Sobolev Spaces and Partial Differential Equations
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Lecture 83
The Schrodinger equation

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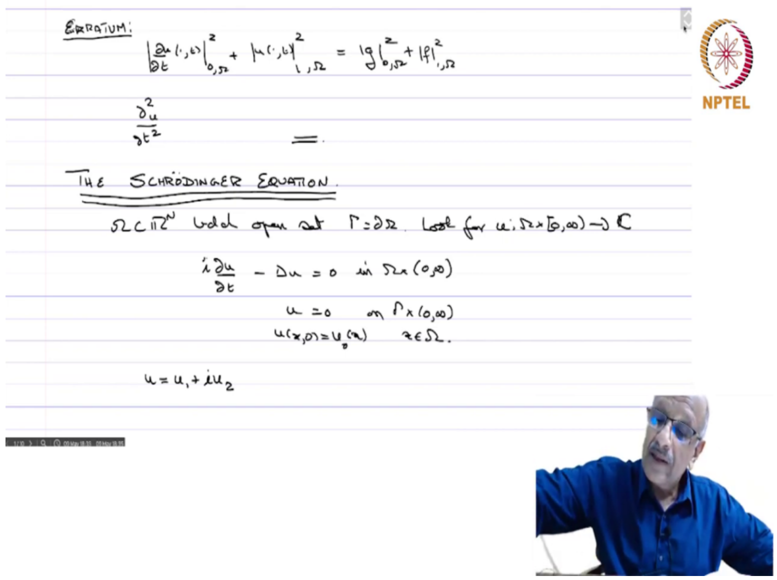

Before, I begin let me fix ERRATUM. So, in the theorem on the wave equation. I wrote the conservation of energy $\left|\frac{\partial u}{\partial t}(.,t)\right|^2_{0,\Omega} + |u(.,t)|^2_{1,\Omega} = |g|^2_{0,\Omega} + |f|^2_{1,\Omega}$. So, the correction is this should be $1,\Omega$ because you the space is for $\frac{\partial u}{\partial t}$ it is an L^2 and this is H^1_0 and this should u correspond to f, $\frac{\partial u}{\partial t}$ correspond to g and that is conserved.

So, this is the thing and another place when writing the equation for $\mathbb{R}^N$ I wrote $\frac{\partial u}{\partial t}$ you would have figured it out it should be $\frac{\partial^2 u}{\partial t^2}$ squared. So, today we will look at one more examples. So, this is the Schrodinger equation. So, this is an equation which is important in quantum mechanics.

So, you take $\Omega \subset \mathbb{R}^N$ bounded open set and $\Gamma = \partial\Omega$ we look for $u: \Omega \times [0, \infty) \to C$ for the first time we are going to deal with something it is complex valued well for few moments and such that,

$$i\frac{\partial u}{\partial t} - \Delta u = 0 \quad in \;\; \Omega \times (0, \infty),$$

$$u = 0 \quad on \;\; \Gamma \times (0, \infty)$$

$$u(x, 0) = u_0(x) \;\; for\, x \in \Omega.$$

So, you have an i here in front otherwise it looks very much like the heat equation. So, we are want to deal with real functions. So, we write

$u = u_1 + iu_2$, the real and imaginary parts. So, we separate the real and imaginary parts in this equation.

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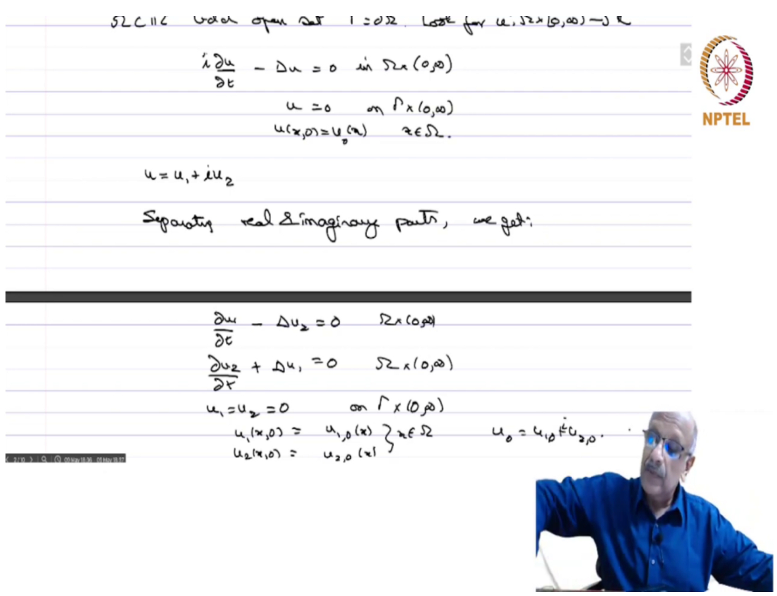

