Our Mathematical Senses
The Geometry Vision
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Lecture - 01
Why Do The Images of Parallel Lines Converge?

Hi, welcome to the Geometry of Vision.

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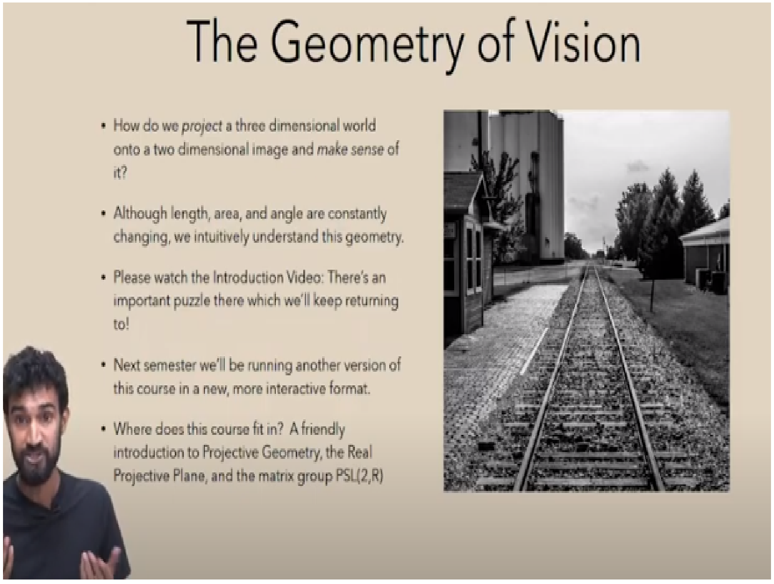

In this four week course, we are going to try and understand how we project the three dimensional world onto two dimensional images, and make sense of them. So in the introduction video, we saw that the length, area and angle within perspective images are constantly changing as we shift perspective. So as you move around, as you are looking around the room, literally the lengths, angles and areas of the objects you are seeing are in constant flux.

And yet somehow, we intuitively make sense of that, it does not throw us off. And despite all the information we are losing, when we are projecting this 3D world onto these 2D images, we somehow maintain tremendous intuition for this visual world around us. So we intuitively understand this geometry of projections.

In this course, we are going to try and understand how that is possible, and what principles we are using intuitively and without knowing it often, to make sense of all of this. So please do watch the introduction video, because there is a very important puzzle there, which is going to keep returning in the course. And in particular today, in today's lecture, we are going to solve that puzzle.

But it would be great if you try it yourself. And also in the video, there are two preliminary puzzles that we solve there, which will be very useful for you to see. So do watch the introduction video. And also next semester, we will be running another version of this course with the same content, but in a new, more interactive format. So do check that out. If for whatever reason, you do not complete this course, do come back and check it out again next semester. And do tell your friends about that.

Also just a quick word on where this course fits in. Because it is not such a standard course. It is a 4-week elective and it's prerequisites are fairly minimal. For the first three weeks of the course, we would not be using anything really except for some basic familiarity with proofs. And which would have come if you have taken a few math classes before this.

In the final week, the fourth week of the class, we will end up using some linear algebra. But mainly just the idea of a matrix as a linear transformation. And some familiarity with both matrices and vector calculus will be useful there. For example, the cross product and the geometric interpretation of the cross product. But beyond that, we are not going to use any very deep linear algebra.

So hopefully, you do not feel held back by prerequisites. And you can stick around and see what this class is about. Another way to describe this class might be a friendly introduction to projective geometry, where we see some of the main results of projective geometry, and get a sense of why they are true and prove some of the basic foundations.

And the other thing that we will get in this class is some hands-on experience with the real projective plane and the matrix group, PSL(2,R). Both of which come up a lot when you study geometry and topology.

So if you are interested in manifolds, or you have never heard of manifolds and you are curious how mathematicians approach geometry in the study of space, or if you have heard the word manifold but you would like to see an example of it in great detail and get familiar with it. If any of those seems to apply, then this will be a great course for you to take.

So let us get started. In this lecture 1A, the first part of lecture 1, we are going to try and address the question, which was raised in the introduction video, why parallel lines appear to converge in our vision.

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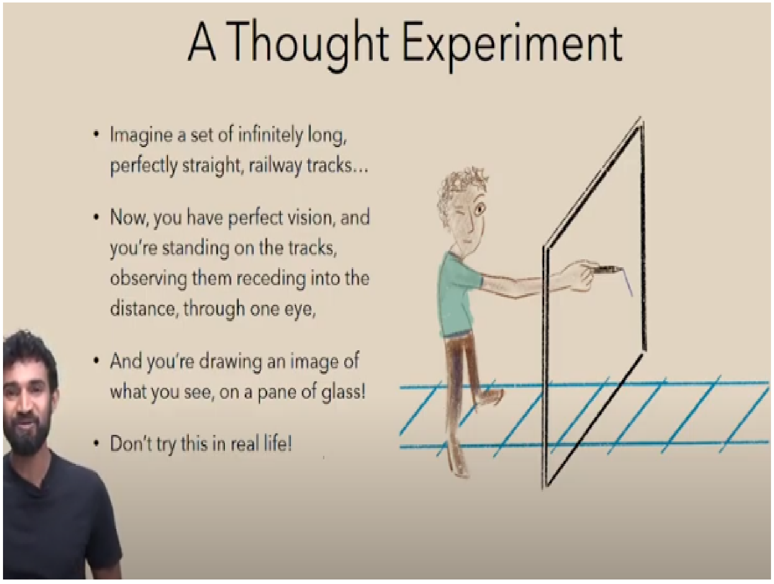

So let us start with a thought experiment. Let us imagine that we are standing on an infinitely long, perfectly straight set of railway tracks. So you are on a flat, infinite plane. Maybe the earth is flat and infinitely large. There is a railway track in front of us, which is straight and infinitely long. And standing on it we also have perfect vision. So we can see as far out onto the tracks as we want.

We are standing here observing the tracks, with just one eye. So close one eye, with your open eye you are observing these infinite railway tracks as they recede into the distance. And as you do that, imagine that you are drawing the image that you see on a glass plane that lies in front of you. So you are drawing these railway tracks as they appear to your one eye on this glass plain.

And clearly this is an experiment that you should not do in real life. It is just a thought experiment.

