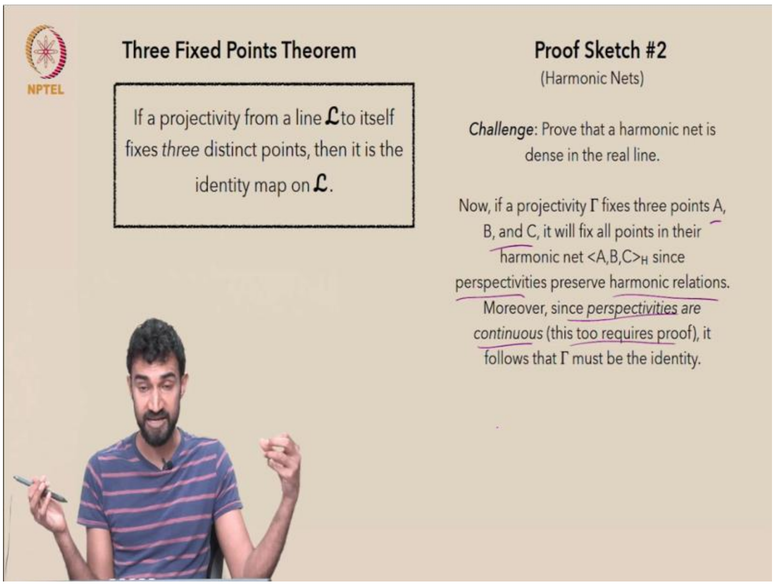

And I claim that this harmonic net that we generate is dense in the real line. Between any two points there is a new point which is harmonically related to A, B and C and that is I am not going to prove that. Proving it does require some work and there are actually many different approaches to this proof. So, it is a fun thing to think about. I will leave it to you to do that, but let us assume that it is dense. Why does that matter?

Well if a projectivity fixes three points A, B and C, we know that it fixes all the points in its harmonic net, since perspectivities preserve harmonic relations that is what we saw last week. So, harmonic nets are fixed under, a projectivity goes from a line to itself and fixes three points. It is going to fix an entire dense harmonic net and that is kind of a big deal because perspectivities are continuous. That too you have to prove we have not actually proven that.

But if we believe that then we have a continuous map fixing a dense subset of the real line which then has to be the identity. So, that is a completely different type of proof of the three fixed point theorem using harmonic nets and harmonic tetrads.

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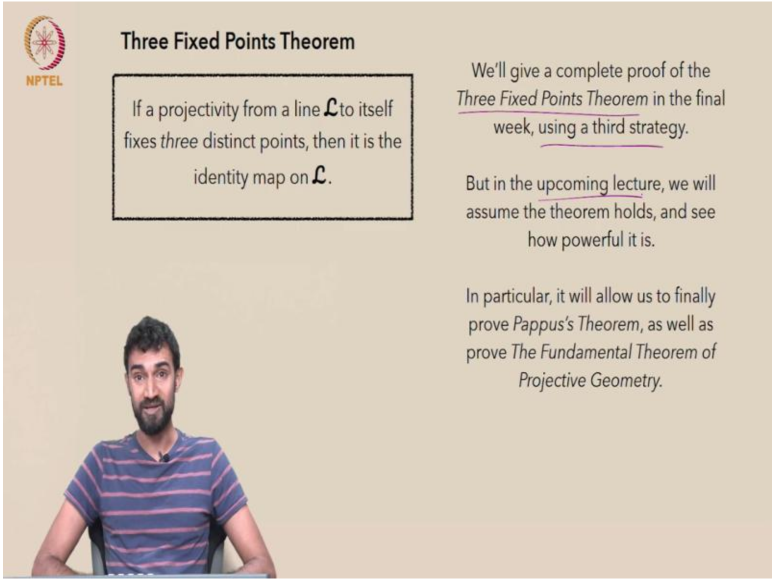

And as I said we are going to give a complete proof of the three fixed point theorem in the final week of the course. So not in the next class, but the following class using a third strategy, in particular using the cross ratio. But in the upcoming lecture we are going to assume that this theorem holds and we are going to see how powerful it really is. In particular, where it is going to allow us to finally prove Pappus's Theorem which we learned in the first week of the course.

And we are also going to learn a theorem called the fundamental theorem of projective geometry which is going to really give us a better sense of what changes when we shift perspective, how many things are changing and how much control we have over those changes. So thank you and see you in the next class.