Sobolev Spaces and Partial Differential Equations

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Lecture 85

Exercises - 14

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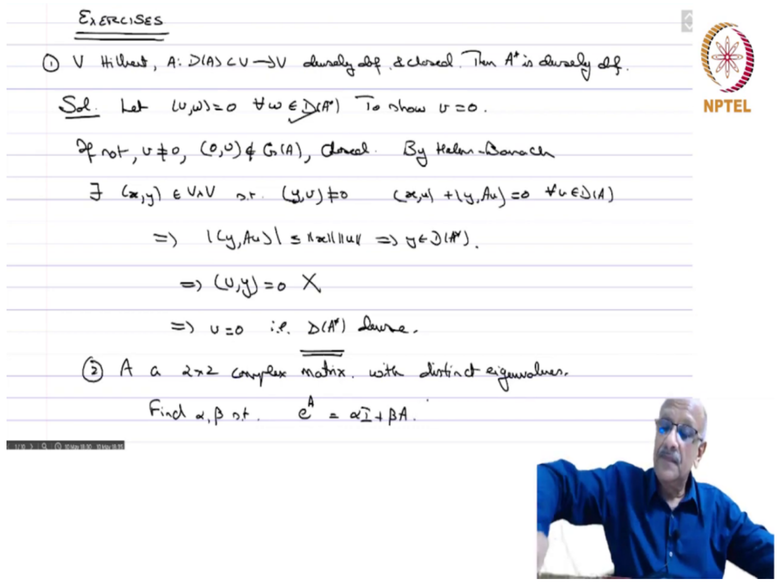

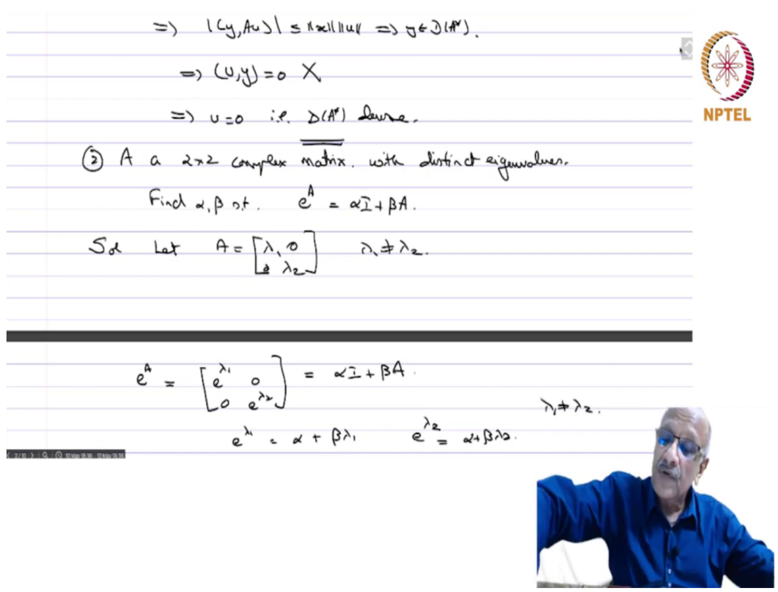

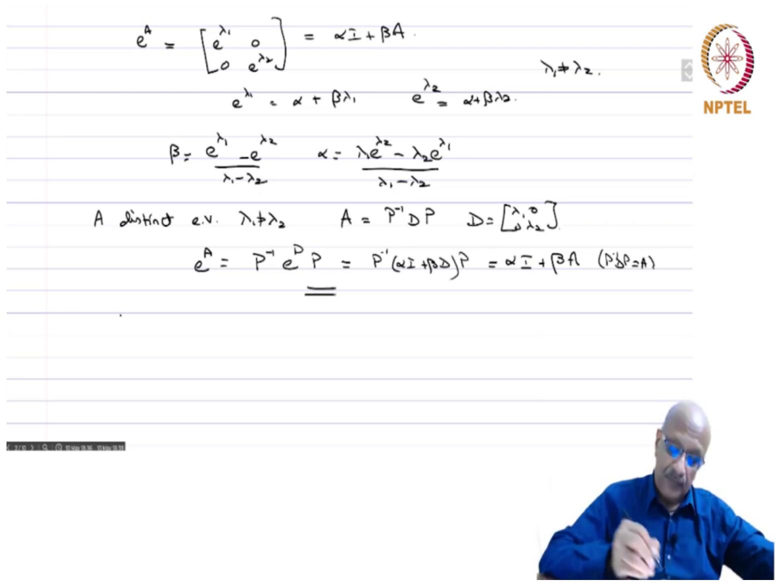

Will now do some **exercises**. First one

Exercise 1: V Hilbert and $A: D(A) \subset V \rightarrow V$ densely defined and closed. So, if it is densely defined you can define that joint and since V is a real Hilbert space we always deal with real unless otherwise mentioned so we can or even complex does not matter but in this case real we can define the joint from V into V itself. So, then A^* is densely defined. We know that whatever may be a A^* is always closed if A is densely defined and close then A^* is also densely defined and closed. So,

Solution: so we so let $(v, w) = 0$ for all $w \in D(A^*)$. So, we want to show $v = 0$.

