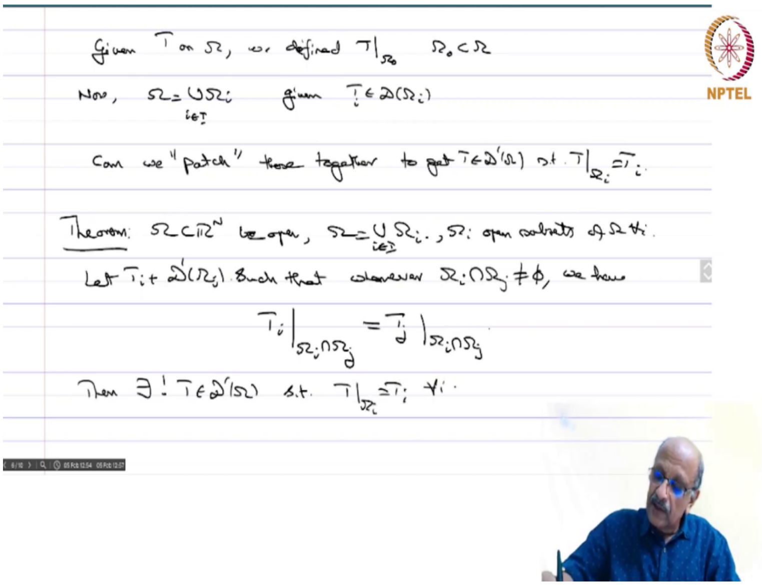
Sobolev Spaces and Partial Differential Equations Differential Equations
Professor. S Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Distribution with compact support singular – Part 2

(Refer Slide Time: 0:23)



So, in the beginning of this discussion when we started about supports, what we did was given T, on Ω we defined $T|_{\Omega_0}$, $\Omega_0 \subset \Omega$. So, this is localizing the distribution to a smaller open set. Now, we want to do the reverse process.

So, now, if $\Omega = \bigcup_{i\in I} \Omega_i$, $given\, T_i \in D(\Omega_i)$, can we patch these together to get $T \in D'(\Omega)$ such that $T_{\Omega_i} = T_i$.

So, we have the following theorem for that. So, under what condition can we do this?

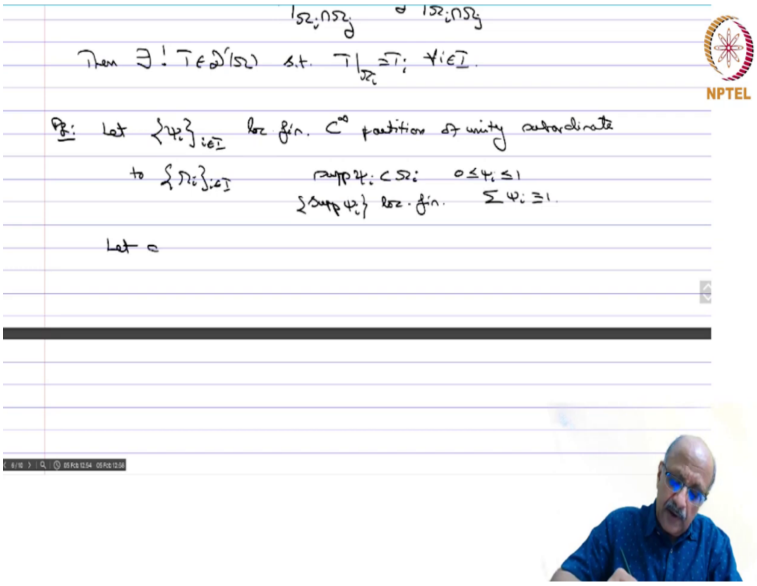
Theorem: Let $\Omega \subset \mathbb{R}^N$ open set, $\Omega = \bigcup_{i\in I} \Omega_i$, $given\, T_i \in D'(\Omega_i)$, Ω_i open sets for all i. Let