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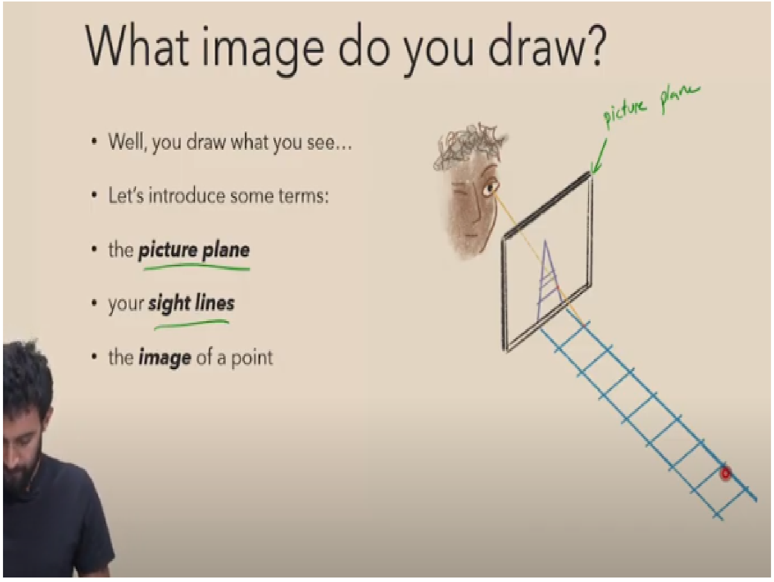

So what image do you draw on the plane of glass? Well, in order to understand that, I want to quickly introduce some terms. So first of all, I want to introduce the term picture plane, which is my technical word for this plane of glass. But it is also a little bit more abstract than the plane of glass. So in this case, this is the picture plane.

And it is just a plane in space on which an image is projected. So in this case, as you are viewing the tracks, the image that you see, we can think of that as the image that is drawn by your sight lines on the picture plane. And what are your sight lines? Well, that is the next term I want to define. They are just your lines of sight. They are straight lines that extend from your eye to various points in space, like that point there.

Or this yellow sight line that we have already drawn here. And your sight lines, you can imagine them painting out a scene on the picture plane. In this case, the object that you are painting with your sight lines is these railway tracks. And how do you paint them? You look at a point on the railway track like this point here. And with your sight line, you paint an image point on the picture plane there.

So, as you look along this railway track, your sight lines are tracing out more and more points on the picture plane, and that is determining the image that you see. This is our basic framework for discussing vision and its geometry. This setup with picture plain, sight lines and images on the picture plane of points in space, which your sight lines trace out.

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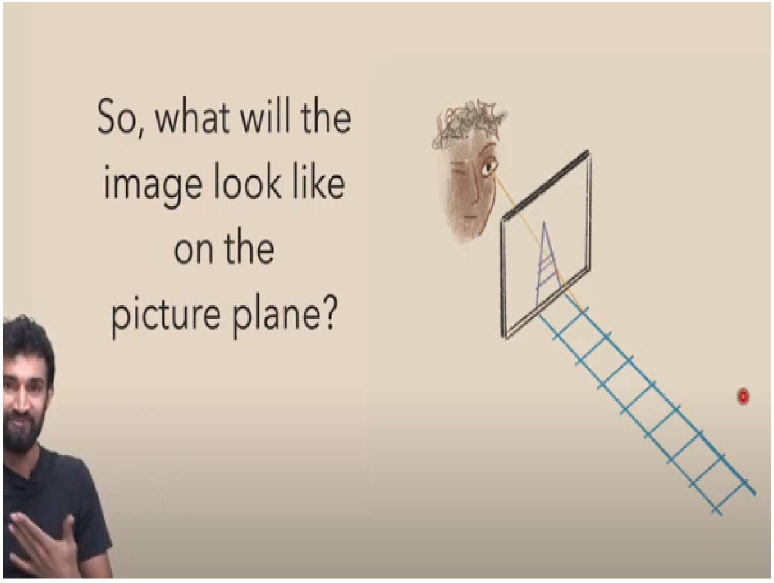

So the question I want to ask in this thought experiment, what will the image look like on the picture plane that you trace out with your sight lines? In other words, what do you see? You are standing on these infinite railway tracks looking at this infinite, looking out into space, you have perfect vision, what do you actually see in front of you?

I mean, you can imagine painting on glass, or you can imagine literally just what you see. Or another way to see the same thing, if you took a photograph with a camera, which had perfect vision, what would it see?

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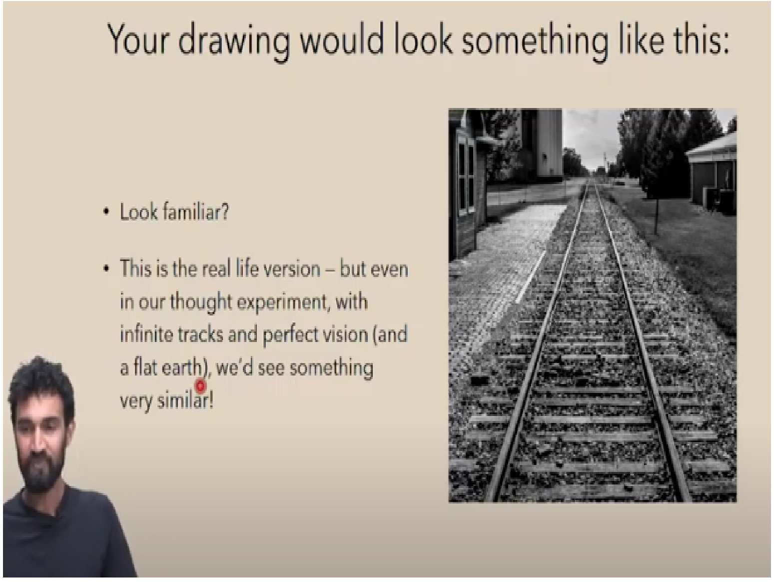

And it would look something like this, which is kind of maybe not so exciting, because this is a very familiar picture. It does not involve anything infinite, or ideal or abstract. It is just regular old railway tracks that we have seen countless times in images that try and show us what perspective is. So it probably looks kind of familiar to you, maybe not this specific picture, but pictures like this.

And this is the real life version of our thought experiment. Our thought experiment which had infinite tracks, and perfect vision and a flat earth, none of those add anything, we see pretty much the same thing that we see in this real life version.