So, this is the Halm-Banach theorem method of showing something is dense so we have $D(A)$ which is a subspace and if you want to show it is dense you have to take a vector v which or a functional in which case by the rays representation theorem is just the inner product with a vector and therefore we have to show that this is equal to 0.

So, if not $v \neq 0$, then you have that $(0, v)$ does not belong to the graph of A because graph of A is what u a u u is in domain of A if the first component is 0 the second component must forcibly be 0. So, if $v \neq 0$ this cannot be the graph of A and graph of A is closed that is given here that is the hypothesis the graph of A is closed therefore you have a closed subspace of $V \times V$ and you have a vector which is not in it. And therefore, by the Halm-Banach there exists $(x. y) \in V \times V$ such that.

So, y acting on V inner product so there exists a pair (x y) and $(x, y) \in V \times V$ here this is the inner product. So, this is not equal to 0 and

$(x, u) + (y, Au) = 0$ for all $u \in D(A)$ that means (x y) and (())(03:46) all the elements of the graph. So, you have a continuous in functional which again a pair of elements in $V \times V$ such that it should not vanish on 0, V so $(y\ v)$ is not equal to 0 and it vanishes on all the elements of the graph.

That means x cube plus but this implies that

$$|(y, Au)| \leq ||x||\ ||u||$$

and this implies that $y \in D(A^*)$ by definition and that implies that $(v, y) = 0$ because that is a definition of v here and but that is a contradiction because we know that (v, y) is not 0 and therefore so this implies that $v = 0$ that is $D(A^*)$ dense.

Exercise 2: A a 2 by 2 complex matrix with distinct eigen values. Find α, β beta such that

$e^A = \alpha I + \beta A.$

Solution: so let

$A = [\lambda_1 , 0 ; 0 , \lambda_2],$

so this diagonal matrix with distinct so $\lambda_1 \neq \lambda_2$. So, then what is

$e^A = [e^{\lambda_1} , 0 ; 0 , e^{\lambda_2}] = \alpha I + \beta A$

this is just straight forward calculation I plus a plus a square by two factorial et cetera if you compute so you will get precisely this thing here so now you want this to be equal to alpha I plus beta A.

So, then you have

$$e^{\lambda_1} = \alpha + \beta\lambda_1 \; and \; e^{\lambda_2} = \alpha + \beta\lambda_2.$$

So, if you saw and $\lambda_1 \neq \lambda_2$, therefore if you solve this pair of equations you will get beta if you subtract you will get

$$\beta = \frac{e^{\lambda_1} - e^{\lambda_1}}{\lambda_1 - \lambda_2} \; and \; \alpha = \frac{\lambda_1 e^{\lambda_1} - \lambda_2 e^{\lambda_1}}{\lambda_1 - \lambda_2}.$$

Now, if A has distinct eigen values $\lambda_1 \neq \lambda_2$ then A can be written as $P^{-1}DP = A,$ is the diagonal matrix $A = \left[\lambda_1 , \; 0 ; \; 0 \; , \lambda_2\right]$. So, this is you can diagonalize the matrix using eigen vectors. Then what is e^{A} is nothing but if you do the calculation it will be $P^{-1}e^{D}P$ and that is equal to

$$e^{A} = P^{-1}e^{D}P = P^{-1}(\alpha I + \beta D)P = \alpha I + \beta A,$$

Similarly, A cube A power 4 and so on and therefore e power A is nothing but $P^{-1}e^{D}P$ and this is equal to this and that is equal to alpha times identity plus beta times $P^{-1}e^{D}P$ since $P^{-1}e^{D}P = A$. So, it is the same constant so you have what you do for the diagonal serves for all the matrix also.