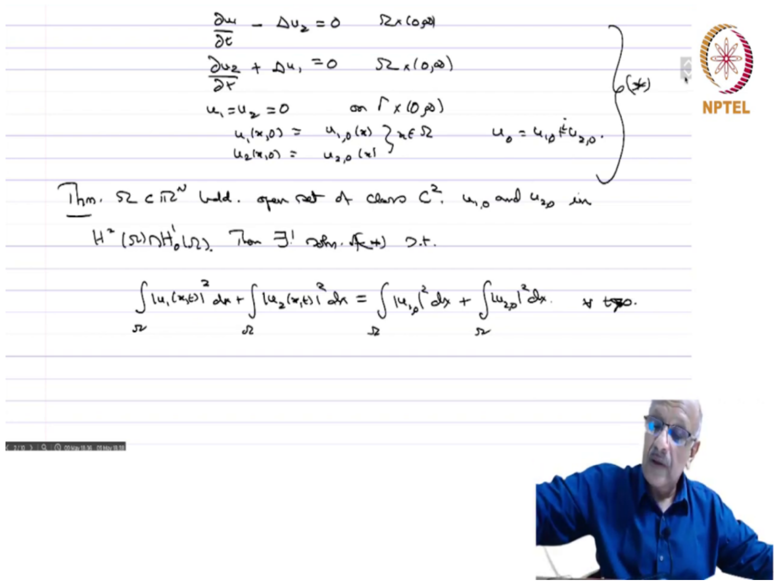

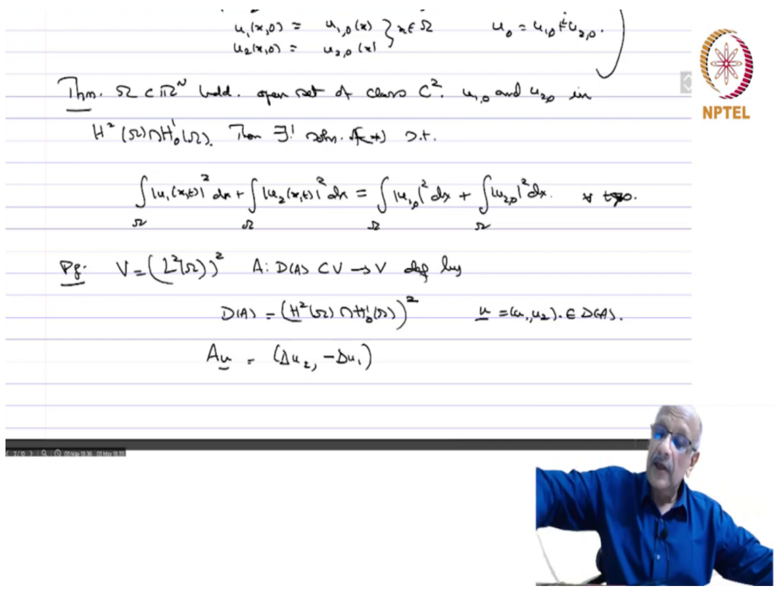

And then so, separating real and imaginary parts, we get the following system of equations

$$\frac{\partial u_1}{\partial t} - \Delta u_2 = 0, \quad in \;\; \Omega \times (0, \infty),$$

$$\frac{\partial u_2}{\partial t} - \Delta u_1 = 0, \quad in \;\; \Omega \times (0, \infty),$$

$$u_1 = u_2 = 0, \quad on \;\; \Gamma \times (0, \infty),$$

$$u_1(x, 0) = u_{1,0}(x) \quad for\ x \in \Omega.$$

$$u_2(x, 0) = u_{2,0}(x) \quad for\ x \in \Omega.$$

So, this is the real and imaginary parts. So, we have a following this system of equations which we want to solve, and now we have the following

Theorem $\Omega \subset \mathbb{R}^N$ bounded open set of class C^2, $u_{1,0}$ and $u_{2,0}$ in $H^2(\Omega) \cap H^1_0(\Omega)$ then there exists a unique solution of star such that.