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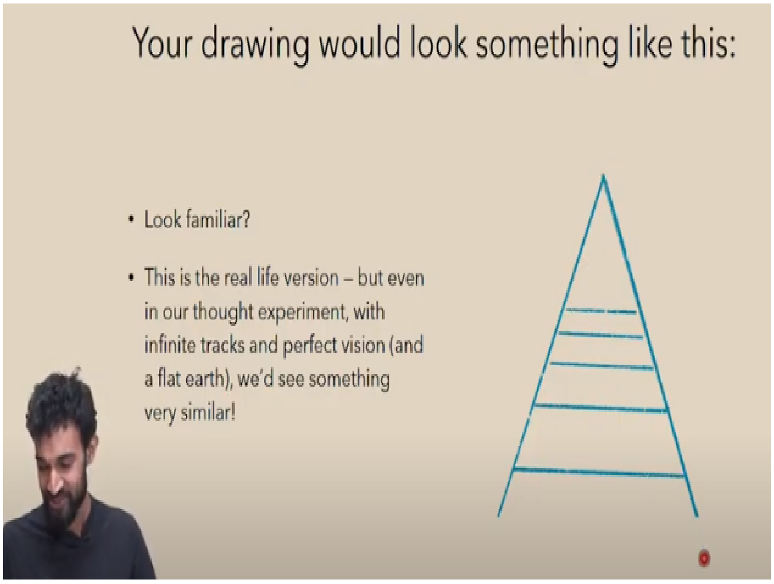

So, if we are actually going back to drawing on glass, maybe drawing on glass would look something like this. Obviously, it is a very simple line drawing. But there is actually a lot we can learn from it. There is a lot to learn from this very simple picture. So you know it is very familiar, there are several features that are actually pretty strange. The first such feature is that the images have infinite lines which terminate.

What do I mean by that? Well, again, I want to return to our thought experiment world where these railway tracks are actually infinitely long. So let us just focus on one railway track, let us focus on this one. It is infinitely long. It’s image, which I have drawn here, is not infinitely long; it terminates here. So that is the first strange thing about this image. Infinite lines have images which terminate.

Second feature, some of these lines terminate at the same point. So in particular, this rail is infinite and terminates here. This other rail is infinite and also terminates. And it terminates at the same point. These two rails terminate at the same point. So that is another kind of surprising feature. These separate lines, not only do they terminate, but also they converge to a single point.

And the final feature we should notice here is that there are also families of lines that do not appear to terminate anywhere. Like these horizontal railway ties, which run between the tracks. They continue looking parallel to each other. And even if we were to extend them out, it is hard to imagine them suddenly stopping.

I do not know why? It just kind of does not seem right. So let us try and investigate all of these three questions and see if we can explain why they are true, why they are holding.

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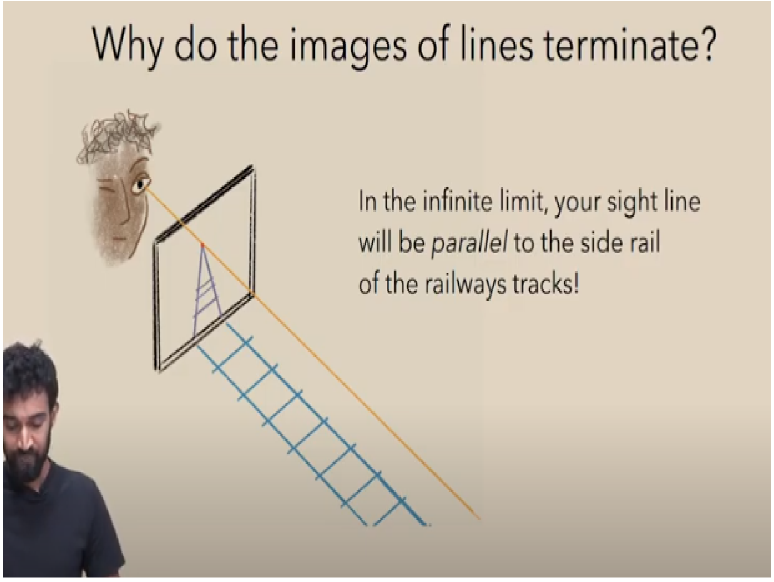

So let us start with the first question. Why do the images of these infinite lines in space terminate in the picture plane? So to see that, let us just go back to the setup. We have our picture plane, we have our sight lines, and we have our railway tracks. And let us just focus on this one side rail here. Let us ignore everything else about this picture and just focus on that.

And let us just imagine that we are looking at this side rail. This is my sight line here, I am looking further and further out along the side rail. So what is going to happen to my sight lines, as I am looking further and further out, along this infinite rail? Well, you can see over here I am looking at this first point on the side rail. As I move here, I have a new sight line, which is now intercepting the picture plane there.

And let us keep going. Okay, our sight line has gotten a little higher. Or maybe we can say that the angle of our sight line with the side rail has actually gotten smaller. But from the perspective of us, from the eye, it is like we are raising our sight line up. And let us keep going. Let us move it all the way down to here. Our sight line is now stretching out to here.

And it is marking out that point in the picture plane. And as we look further and further along, what is going to happen? Well, as you can imagine with this string, as you look further and further along, your sight line will eventually become something like this. In the infinite limits, your sight line is going to become parallel to the side rail of the railway tracks.

So in other words, this sight line, and this original side rail that we are looking further and further down, are actually parallel.

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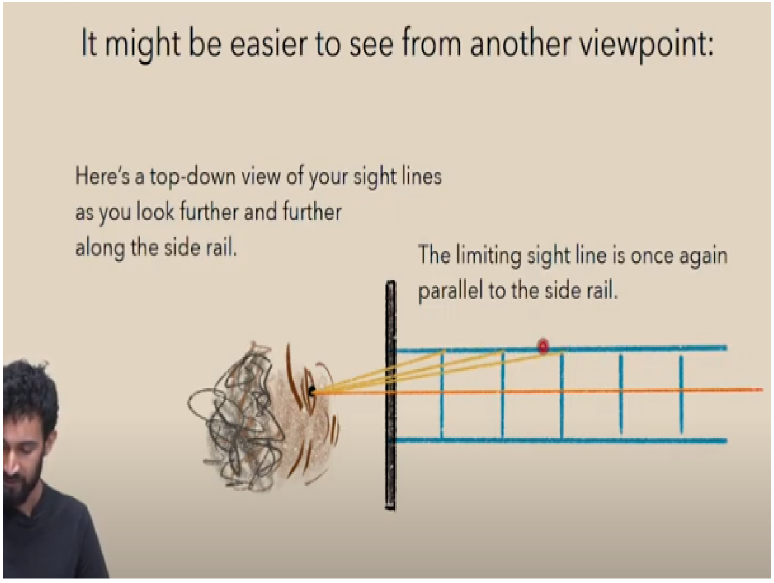

Now it might be easier to see this and believe it from another viewpoint. So let us look at a top down bird's eye view of the same scene. And let us view the sight lines as we look further and further along the side rail. So as you can see here, as we are looking further and further along, our sight lines, indeed in their limit, suddenly become parallel to the side rail.