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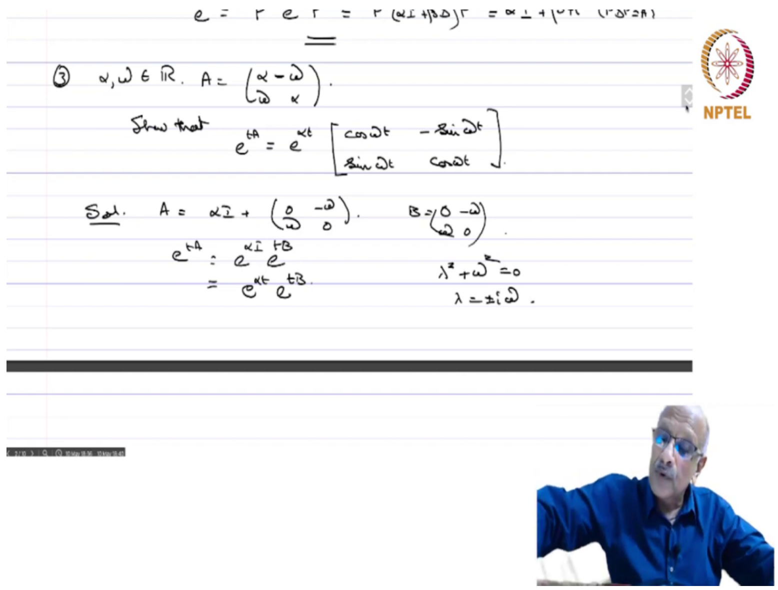

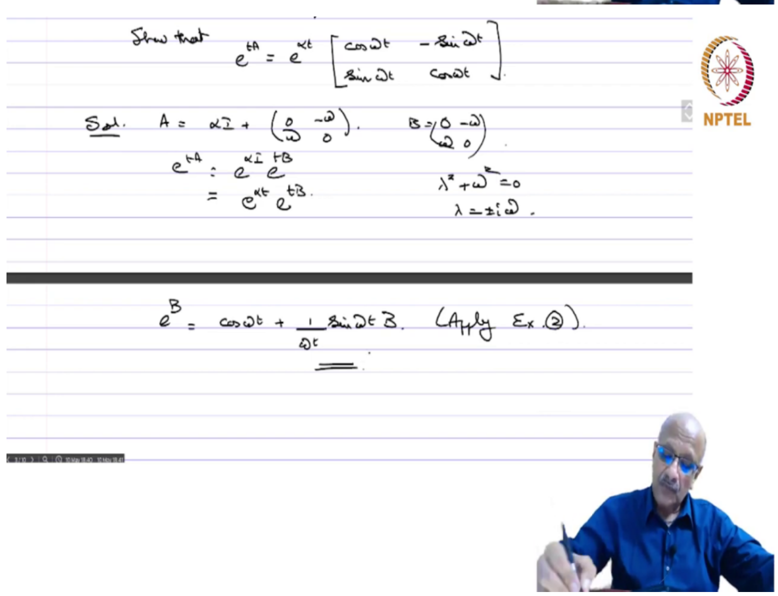

Exercise:3 let $\alpha, \omega \in \mathbb{R}$ and $A = [\alpha\,, \; -\,\omega;\; \omega\,, \;\; \alpha]$ then show that

$e^{tA} = e^{\alpha t}[\,Cos\,\omega t\;,\;\; -\,Sin\,\omega t\;;\; Sin\,\omega t\;,\; Cos\,\omega t\;]$.

Solution: so A can be written as

$A \,=\, \alpha I \,+\, [0\,,\;\; -\,\omega;\; \omega\,,\;\; 0]$.

So, if you took

$$e^{tA} = e^{\alpha tI} e^{tB}$$

so this is sum of two matrices they commute because one of them is identity and therefore e power this will be e power alpha I, e^{tB} where $B = [0\,, \ -\omega;\ \omega\,,\ \ 0]$.

But e power alpha I is nothing but e power alpha t times I. So, that is e power alpha t times e^{tB}. So, now you have what are the eigen values of this matrix. So, if you compute the eigen values of this matrix they are what so lambda square minus trace is 0 plus determinant so that will give plus omega square equal to 0 so lambda equals plus or minus i omega. So, then if you compute the constants then that then you can write

$$e^{B} = Cos\ \omega t + \frac{1}{\omega t} Sin\ \omega t B.$$

So, this apply **exercise 2**. And then you will get whatever you want. So, if you from this you should be able to complete this exercise.