$T_i \in D'(\Omega_i)$ s.t. whenever $\Omega_i \cap \Omega_j \neq \phi$, we have

$$T_i|_{\Omega_i\cap\Omega_j} = T_j|_{\Omega_i\cap\Omega_j}.$$

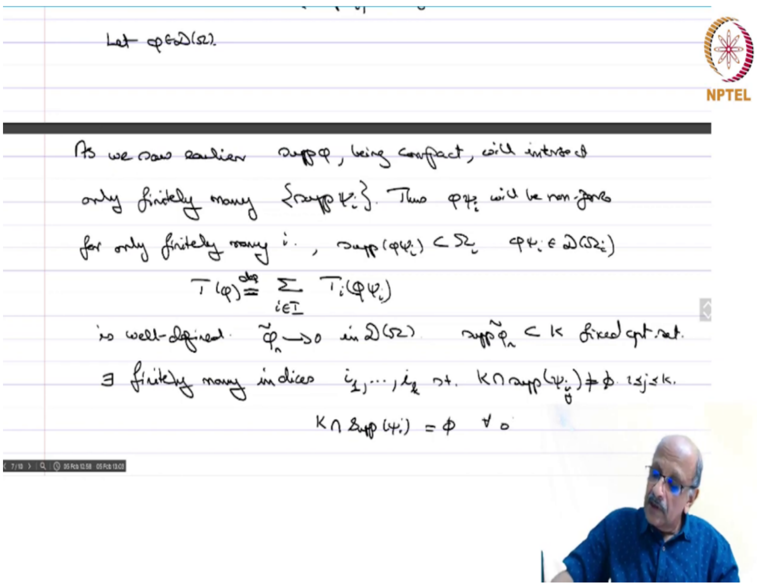
Then there exists a unique $T \in D'(\Omega)$, such that, $T_{\Omega_i} = T_i$ for all i.

(Refer Slide Time: 3:41)



proof. So, let as usual we take the psi i, i in I locally finite C infinity partition of unity subordinate to omega i, i in I, so we know what this means. The support of psi i will be in omega i, so support psi i will be contained in omega i and then 0 less than or equal to psi i, will be less than equal to 1 and then support psi i, is a locally finite family and then sigma psi i, is identically 1. So, these are the four conditions which we have for these functions. So, let phi belong to d omega.

(Refer Slide Time: 4:39)



So, we saw this earlier, as we saw earlier support of phi being compact will intersect only finitely many support of psi is, why, because each point in the support of phi will have a neighborhood which intersects only financially many one of them, these neighborhoods cover support a phi which is compact, so support the phi can be covered by a compact finite number of neighborhoods, each of which will intersect only a finite number of the support of the psi is, and therefore, totally the support of phi itself will intersect only finitely many. This is something which we already used.

So, this $\phi\psi_i$, will be non-zero for only finitely many i. So,

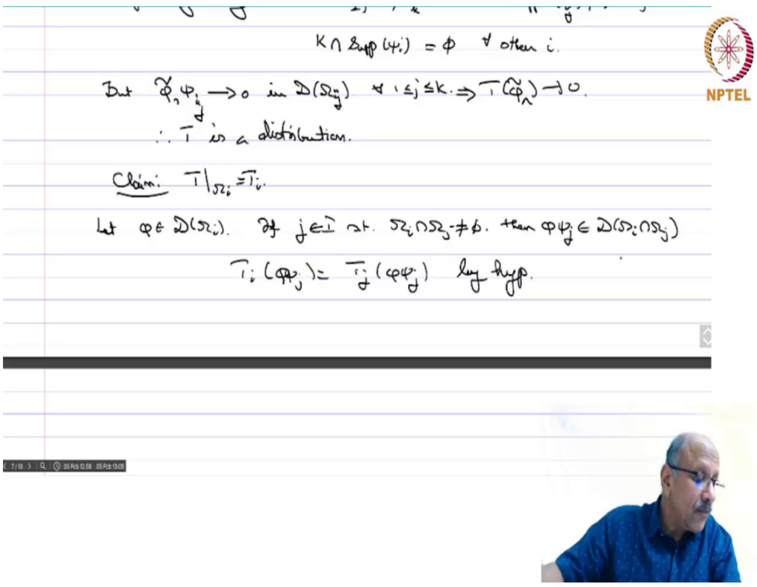
$$T(\phi) = \sum_{i\in I} T_i(\phi\psi_i) \text{ is well defined.}$$

So, now you take $\tilde{\phi}_n \to 0 \; in \; D(\Omega)$, $supp(\tilde{\phi}_n) \subset K$ $-$ fixed compact set, there exists finitely

$i_1, i_2, ..., i_k$ s.t. $K \cap supp(\psi_j) \neq \phi, 1 \leq j \leq k.$

$$K \cap supp(\psi_i) = \phi \text{ for others i.}$$

(Refer Slide Time: 9:00)



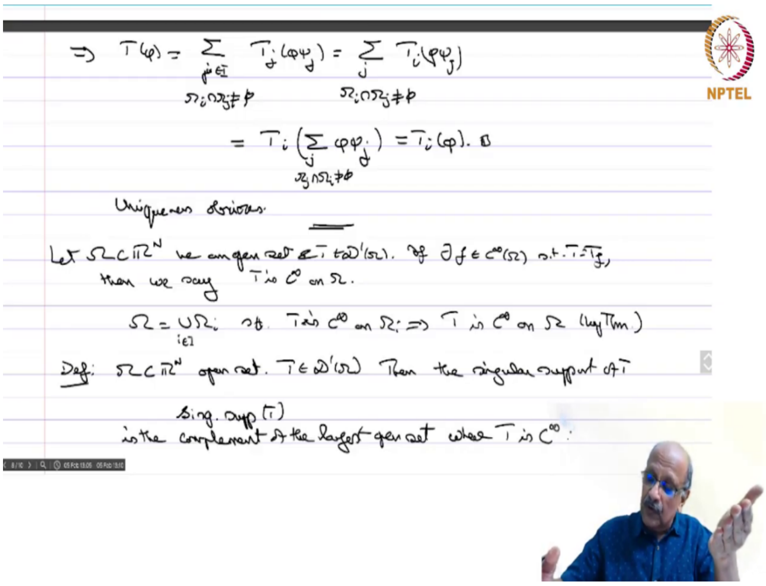
Now, $\tilde{\phi}_n\psi_{i_j} \to 0 \; in \; D(\Omega_j)$, $\forall\, 1 \leq j \leq k \Rightarrow T(\tilde{\phi}_n) \to 0.$

Therefore, T is a distribution.