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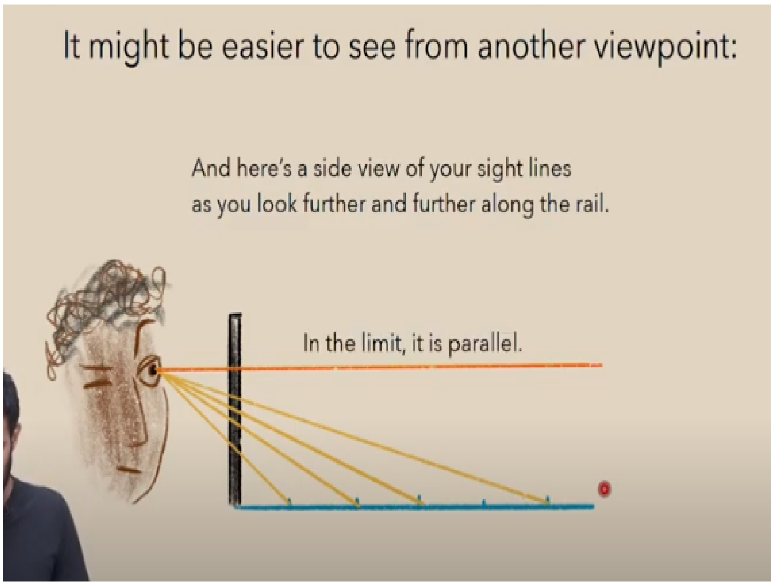

And let us look from another viewpoint, a side view, so that this is even more clear. Once again, our sight lines as we look further along, are rising up. And at the same time the angle of our sight lines with respect to the side rail is actually decreasing, right? Here you can see that these angles of the sight line with respect to the side rail? Those are actually decreasing as our sight lines rise up.

And in the infinite limits, you know our sight lines would have had to pass along every one of the points of this railway side rail to get to this infinite limit. But eventually in this infinite limit, our sight line is horizontal. It is parallel to the side rail. And this final point where it intersects the picture plane will be at eye level. So in other words, the side rail will appear to terminate at a point at our eye level. That is just another interesting thing to note.

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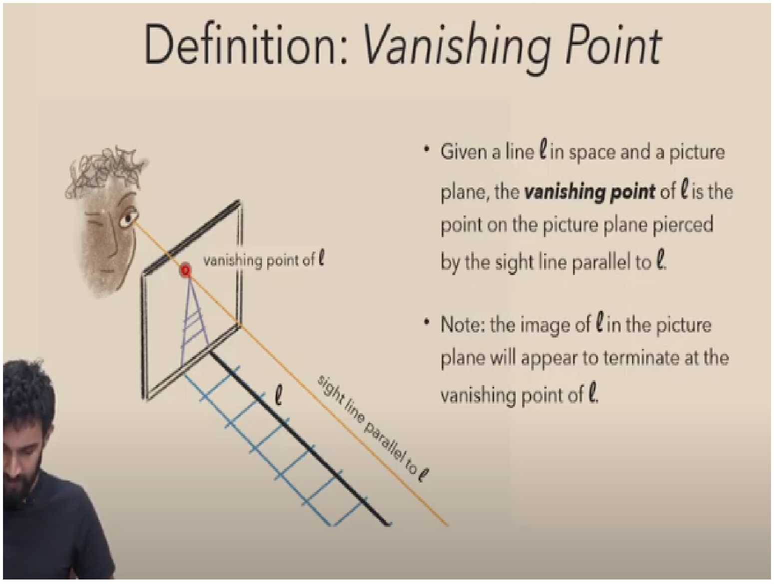

So I want to define something very important called the vanishing points. This is going to be very crucial for all of our discussions from here on. And the vanishing point is precisely just that point of termination of the side rail. It is the point on the picture plane where the image of the line appears to terminate.

So more precisely, given a line L in space, and a picture plane, the vanishing point of L is the point on the picture plane that is pierced by the sight line parallel to L. So given a line L in space, and given a picture plane, the vanishing point of L, we can find it on the picture plane by taking the sight line parallel to L and seeing where it intercepts the picture plane.

And you can see from the picture, that as we look further and further along L, we are seeing points on the picture plane along this image line here. And finally, in the end, we will get closer and closer and closer to this vanishing point here. So in other words, we have shown that infinite lines in space terminate, appear to terminate in the picture plane. And in particular, their images terminate at the vanishing points of those lines.

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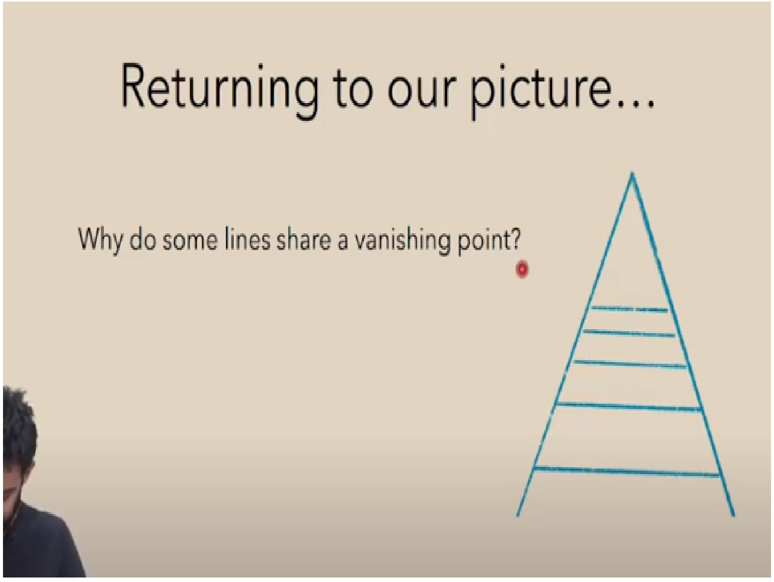

So returning to our picture, let us look at the second feature that we mentioned, which is that not only do these infinite lines terminate, but also these two infinite lines share a single point of termination. They share a single vanishing point. So let us see why that is true.