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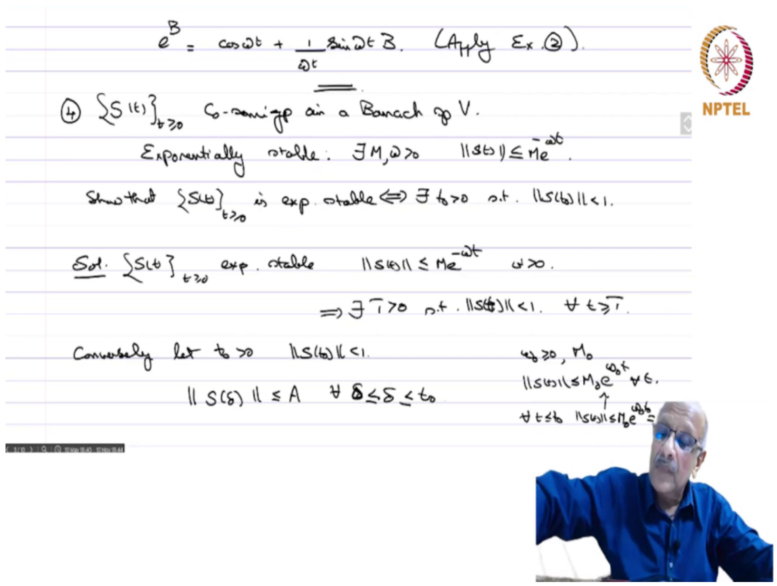

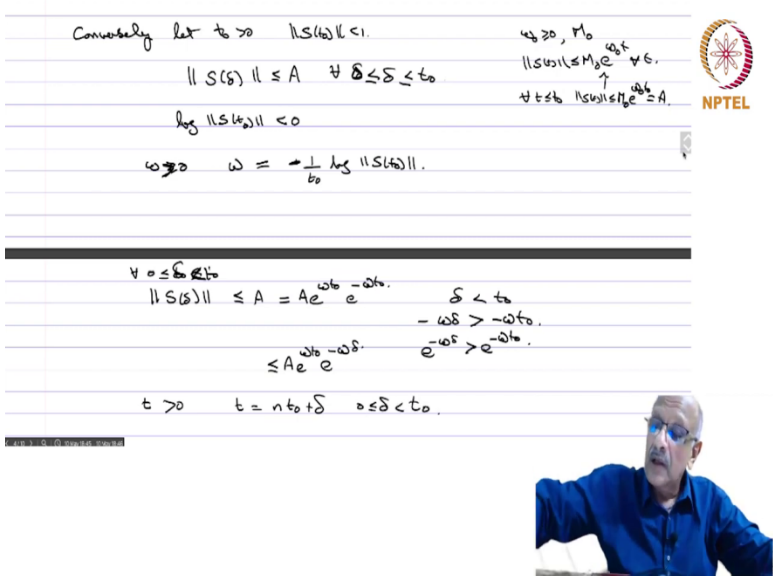

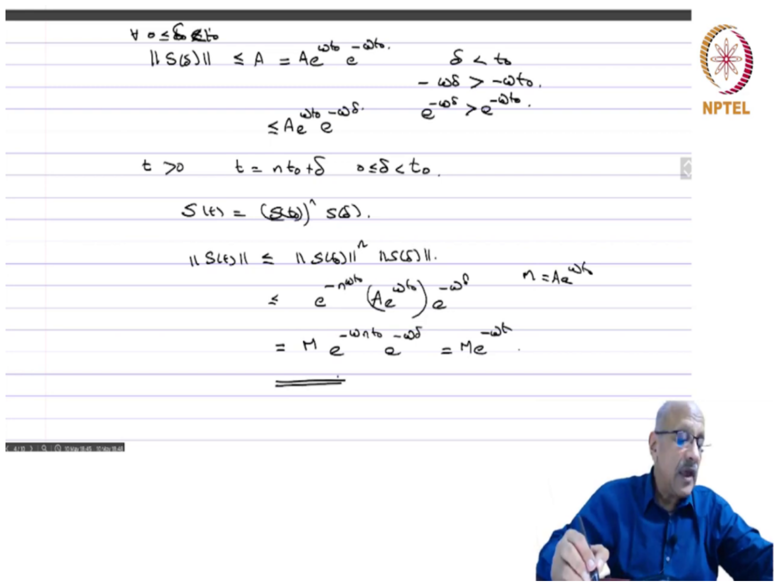

Exercise 4: $\{S(t)\}_{t\geq 0}$, C_0 semi group on a Banach space so what do you mean by exponentially stable that means that exists $M, \omega > 0$ such that

$$||S(t)|| \leq Me^{-\omega t}$$

this is then it decays exponentially the norm so.

Solution: So, show that $S(t)$is exponentially stable if and only if there exist t_0 positive such that

$||S(t_0)|| \leq 1$ so this is a very beautiful characterization so if you have one element of the semi group which has a norm strictly less than 1 then the semi group norms decrease exponentially. So, if so solution so $S(t)$, t greater than equal to 0 exponentially stable that means $||S(t)|| \leq Me^{-\omega t}, \ \ \omega > 0$.

