Sobolev Spaces and Partial Differential Equations
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Lecture 82
The Wave Equation

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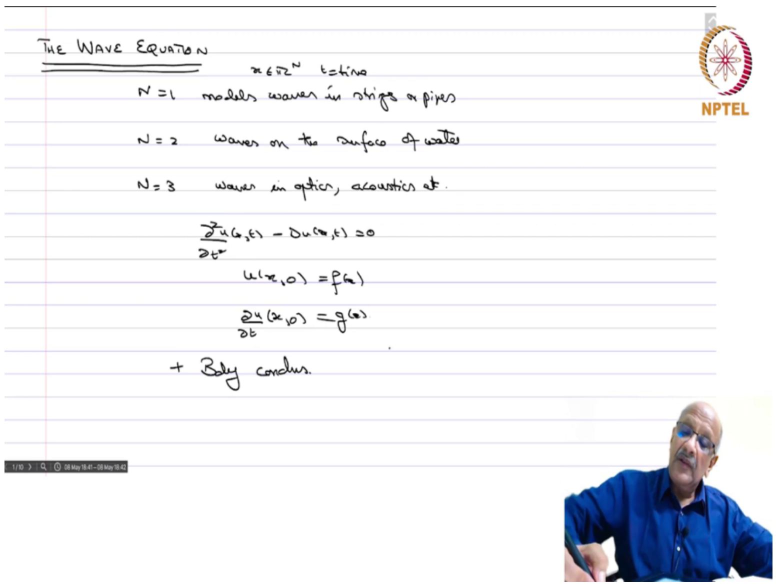

Our next example is the wave equation. So, the wave equation is the simplest example of a hyperbolic differential equation of second order so, if $x \in \mathbb{R}^N$, and t is the time variable than the wave equation. So, if $N = 1$ so, $x \in \mathbb{R}^N$ and t equals time so, if $N = 1$ then models waves in strings or pipes.

If $N = 2$ it waves on the surface of water and if $N = 3$ you have waves in optics acoustics, et cetera. So, again so, what is equation you have

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 \quad in \ \Omega \times (0,\infty)$$

$$u(x,t) = 0 \ \ on \ \Gamma \times (0,\infty),$$

$$u(x,0) = f(x) \quad for \ \ x \in \Omega,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad for \ \ x \in \Omega,$$

And Laplace is again the boundary thing and if you have you may have boundary conditions if this setting.

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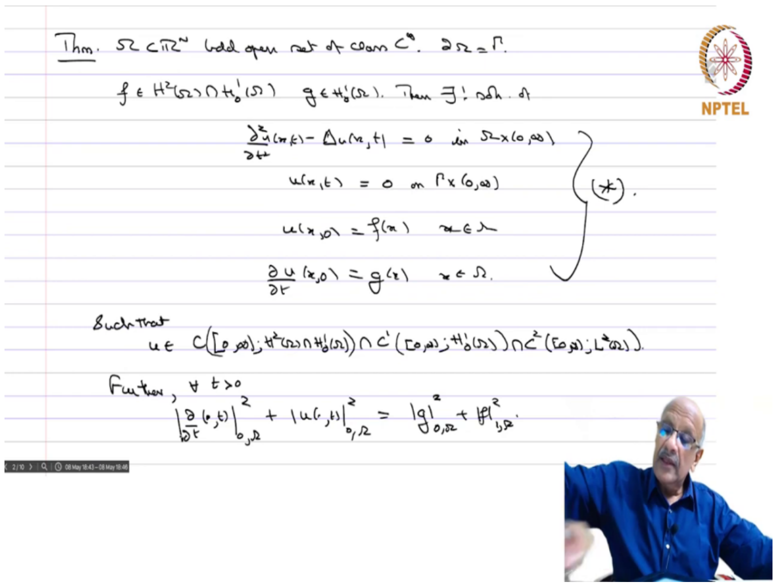