So, $\int_\Omega |u_1(x, t)|^2 dx + \int_\Omega |u_2(x, t)|^2 dx = \int_\Omega |u_{1,0}|^2 dx + \int_\Omega |u_{2,0}|^2 dx$

So, this is for all t greater than 0. So, this is the theorem which we have. So,

Proof. So, we take $V = (L^2(\Omega))^2$ and $A: D(A) \subset V \to V$ to be defined by

$$D(A) = (H^2(\Omega) \cap H^1_0(\Omega))^2,$$

$$Au = (\Delta u_2, -\Delta u_1),\ u = (u_1, u_2) \in D(A).$$

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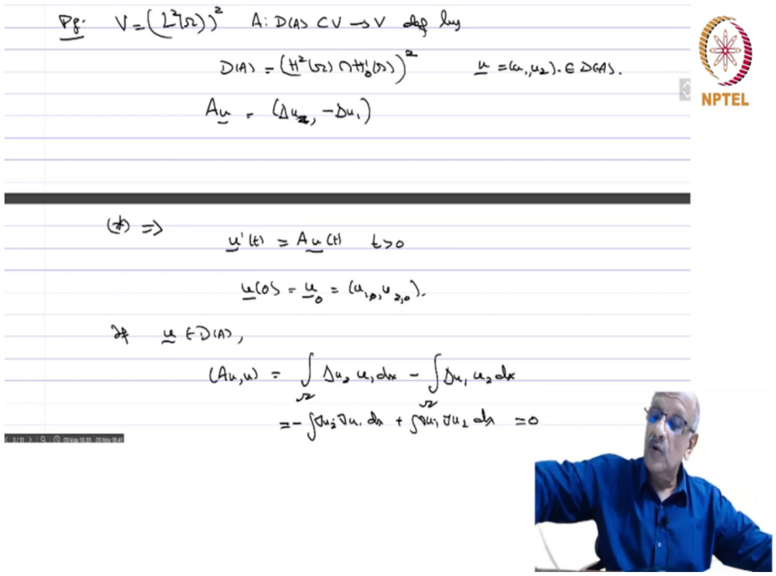

So, this is the thing then star implies that u dash t equals $Au(t)$. So, du1 by dt du_2 by dt that is u dash t minus delta u_2 plus delta u_1 and so, that is equal delta u_2 minus delta u_1 when you take it to the other side and therefore, that will give you precise the d^2u by dt t greater than 0 and u of 0 equals u_0 equals $u_{1,0}$, $u_{2,0}$.

So, if you belongs to $D(A)$ then you have a

$$(Au, u) = \int_{\Omega} (\Delta u_2) u_1 \, dx - \int_{\Omega} (\Delta u_1) u_2 \, dx = 0.$$

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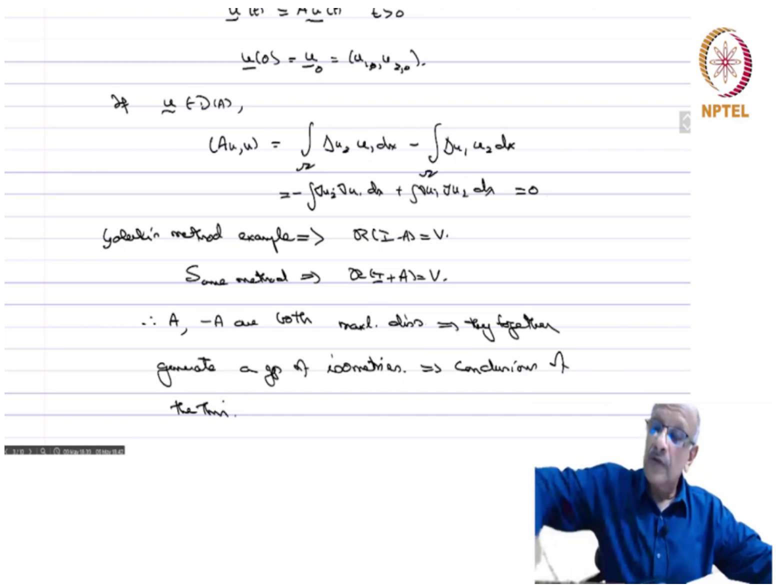

