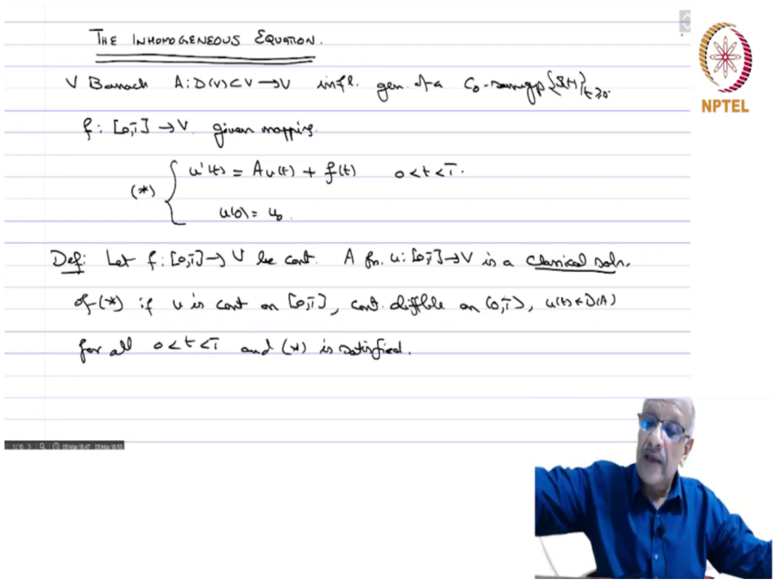
Sobolev Spaces and Partial Differential Equations
Professor S Kesavan
Department of Mathematics
Institute of Mathematical Sciences
Lecture 84
The Inhomogeneous Equation

(Refer Slide Time: 00:25)



The Inhomogeneous Equation

So, up to now, we have been looking at the equation $u^{'}(t) = Au(t)$. So, now we will look at the inhomogeneous equation. So, V Banach space, $A: D(V) \subset V \rightarrow V$ infinity symbol generator of a C^0 semi group $\{S(t)\}_{t\geq 0}$. So, now you we consider a function $f: [0, T] \rightarrow V$ given mapping.

So, we investigate the solutions of the following equation

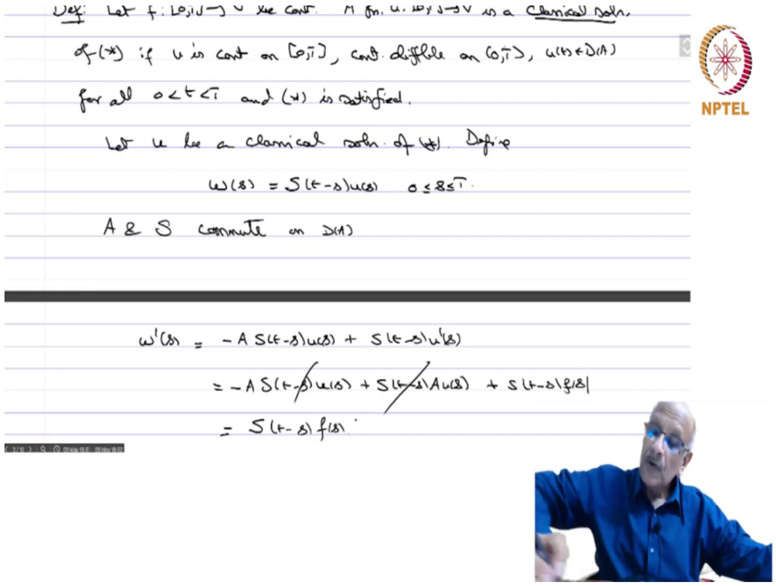
$$u^{'}(t) = Au(t) + f(t), \ \ 0 < t < T,$$