So, this implies there exists a T sufficiently large such that norm of $S(t)$ is less than one for the in fact $||S(t)|| \leq 1$ for all $t \geq T$. So, then trivially so conversely let t naught be greater than 0 with norm $S(t)$ t naught strictly less than 1 then norm of S delta is lessening to some A for all delta 0 less than equal to delta less than equal to t naught.

Because you know there exists omega naught greater than equal to 0 and M naught such that norm of $S(t)$ is less than to M naught e power omega naught t for all t given any semi group you have this and therefore if you put. This increasing function exponential montana increasing function therefore for all t less than equal to t naught you have norm of $S(t)$ is less than equal to M naught e power omega naught t naught which is I call as A.

So, we have some elements here. So, now you have that log norm of S t naught is negative because S t naught is less than 1. So, now you put omega greater to 0 omega equals minus 1 by t naught log norm S t naught. So, this will be positive. So, norm of S delta for all 0 less than equal to delta less than t naught norm of S delta is less than equal to A which is equal to Ae power omega t naught e power minus omega t naught.

Now, delta is less than t naught so minus omega delta will be greater than minus omega t naught so e power minus omega delta will be greater than e power minus omega t naught. So, this is less than equal to Ae power omega t naught e power minus omega delta. Now, given any t, t greater than 0 we can write t as n times t naught plus delta where 0 less than equal to delta less than t naught the usual trick which we did later on.

So, $S(t)$ by the semi group property is $S(t_0)$ power n times $S(\delta)$. Now therefore

$$||S(t)|| \leq ||S(t_0)||^n \, ||S(\delta)|| \leq e^{-n\omega t_0}(Ae^{\omega t_0})e^{-\omega\delta} = Me^{-n\omega t_0}e^{-\omega\delta} = Me^{-\omega t}$$

And norm of $S(\delta)$ is less than equal to A e power omega t naught times e power minus omega delta. So, this is equal to M so M is equal to A e power omega t naught and then e power minus omega n t naught and e power minus omega delta which is equal to M e power minus omega t. So, that proves the exponential d k.

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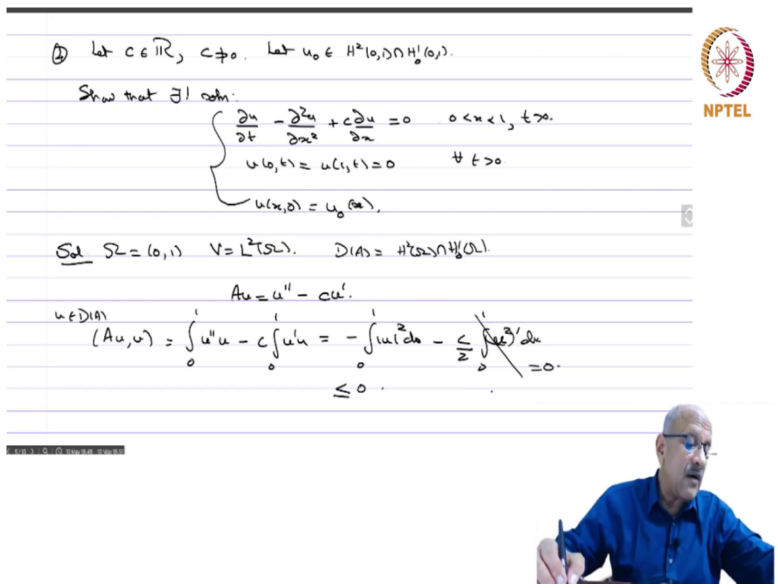