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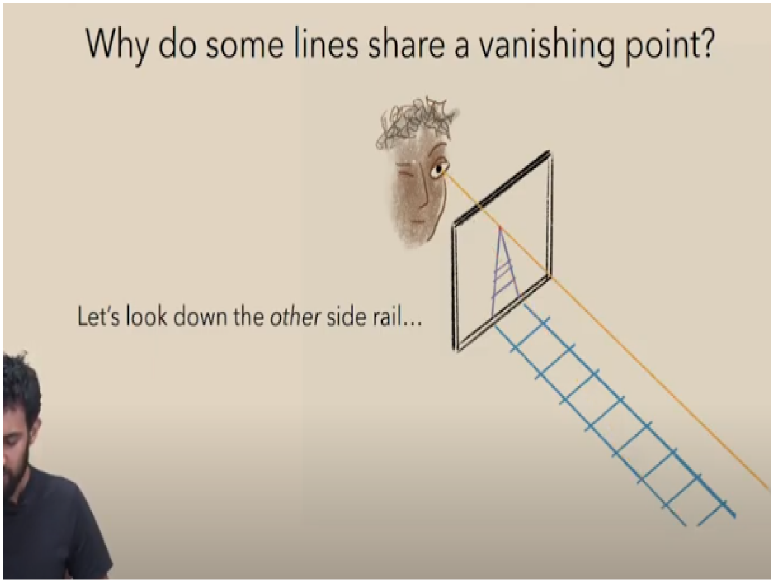

So to see that, let us actually look down the other side of the rail here. We have already looked at this first side rail. Now let us turn our attention to this other side rail, which is parallel to the first one. And as we look down at it, here I am looking at this first point, I am getting an image point over here.

Now I am looking further down, getting another image point and so on. And I am looking further and further along, getting image points higher and higher up along this line.

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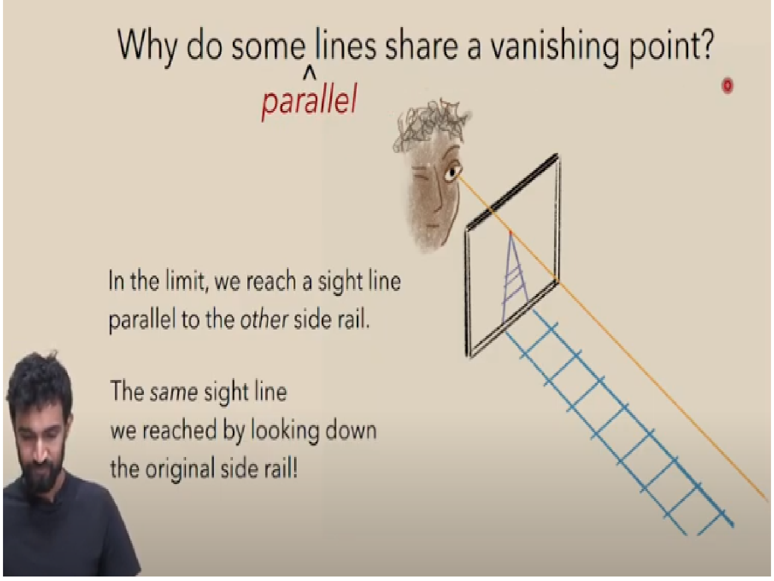

Until, once again, in the infinite limit, we reach a sight line, which is parallel to the other side of the rail. But it is the same sight line that we reached by looking down the original side rail. In both cases, we reach the same limiting sight line. And why is that? Well, it is because the sight line that we reach is the sight line, which is parallel to that line in space. And these two lines in space are parallel to each other.

So in both cases, we are going to end up at the unique line originating at your eye, which is parallel to this family of parallel lights. So that is why these two side rails share a vanishing point. So really, the question we should have asked is, why do some parallel lines share a vanishing point? So the final question is, why do we have this “some” here?

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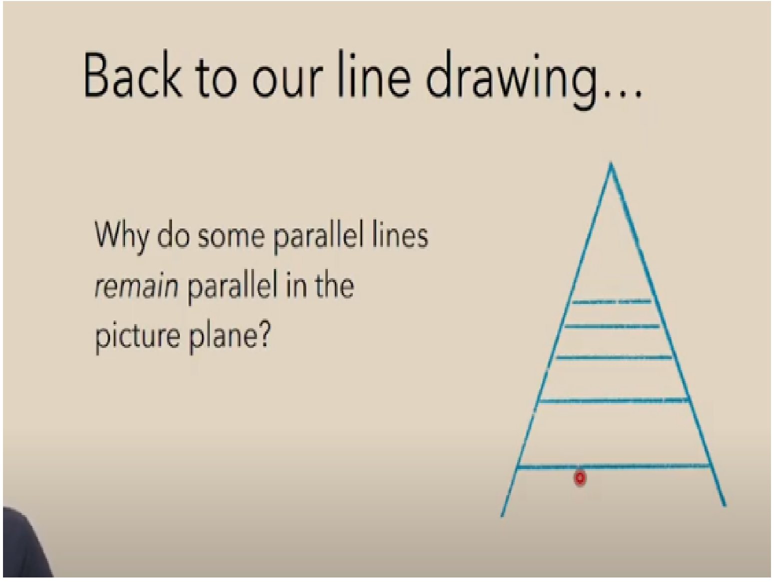

So back to our line drawing, we also saw that some parallel lines remain parallel in the picture plane. So in particular, these horizontal railway ties running between the rails, they continue to appear parallel in the picture plane. Their images are still parallel to one another. So why is that? What is special about this family of parallel lines?