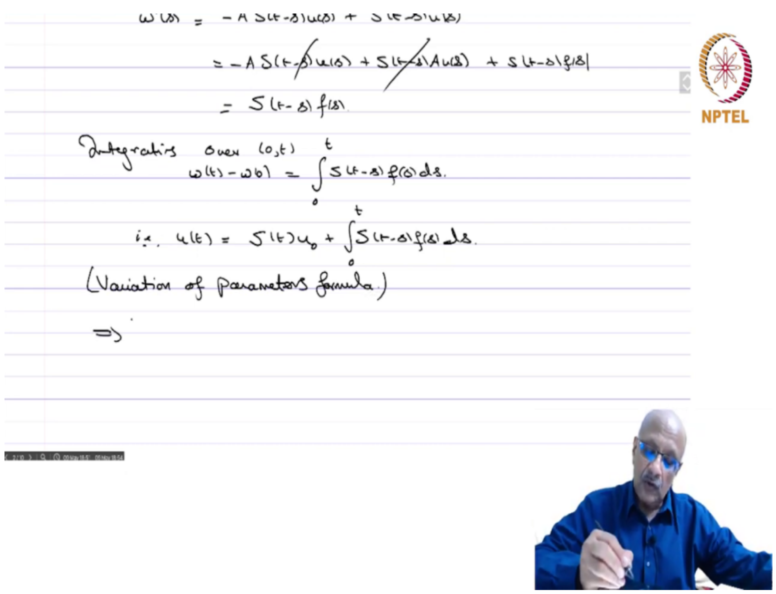
$$u(0) = u_0$$

So, this is equation which we want to look at. So,

Definition: let $f: [0, T] \to V$ be continuous. A function $u: [0, T] \to V$ is a classical solution of star if u is continuous on $[0, T]$ continuously differentiable on the open interval $(0, T)$, $u(t) \in D(A)$ for all $0 < t < T$ and star is satisfied. So, this is called a classical solution that means everything is fine and you can solve the solution exactly.

(Refer Slide Time: 03:08)





So, now, let us assume that let u be a classical solution of star. So, we will get a representation for u. So, now consider define

$$w(s) =- S(t - s)u(s),\ \ 0 \leq S \leq T.$$

Now, A and S(.) commute we know on $D(A)$ of course obviously and therefore, we have

$$w^{'}(s) =- AS(t - s)u(s) + S(t - s)u^{'}(s),$$

$u \in D(A)$.

So, when you differentiate this you get $-\ AS(t\ -\ s)u(s)$ and therefore, and A will come here A of us you will get but then A and S commute plus S of t minus s u dash s. So, I have just used the product rules and that is

$$=-\ AS(t\ -\ s)u(s)\ +\ S(t\ -\ s)Au\ (s)\ +\ S(t\ -\ s)f(s)$$

$$=\ S(t\ -\ s)f(s).$$

So, these two will get cancelled because of the commute activity. So, that will give you $S(t\ -\ s)f(s)$. So, if you integrating you get

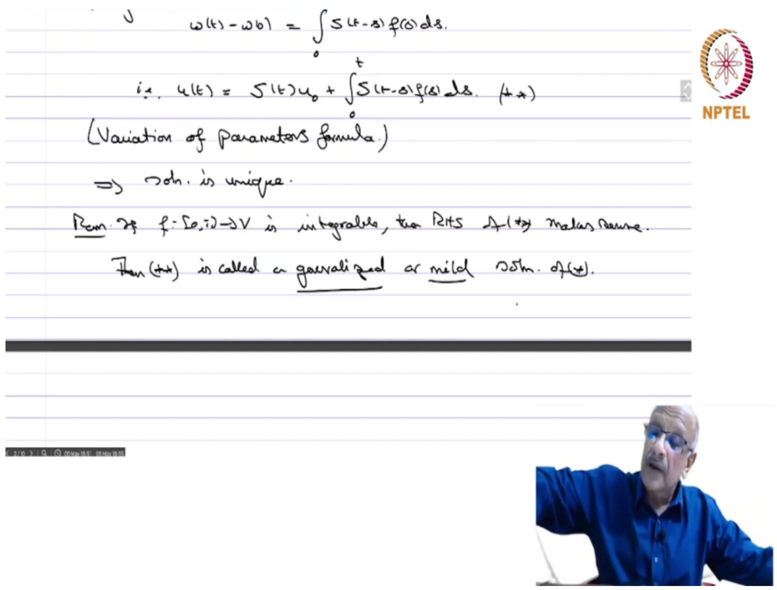
$$w(t)\ -\ w(0)\ =\int_0^t S(t\ -\ s)f(s)\, ds$$