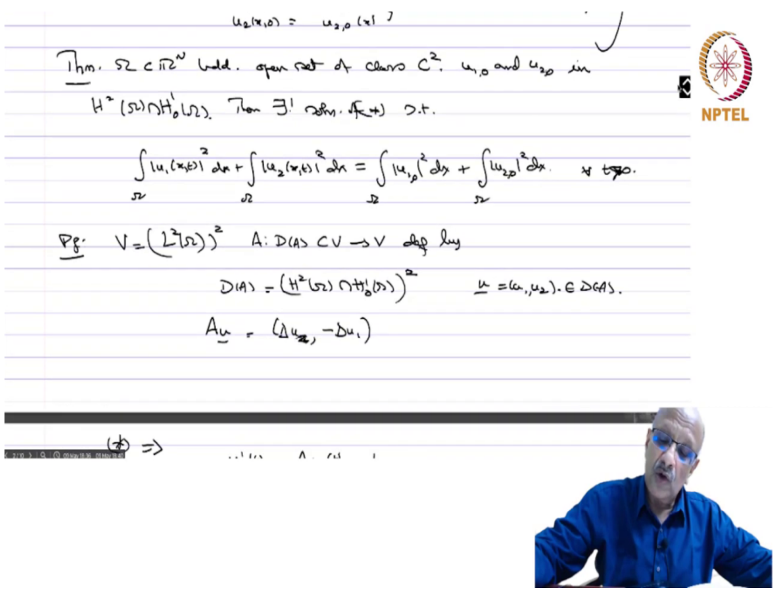

Now, if you go back to the Galerkin method example. This implies if you go and look at that it precisely says $R(I - A) = V$ and same method will implies $R(I - A) = V$ therefore, A and $- A$ are both. So, you just go to the previous chapter where they gave you an example of the Galerkin method I mentioned that will be useful in the solution Schrodinger equation and that is exactly if you look at right range (())(09:22) that is exactly what we proved there exists a unique solution to that we did not use a Laxman lemma we use the Galerkin method instead.

So, A and $-A$ are both maximal dissipative implies they together generate a group of isometries and this implies conclusions of the theorem. So, if you notice in the domain which it is which we are given u_0 is in the domain. So, $u_{1,0}$ and $u_{2,0}$ or both in $H^2 \cap H^1_0$. So, u_0 is in the domain and then you have and this is a condition which says that it is an isometry it preserves the norm of the initial value throughout the thing. So, this represents this solution. So, therefore, you have a unique solution for the Schrodinger equation by through its real and imaginary parts.

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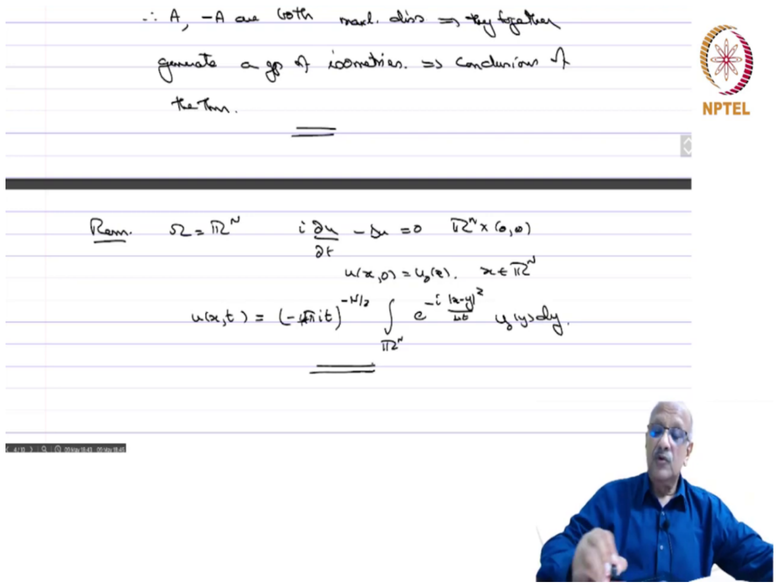

So, the Schrodinger equation looks like the heat equation, but it solves use methods like the wave equation here. And

Remark: If $\Omega = \mathbb{R}^N$ then you have

$$i\frac{\partial u}{\partial t} - \Delta u = 0 \ \ in\ \mathbb{R}^N \times (0, \infty),$$

$$u(x, 0) = u_0(x) \ \ for\ x \in \mathbb{R}^N.$$

no boundary terms now. So, then you can write as using the Fourier transform you can then show that

$$u(x,t) = (-4\pi i t)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{-i\frac{|x-y|^2}{4t}} u_0(y)\ dy.$$

So, this is the solution of the Schrodinger equation in $\mathbb{R}^N$ and again you will have that u of L^2 norm will be preserved. So, this is about the Schrodinger equation and we will. So, that brings us to the end of the examples which I wanted to talk about.