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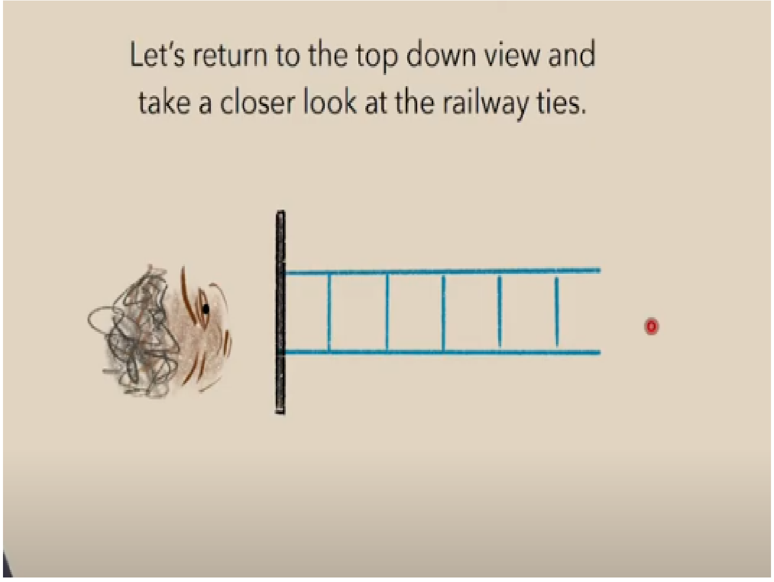

So to see that, let us return to the top down view and take a closer look at the railway ties from this view.

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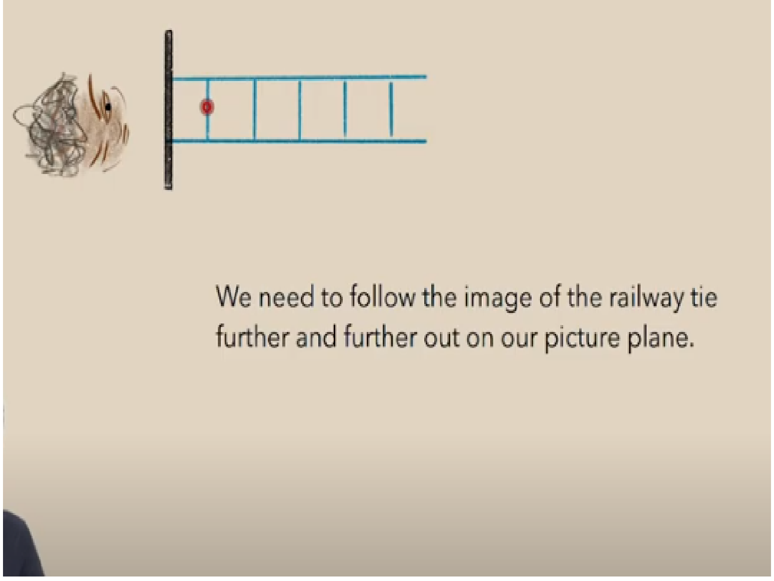

So we want to follow the image of this first railway tie further and further out on our picture plane.

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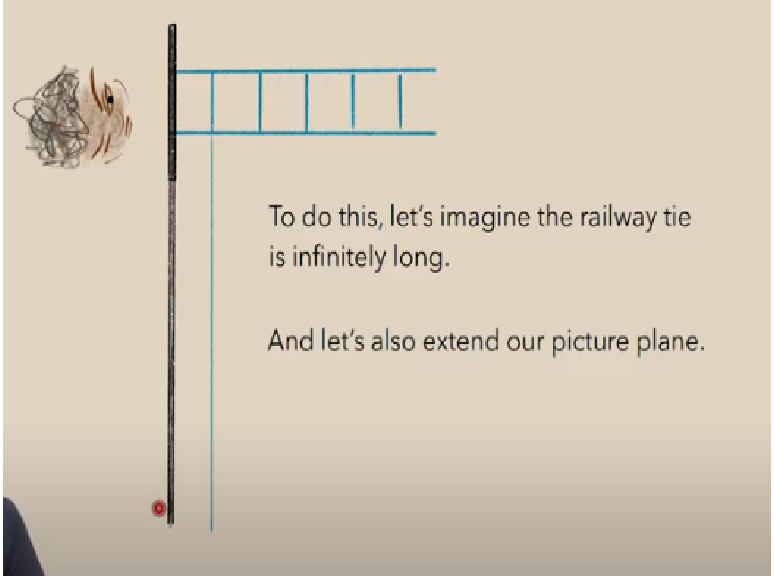

So to do that, let us extend this railway tie and make it infinitely long. Let us also extend our picture plane or make it the abstract plane I mentioned at the beginning. Let us actually think of it as an infinite plane in space that we are projecting information onto.

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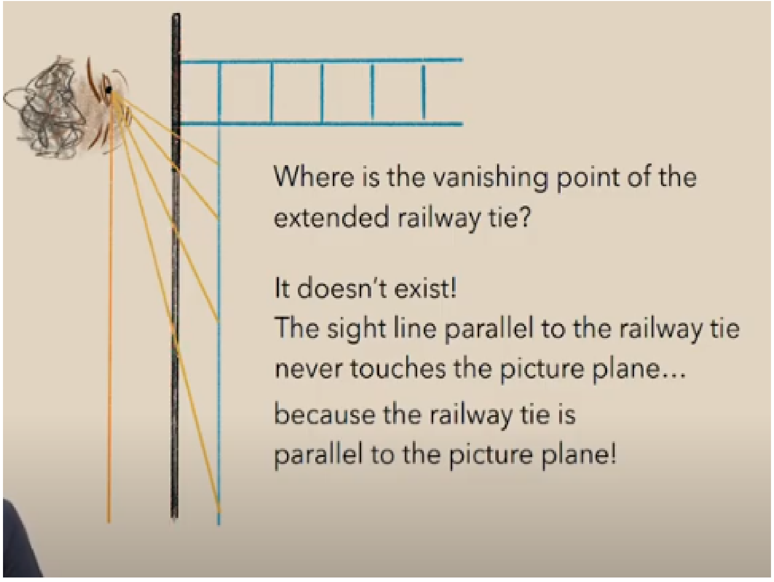

And where is the vanishing point of the extended railway tie going to appear in this newly extended picture plane? That is my question. So where do you think the vanishing point of this extended railway tie will lie on the picture plane? Well, to find it we can play the same game. We can look further and further along the railway tie and see where the image points.

So when we see this point, we are creating an image point here. From here, we are getting an image point here. From here, we are getting an image point here and so on. And we can see where our image point is going. Where are they converging? Well, what is our limiting sight line in this case? As we look further and further along, where do we get? Well, we get to this sight line here.

And that sight line does not pierce the picture plane at all. This sight line is parallel to the picture plane. So the vanishing point does not exist, it does not exist on the picture plane at all. And the sight line parallel to the railway tie, in other words this sight line here, is never going to touch the picture plane. And the reason is that the railway tie is parallel to the picture plane. So in this case, we simply do not have a vanishing point.