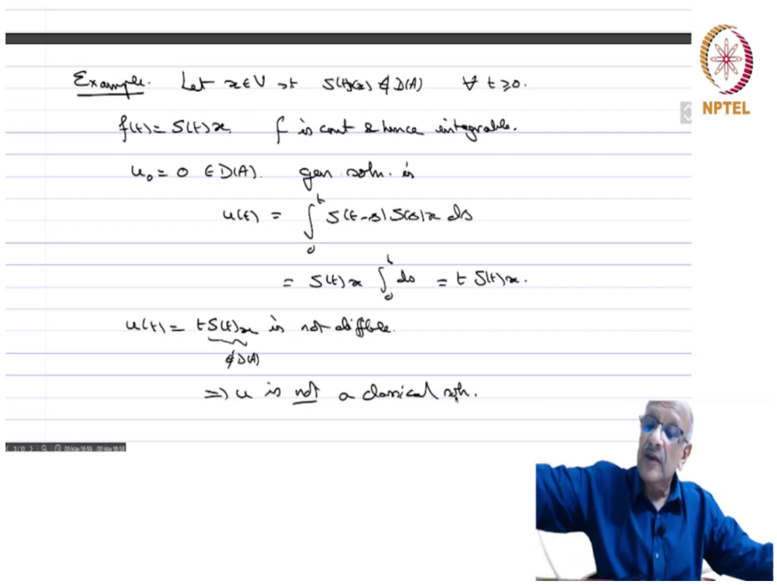
and therefore so, w(t) is what? So, $w(t)\ -\ w(0)$. So, that we get

$$u(t)\ =\ S(t)u_0 + \int_0^t S(t\ -\ s)f(s)\, ds.$$

This is called the variation of parameters formula and this is similar to whatever you have done for linear equations in ordinary differential equations this is exactly how you would have written you have the integrating factor and then which is e power something and that is exactly how you get this. So, this is the generalization of the usual variation parameters formula which you have in linear ordinary differential equations. So, this is called so, solution can be written in terms of this in fact

(Refer Slide Time: 06:54)





So, the solution is given like this therefore, implies solution unique since you have a representation formula. So,

Remark: if $f: [0, T] \to V$ is such that is integrable then RHS of double star makes sense and therefore, you then this is called a generalized solution or mild this different terminology used in different books and therefore generalized or mild solution of star.

So, then let us say instead of this let us say double star is called a generalized or mild solution of this thing. So, a generalized solution always exists big if you have integrability but it may or may not be a classical solution. So, let us give an

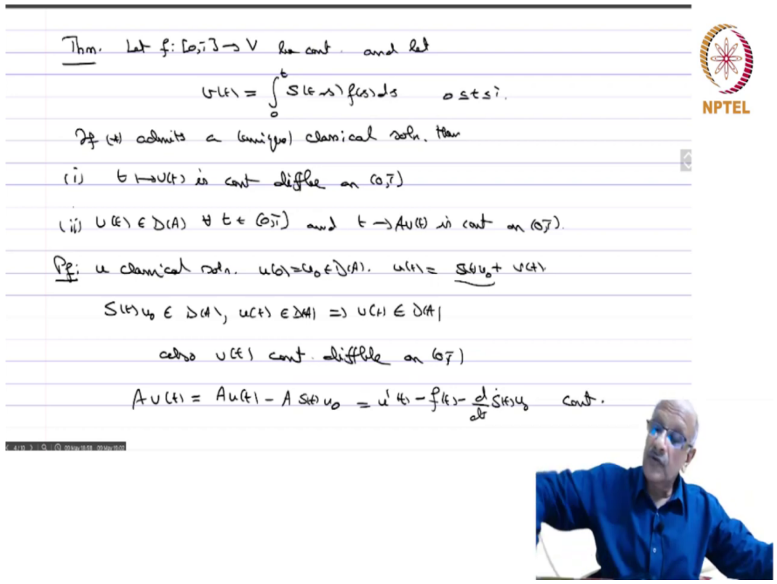
Example: of this. So, let $x \in V$ such that $S(t)x \notin D$. So, if $x \notin D(A)$ this can happen.