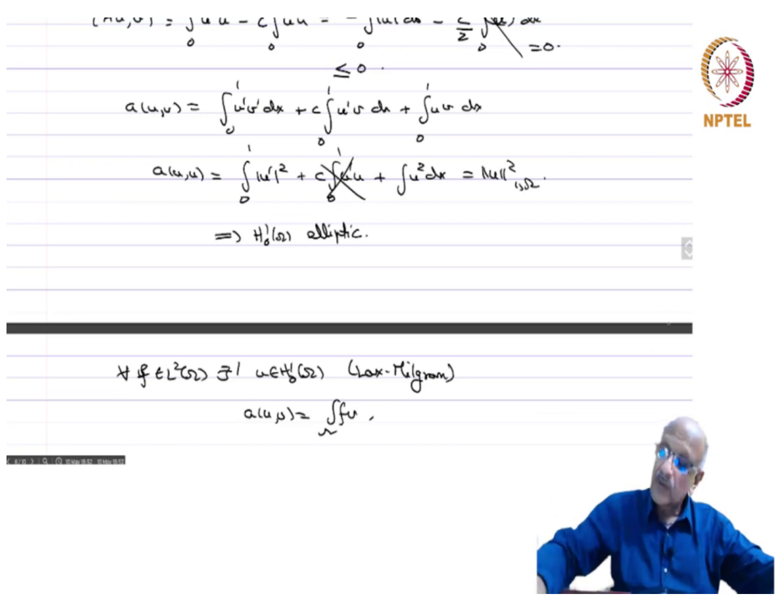

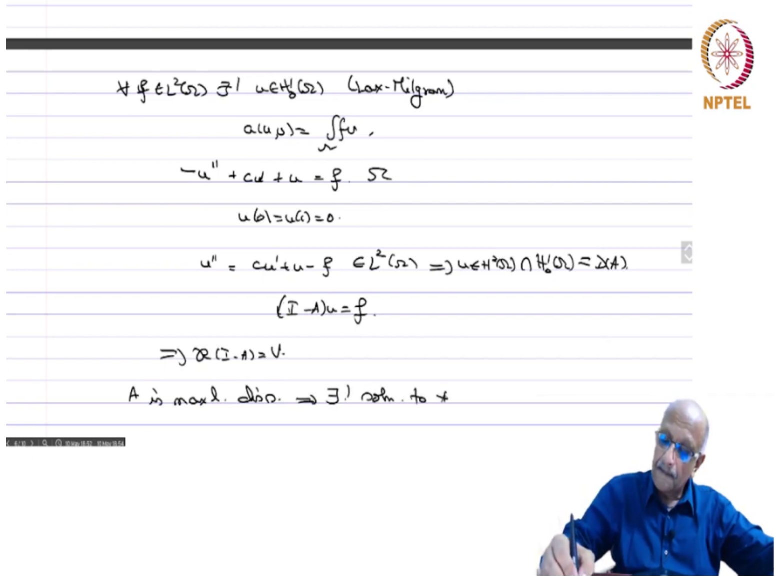

Exercise 5: let $c \in \mathbb{R}$, $c \neq 0$. Let $u_0 \in H^2(0,1) \cap H^1_0(\Omega)$. Show that there exists a unique solution of

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x} = 0, \quad 0 < x < 1,\ t > 0.$$

$$u(0,t) = u(1,t) = 0, \quad t > 0,$$

$$u(x,0) = u_0(x).$$

Solution: so we set $\Omega = (0\,,1)$ and we set $V = L^2(\Omega)$ and we define

$$D(A) = H^2(\Omega) \cap H^1_0(\Omega)$$

and set $Au = u'' - cu'$,

so then what is $(Au\,,u) = \int_0^1 u''u - c\int_0^1 u'u = -\int_0^1 |u|^2\,dx.$

And minus c by 2 integral u dash u is u square prime dx 0 to 1. Now, if you evaluate this at the end points this because it is a prime and therefore you will get it is 0 at the two end points by the boundary condition and therefore this term equal to 0 and therefore this is less than or equal to 0. So, you have that A is a dissipative operator.

Now, you look at this $(Au\,,u) = \int_0^1 u''u - c\int_0^1 u'u = -\int_0^1 |u|^2\,dx$. So, then $(Au\,,u)$ in this case is equal to integral mod u dash square 0 to 1 plus c integral 0 to 1 u dash u again this is 0 by the calculation which we did just now plus integral u square d x which is equal to norm u square 1 omega. And therefore, this is $H^1_0(\Omega)$ elliptic therefore for every if $f \in L^2(\Omega)$ there exists a unique u in $H^1_0(\Omega)$ by Lax-Milgram.

Such that integral a u v equals integral f v on omega that is and then if you look out what you say it is minus u double dash plus c u dash plus u equal to f in omega u equal to u_0 equals u_1 equal to 0 and u double dash is equal to c u dash plus u minus f and that belongs to $f \in L^2(\Omega)$. So, this implies that u is in $H^2\ (\Omega)$ intersection $H^1_0(\Omega)$ which is nothing but $D(A)$. And what is the differential equation say I minus A u a is what u double dash minus c u dash. So, this implies $R(I - A) = V$ therefore A is maximal dissipative implies there exists a unique solution to star.