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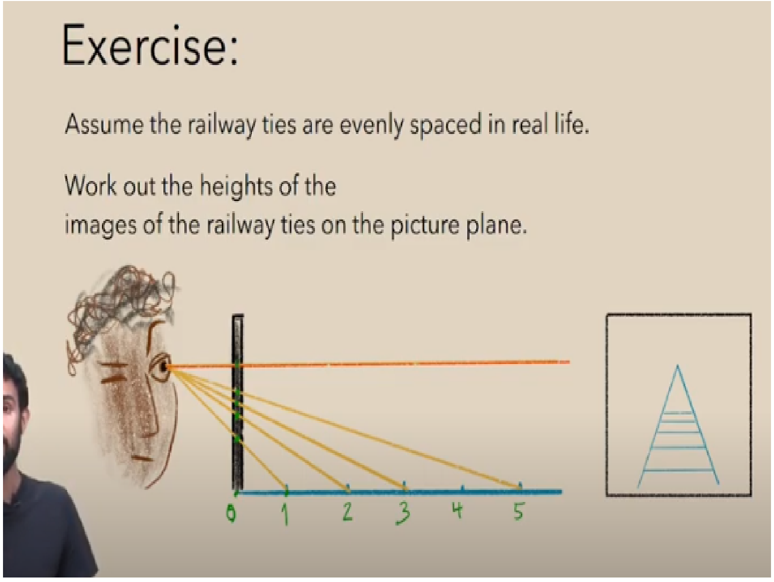

And a nice exercise to try and think about, which will be part of the homework is to assume that these railway ties are evenly spaced in real life. And let us work out the heights of the images of the railway ties on the picture plane. So let us assume that this is the origin in the plane and this railway tie is at 1 unit on the x axis.

This is point 2 on the x axis, point 3, 4, 5, and so on. And can you calculate the locations of these image points in the picture plane along the y axis, the heights of these image points? So on the homework, you will calculate those. And just see what kind of progression these numbers are for.

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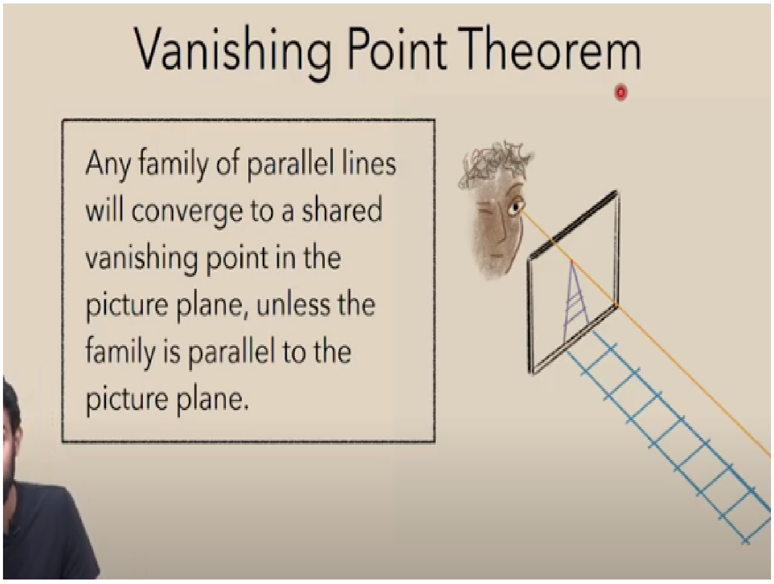

Let us put all these three observations together. We get what is known as the vanishing point theorem. That is, any family of parallel lines will converge to a shared vanishing point in the picture plane, unless that family is parallel to the picture plane.

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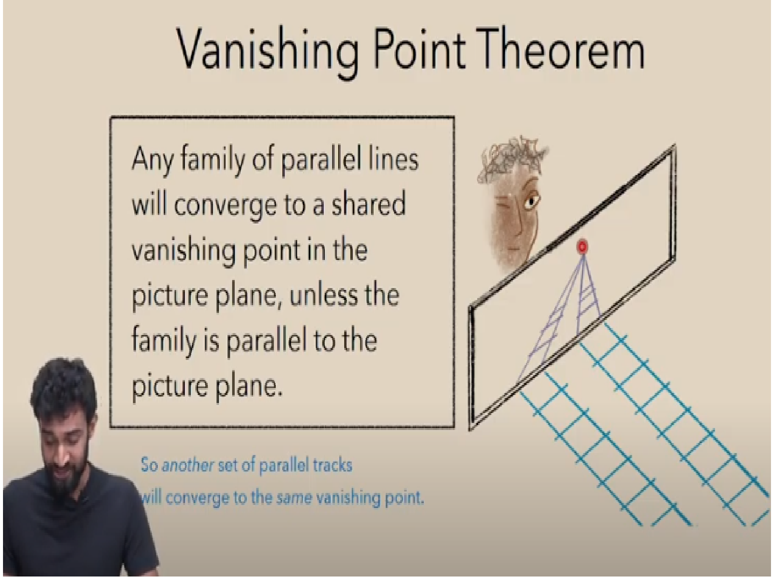

So in particular, we can imagine another set of parallel tracks. That means another set of railway tracks which are parallel to our first set of railway tracks. What will its image look like? Well, it will also be guaranteed to converge to the same vanishing point in the picture plane. Because all of these lines are in the same family of parallel lines. And they are not parallel to the picture plane.

So they are guaranteed to converge to a shared vanishing point in the picture plane. Although it relies on just these kinds of first principle observations, it is actually a very powerful result.