So, otherwise $S(t)x \in V$. So, for all t greater than equal to 0 now you take f of t equals St of x then f is continuous and hence integrable. So, now you take $u_0 = 0$ which of course belongs to the domain of A and then the generalized solution is,

$$u(t) = \int_0^t S(t - s)f(s)\, ds, \ \ 0 \le t \le T.$$

So, $S(t - s)$ of s that will give you composition will give you S of t $S(t)x$ which will become out of the integral and therefore, this will be giving you so this equal to integral 0 to t S of t minus s $S(t)x$ ds which is equal to $S(t)x$ integral 0 to t ds which is equal to t times $S(t)x$ but u t equals t times $S(t)x$ is not differentiable since you have $S(t)x$ does not belong to $D(A)$ and therefore, this is not differentiable we know and therefore, you have that implies u is not a classical solution. So, generalized solution will always exist and but it may not be a classical solution. So, when will we have a generalized solution is a classical solution this is a question which we want to ask. So, we now have the following theorem.

(Refer Slide Time: 10:48)



Theorem:

Let f: $[0, T] \rightarrow V$ be continuous and let

$$v(t) = \int_0^t S(t - s)f(s)\, ds, \;\; 0 \leq t \leq T.$$

If star admits a classical unique classical solution then one you have t going to v(t) is continuously differentiable on (0, T). And 2 $v(t) \in D(A)$, $t \in (0, T)$ and t going to $Av(t)$ is continuous on $(0, T)$ proof this very simple.

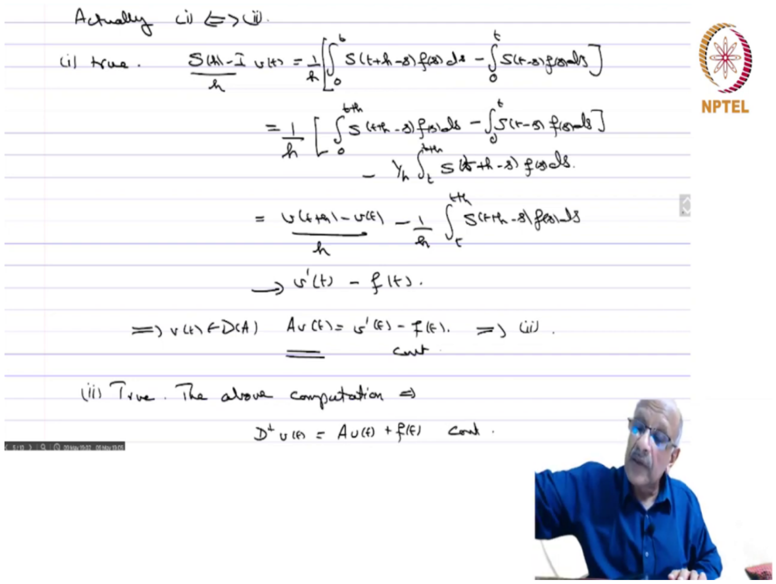
Proof: So, u is a classical solution and $u(0) = u_0 \in D(A)$ then $u(t) = S(t)u_0 + v(t)$. Now, $S(t)u_0 \in D(A)$ ut is also in $D(A)$ because you have a classical solution. So, this implies that v(t) belongs to $D(A)$ and also you have this is St of u_0 is also in $D(A)$ v(t) is continuously differentiable.

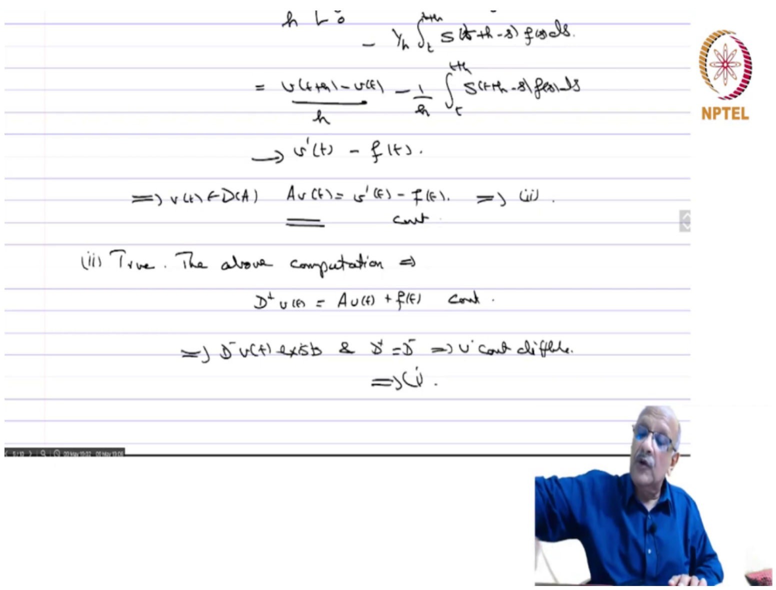
Because u(t) is continuously different (())(13:14) u0 is in domain of A and therefore St u_0 is also continuously differentiable and therefore v(t) has to be continuously differentiable. So, that shows the first one is v(t) is continuously differentiable and also v(t) is in $D(A)$. And now finally, we have