So, now claim: $T_{\Omega_i} = T_i$

So, let phi belong to D of omega i. So, what do we have to show? We have to show T of phi i is the same as Ti of phi. So, if j in i such that omega i intersection omega j is non-empty, then phi psi j belongs to D of omega i intersection omega j, because this has compact support and it is contained in both these sets. So, this has to be this in the set. And therefore, we have Ti of phi, psi j is the same as Tj of phi, psi j by hypothesis.

(Refer Slide Time: 11:15)



So, this implies that T of phi, how did we define it, this is sigma overall i, all j in i, such that, omega i intersection omega j is non-empty not equal to empty set only then phi it will show that will be Tj times phi psi j but this is equal to sigma over j omega i intersection omega j not equal to empty set. Tj phi psi j is same as Ti phi psi j so, that is equal to Ti acting on sigma over j omega j intersection omega i non-empty of phi psi j and that is equal to Ti of because psi j is a partition of unity.

And because of this, why did this come out of the summation because this is essentially only a finite sum and because the support of phi will intersect only finitely many of the supports of the psi j, and therefore, this is essentially a finite sum. So, it comes out and consequently you have this and we have completed it. So, uniqueness is obvious, because if you want T to be equal to Ti, on the sub domain, you have to define it only this way you cannot define it by any other method.

So, now, let $\Omega \subset \mathbb{R}^N$ open set and $T \in D'(\Omega)$. So, if there exists an $f \in C^{\infty}(\Omega)$ such that $T = T_f$, then we say T is C^{∞} on Ω?

Now, you can do it for any subdomains also. Therefore, if you have an omega equal to union of omega i, such that T equal to T is C infinity on omega i, so then that means it is given by a C infinity function. Then automatically the function will have to coincide on the intersections and therefore, this implies by the above theorem T is C infinity on omega by theorem.

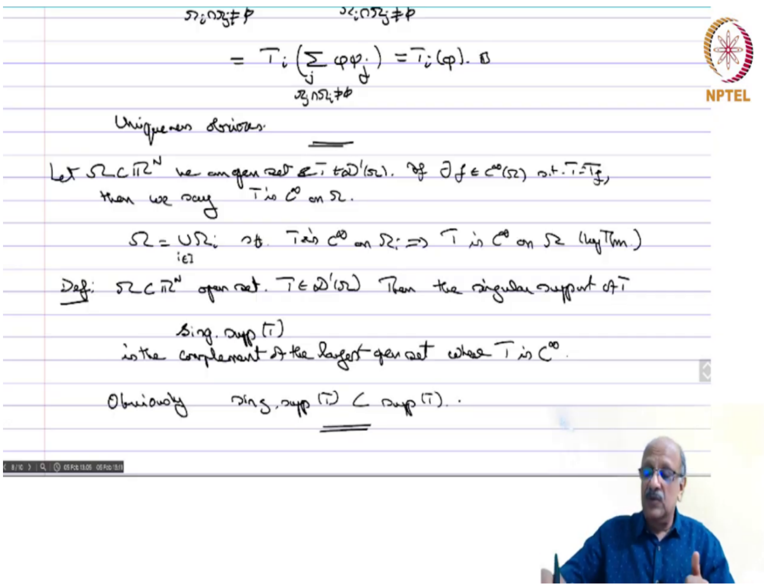
And consequently, we can talk of the largest open set where the function is C infinity. So, that leads us to the following definition.

Definition: Let $\Omega \subset \mathbb{R}^N$ open set and $T \in D'(\Omega)$. Then the singular support of T, so you write sing.supp (T) is the complement of the largest open set where T is C^{∞}.

So, you take wherever T is C infinity open set, then you put them all together obviously, they will patch up because they are now made up of functions which are C infinity.

So, they are C^{∞} at the intersection as well. So, f_i will have to be equal to f_j and then by the previous theorem you can make T through for the whole domain and that will give you the function which will, functions will also patch up and so, you will have this thing.

(Refer Slide Time: 16:00)



So, obviously, sing.supp(T)⊂supp(T), because on the complement of supp(T), T is 0 and let us say automatically C^{∞} functions. So, therefore, the singular support will always be contained in the support. So, for the Dirac Distribution and its derivatives etc the support itself is the origin. So, the singular support is also the origin because outside everything is 0.