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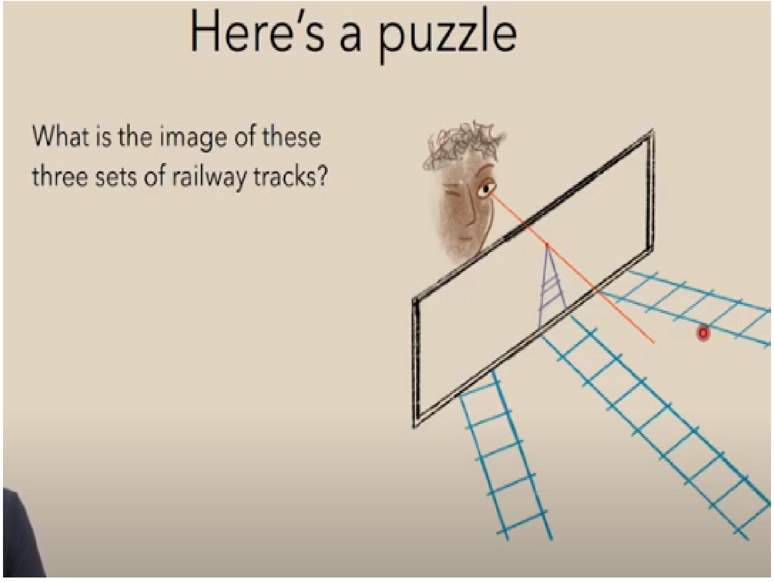

To understand it a little better let us do a little puzzle. So now imagine you have three sets of railway tracks. And this is our original one. And we have added these two new sets of railway tracks. So what do you think the image of these three sets of railway tracks will look like in the picture plane? I do not want you to sit and draw this right now.

This is more just to see if you can make a guess of what they look like. How many vanishing points will there be? Are there any parallel lines in real life whose images remain parallel, or will they all end up converging? So just based on the vanishing point theorem, which we just proved, what do you think the image of these sets of railway tracks is going to look like? So this is more of just taking a guess and see if you are right or not.

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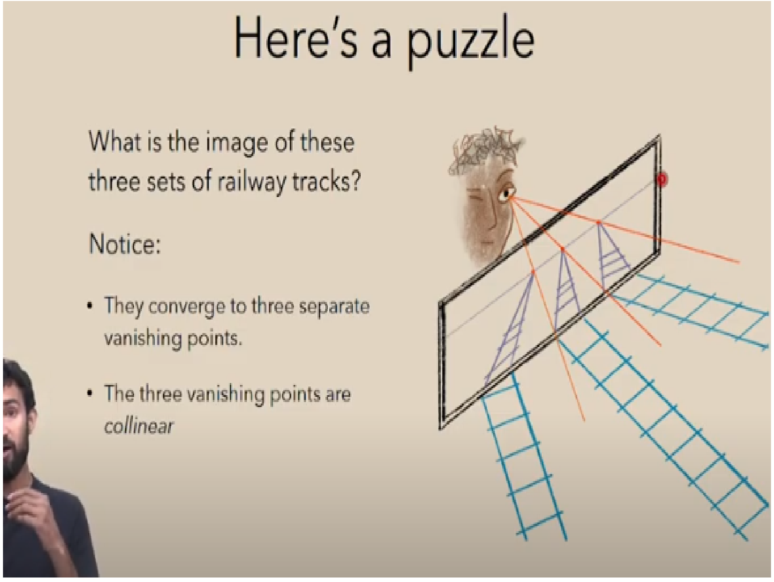

So here is the answer. It will look something like this. There are going to be three separate vanishing points. Because as you can see from the picture, these two new sets of railway tracks are not parallel to the first set of railway tracks. And they are also not parallel to each other. So these are three independent families of parallel lines. And they will each yield three distinct vanishing points in the picture plane.

So that is the first thing to notice about this kind of sketch, this kind of diagram that I have drawn. Now they converge to three separate vanishing points. By the way, this is not something we will get back to later. But maybe I will just point out, in our first railway track, we had that the images of these horizontal railway ties were horizontal and they are parallel to each other.

In this new set here, and this new set over here, are the images of horizontal railway ties still going to be parallel to each other? No, because in real life they are no longer parallel to the picture plane. So I will leave that for you to think about a little more. They are not going to be parallel to each other, which means that they are going to have to converge to a vanishing point somewhere.

So these ones will converge to some vanishing point. And these ties will converge to some other vanishing point. So that is just a little observation, we will come back to that later. But right now I want to focus on something much simpler, which is just that we have three distinct vanishing points for these three sets of parallel railway tracks.

Now the other thing I want to point out, is that these vanishing points are not just three arbitrary points on the picture plane. But they are collinear, meaning that they all lie on a single line. And that is actually quite significant, because it is not just happening by chance.

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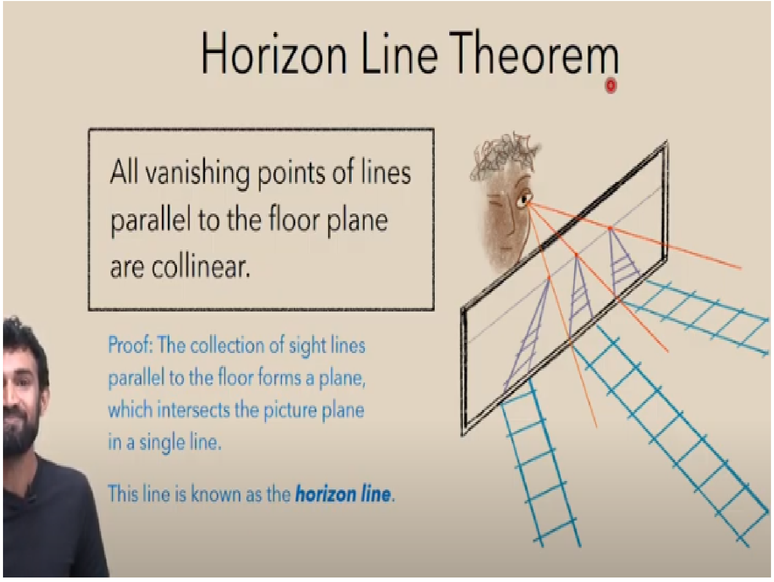

So in general, it is going to be the case that all vanishing points of all lines that are parallel to the floor plane, are going to be collinear. So it is not just the vanishing points of these three sets of railway tracks. Any family of parallel lines which is parallel to the floor, is going to converge in the picture plane to a vanishing point on this line here.

All of the vanishing points of lines parallel to the floor, are going to be collinear to each other. We will call this result the horizon line theorem, for a reason that we will see in just a minute. First, let us just try and convince ourselves that this theorem holds. So why is it the case that all vanishing points of lines parallel to the floor plane are collinear to each other?

So the proof actually turns out to be quite simple. If you take the collection of all sight lines that are parallel to the floor, they are going to form a plane in space. And this plane is going to intersect the picture plane in a single straight line.

By construction, that line will contain all the vanishing points of all lines parallel to the floor. This line is known as the horizon line. So that is why this theorem is called the horizon line theorem. Let us keep exploring and see what we uncover. Thanks.