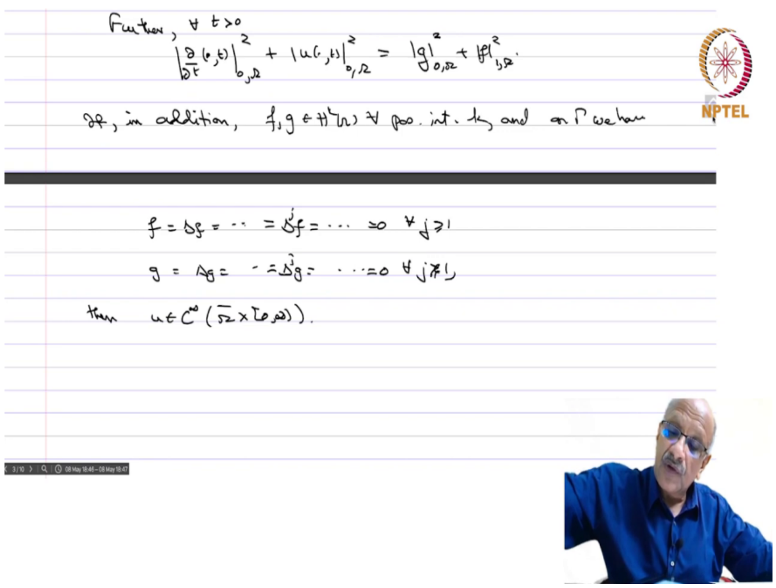

So, we once again will look at the existence of and uniqueness, et cetera the solution of the wave equation using the theory of semi groups. So, we have the following theorem. So,

Theorem: $\Omega \subset \mathbb{R}^N$ bounded open set of class C^∞ we are assuming maximum so, as we do not have to worry about various things will have in of regularity.

So, $f \in H^2(\Omega) \cap H^1_0(\Omega)$ and $g \in H^1_0(\Omega)$ then there exists a unique solution of

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 \quad in\ \Omega \times (0,\infty)$$

$$u(x,t) = 0\ on\ \Gamma \times (0,\infty),$$

$$u(x,0) = f(x)\ \ for\ x \in \Omega,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)\ \ for\ \ x \in \Omega.$$

So, this is our basic equation such that $u \in C([0,\infty); H^2(\Omega) \cap H^1_0(\Omega)) \cap C([0,\infty); H^1_0(\Omega)) \cap C^2([0,\infty); L^2(\Omega))$. Further for all t greater than 0

$$\left|\frac{\partial u}{\partial t}(.,t)\right|^2_{0,\Omega} + |u(.,t)|^2_{1,\Omega} = |g|^2_{0,\Omega} + |f|^2_{1,\Omega}.$$

Then if in addition $f, g \in H^k(\Omega)$ for all positive integers k and on Γ we have

$$f = \Delta f = \ldots = \Delta^j f = \ldots = 0, \quad for\ all\ j \geq 1,$$

$$g = \Delta g = \ldots = \Delta^j g = \ldots = 0, \quad for\ all\ j \geq 1,$$

then $u \in C(\overline{\Omega} \times [0,\infty))$. So, this is the sub proof.

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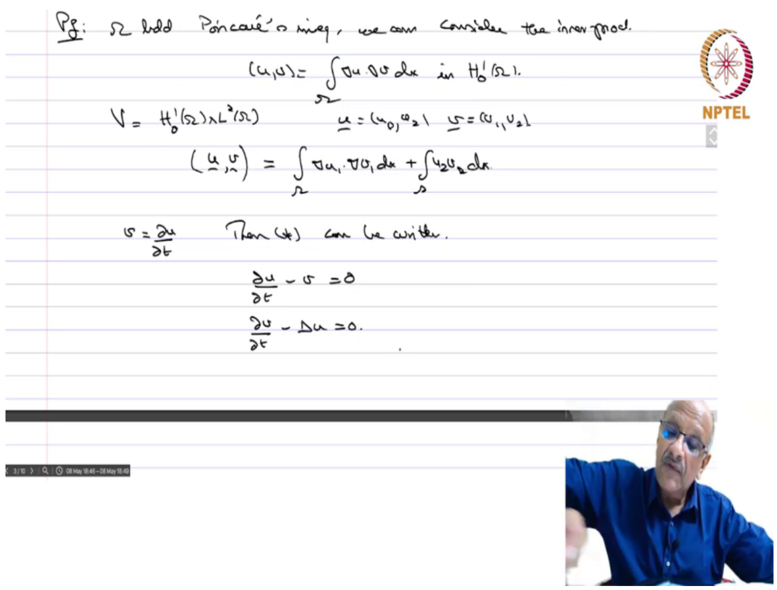