$$Av(t) = Au(t) - AS(t)u_0 = u'(t) - f(t) - \frac{d}{dt}S(t)u_0$$

and this is continuous. So, A of v(t) is also continuous and that completes the proof.

(Refer Slide Time: 13:59)





Now, we though we said one and two, one if and only they are equivalent statements. So, assume one true then look at

$$\frac{S(h)-I}{h}v(t) = \frac{1}{h}\left(\int_0^t S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\right)$$

$$= \frac{1}{h}\left(\int_0^{t+h} S(t+h-s)f(s)\,ds - \int_0^t S(t-s)f(s)\,ds\right) - \frac{1}{h}\left(\int_t^{t+h} S(t+h-s)f(s)\,ds\right)$$

$$= \frac{v(t+h)-v(t)}{h} - \frac{1}{h}\left(\int_{t}^{t+h} S(t + h - s)f(s)\, ds\right).$$

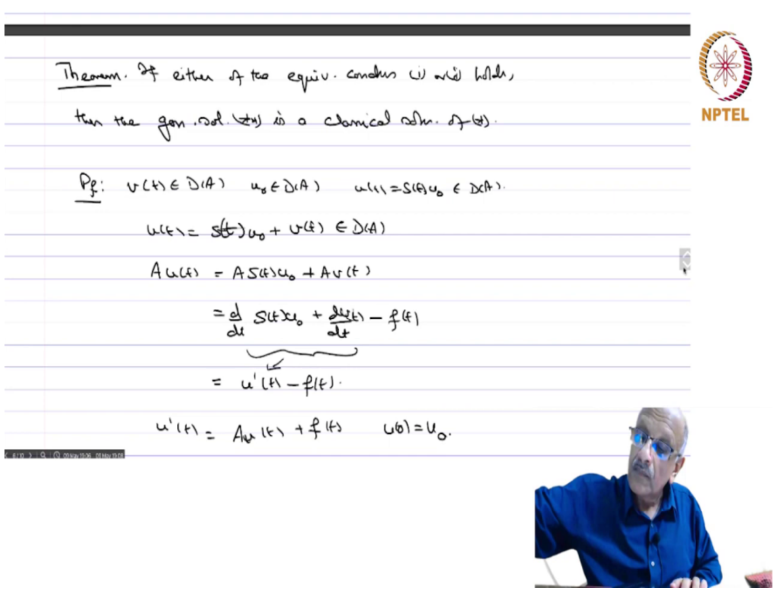
And now we know v is differentiable, one is true. So, this goes to $v'(t)$ and then this goes should go to the lower value with $h = 0$ and therefore, t equals s this s of 0 minus f of t. So, this implies that $v(t) \in D(A)$ and Av(t) equals $v'(t) - f(t)$ and therefore, you have that you have this also.

So, this implies two which is continuous therefore, this implies two. Now, two is true conversely then what happens you have that the above computation implies that

$$D^{+}v(t) = Av(t) + f(t).$$

that is what the same computation you do now, you know that this is in $D(A)$ so, you get Av(t) take v(t) to is and this will give you just $D^{+}v(t)$ instant and this is continuous. So, $D^{+}v(t)$ is continuous implies $D^{-}v(t)$ exist we have seen these many times before and $D^{+}v(t) = D^{-}v(t)$ and therefore, v is continuously differentiable and therefore, implies one. So, one and two are equivalent to each other though we prove them separately.