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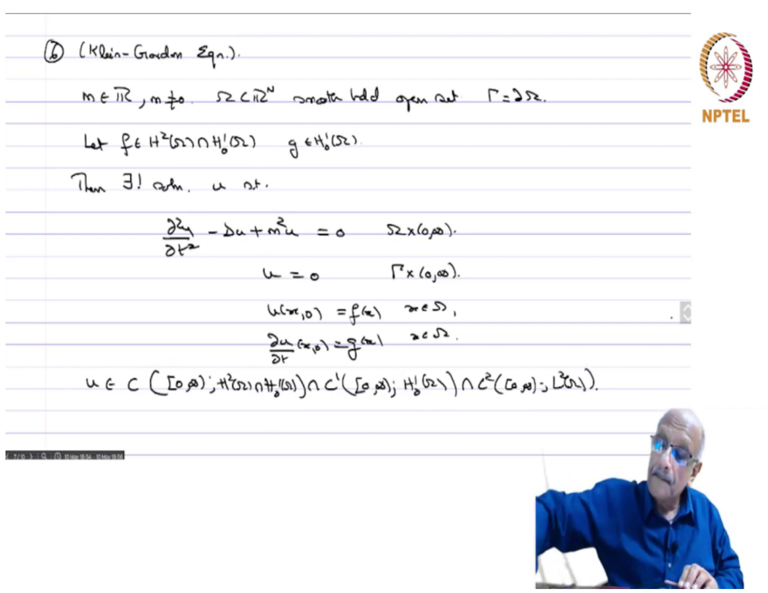

(6) Klein Gordon equation. So, let $m \in \mathbb{R}$, $m \neq 0$ and $\Omega \subset \mathbb{R}^N$ smooth bounded open set $\Gamma = \partial\Omega$ let $f \in H^2(\Omega) \cap H^1_0(\Omega)$ and $g \in H^1_0(\Omega)$ then there exists a unique solution u such that

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + m^2 u = 0, \quad \Omega \times (0, \infty)$$

$$u = 0 \quad \Gamma \times (0, \infty)$$

$$u(x, 0) = f(x), \quad x \in \Omega$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \quad x \in \Omega.$$

and $u \in C\left([0, \infty); H^2(\Omega) \cap H^1_0(\Omega)\right) \cap C^1\left([0, \infty); H^1_0(\Omega)\right) \cap C^2\left([0, \infty); L^2(\Omega)\right)$.

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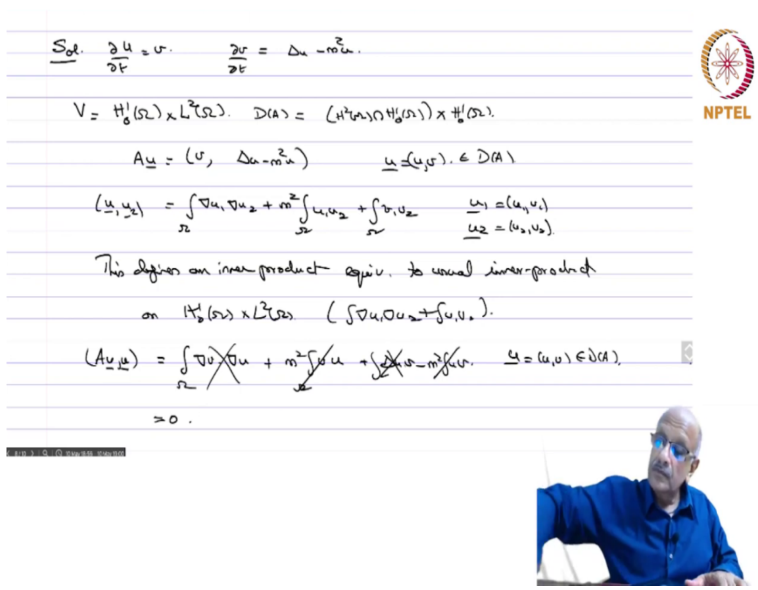