(Refer Slide Time: 18:06)



So, now we have the following

Theorem: If either of the equivalent conditions one or two holds then the generalized solution double star is a classical solution of star. So, we were asking the question when is the generalized solution a classical solution. So, if one of these conditions is true, then that means both are true then you have the generalized solution is a classical solution. So,

Proof. so, we have $v(t) \in D(A)$ and if $u_0 \in D(A)$ when

$u(t) = S(t)u_0 + v(t) \in D(A)$ and therefore, you have $u(t) = S(t)u_0 + v(t) \in D(A)$ and you have

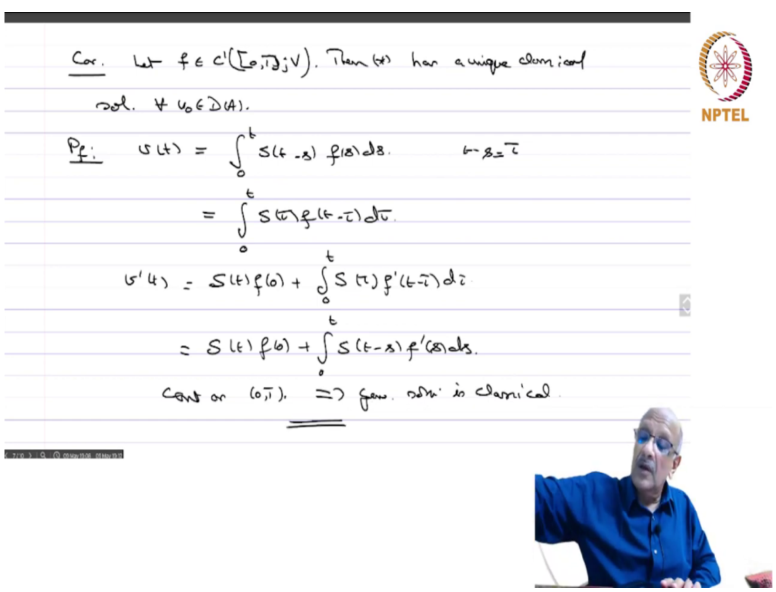
$$Au(t) = AS(t)u_0 + Av(t)$$

$$= \frac{d}{dt}S(t)u_0 + \frac{dv}{dt}(t) - f(t)$$

$$= u'(t) - f(t)$$

So, $Au(t) + f(t),\ u(0) = u_0$, and this completes the proof.

(Refer Slide Time: 20:45)



So, you will give a

Corollary: of this concrete application of this result let $f \in C^1([0, T]; V)$. with values in V then star has a unique classical solution for every $u_0 \in D(A)$. So, we just have to show that one of those conditions is true. So,

Proof: so, we write v(t), we have to show that this is continuously differentiable, that is all we have to show.

So,

$$v(t) = \int_0^t S(t - s)f(s)\, ds.$$

Now, I can also write this by change of variables to 0 to $S(t - s)f(s)$ ds. I am going to put $t - s = \tau$ and therefore,

$$= \int_0^t S(\tau)f(t - \tau)\, d\tau$$

and therefore, I have this.

So, if you like this just change of variable. So, if you do not like s here, put it τ. So, now I am going to try to differentiate this we differentiate the under the integral sign. So, you have $v'(t)$ is equal to so you first have to evaluate this t and then differentiate t which is give you one. So,

$$v'(t) = S(t)f(0) + \int_0^t S(\tau)f'(t - \tau)\, d\tau$$

$$= S(t)f(0) + \int_0^t S(t - s)f'(s)\, ds$$

this just in differentiation under the integral sign is well defined because f in C^1. So, this integral can be differentiated and therefore, this is what you have and therefore, this is continuous and now, you can rewrite it again if you change the variable once more this will give you S of t minus s f dash s ds.

So, once again you make a change of variable and you get this. So, now, this is continuous on $(0, T)$. So, v dash v is a continuously differentiable so the first condition is satisfied and therefore, the implies generalized solution is classical. So, this is about the inhomogeneous equation. Now, this in homogeneous equation is very important in the study of control theory this is where the control theory begins and therefore, you have this equation is very important