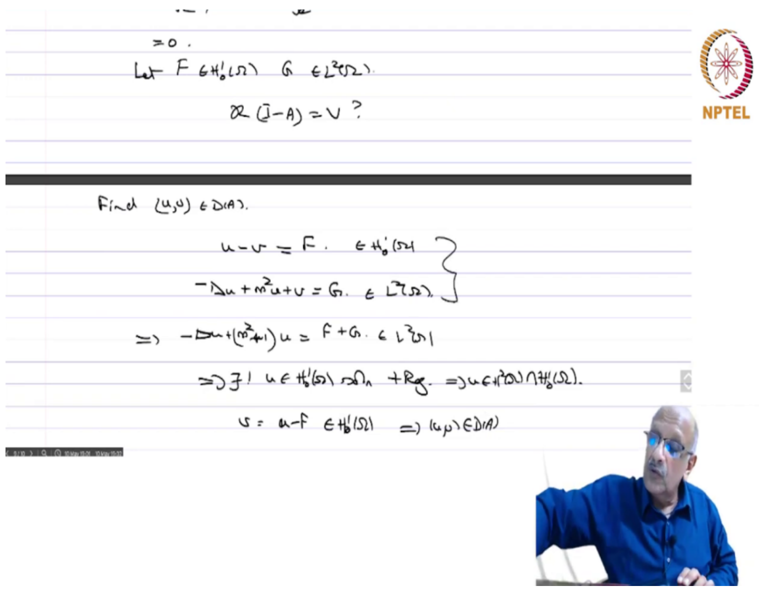

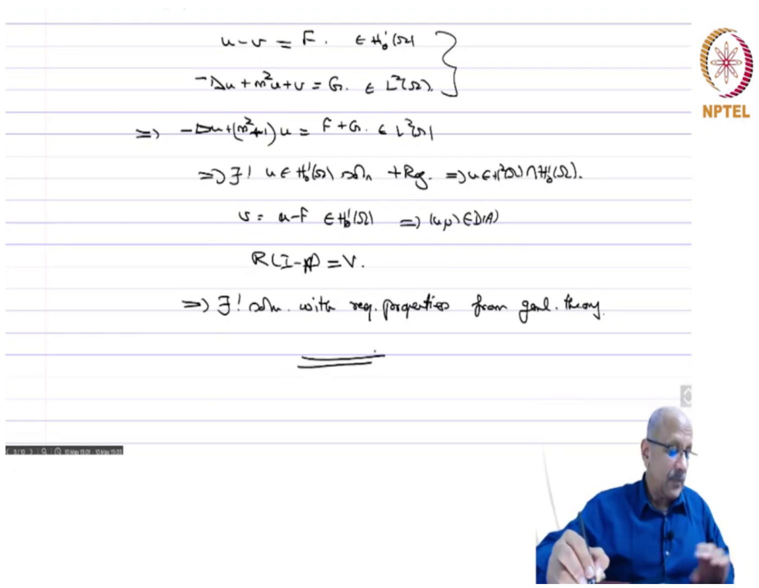

Solution: So, we set $\frac{du}{dt} = v$ and then you get

$\frac{dv}{dt} = \Delta u - m^2 u$

so you said v is equal to like in the wave equation $H^1_0(\Omega) \times L^2(\Omega)$ and $D(A) = \left(H^2(\Omega) \cap H^1_0(\Omega)\right) \times H^1_0(\Omega)$ and you write $Au\_ = (v, \ \Delta u - m^2 u)$. So,

$u_{\_} = (u, v) \in D(A)$

and now you consider

$$(u_{-1}, u_{-2}) = \int_{\Omega} \nabla u_1 . \nabla u_2 + m^2 \int_{\Omega} u_1 u_2 + \int_{\Omega} v_1 v_2, \quad u_{-1} = (u_1, v_1), \; u_{-2} = (u_2, v_2).$$

So, this is in L^2 and this is in $H^1_0 \cap H^2$.

So, then this is defines an inner product equivalent to usual inner product on $H^1_0(\Omega) \times L^2(\Omega)$ is the same so this you are taking a thing and that the L^2 norm is less than the H^1_0 norm by the punch array inequality and therefore this is less than the constant times the usual norm and the usual norm is of course less than this and therefore you have that these two are equivalent.

So, this usual inner product which is $\int_{\Omega} \nabla u_1 . \nabla u_2 + \int_{\Omega} v_1 v_2$. Now (Au, u) in this new inner product is equal to integral what is A, A is v this so you will have grad v dot grad u plus m square integral v u so u equals $(u, v) \in D(A)$. And then plus delta u v minus m square u v so these two get cancelled and then these two also get cancelled and therefore this is equal to 0.

So, now let $F \in H^1_0(\Omega)$ and $G \in L^2(\Omega)$. So, we want to show $I - A = V$ so we want to solve the following equations so find $(u, v) \in D(A)$. So, such that $u - v = F$ and minus Laplacian u plus m square u plus v equal to G and then that will give you if you add these two minus Laplacian u plus m square u. So, m square plus 1 u equals F plus G and then this can all always be solved so F so this belongs $H^1_0(\Omega)$ and this belongs to $L^2(\Omega)$.

So, this belongs to $L^2(\Omega)$ implies there exists unique $u \in H^1_0(\Omega)$ solution and plus regularity implies $u \in H^2 \ (\Omega) \times H^1_0(\Omega)$. And now you can write $v = u - F$ and this belongs $H^1_0(\Omega)$ so this implies $(u, v) \in D(A)$ and it solves the system of equations so you have that $R(I - A) = V$ and this implies there exists unique solution with required properties from general theory.

Because you have little more than dissipative you have $(Au, u) = 0$ in fact like in the wave equation and therefore you can you have this. So, with this I will stop these exercises and this course has also come to an end. I do hope you did enjoy it and you got something out of it and that it was a useful learning experience for you thank you for your attention.