Proof:

So, Ω is bounded so, by **Poincare's inequality** we can consider the inner product

$$(u, v) = \int_{\Omega} \nabla u. \nabla v \ dx.$$

in $H^1_0(\Omega)$. So, now we said $V = H^1_0(\Omega) \times L^2(\Omega)$ so, the inner product so, now

$$(u, v) = \int_{\Omega} \nabla u_1. \nabla v_1 \ dx + \int_{\Omega} \nabla u_1. \nabla v_2 \ dx.$$

So, the first components will be in $H^1_0(\Omega)$.

So, now we set $v = \frac{\partial u}{\partial t}$ then star can be written as du by dt minus v equal to 0 and dv by dt minus Laplacian u equal to 0 so, the standard thing if you have a second order the equation you write it as a system of first order equations this is a standard technique which we have.

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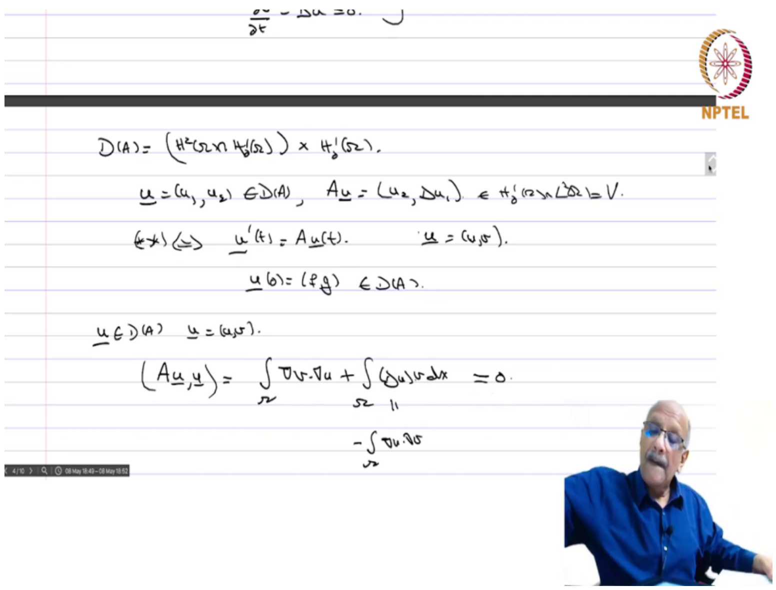

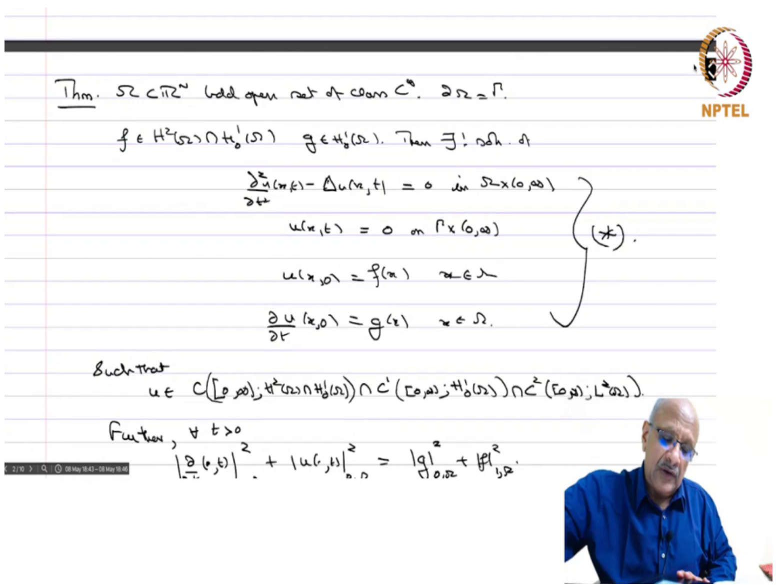

So, now we will look at $D(A) = (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ for the first component cross $H^1_0(\Omega)$ and for any $u = (u_1, u_2)$ in D(A) we will define

$$AU = (v, \Delta u),\ \ U = (u, v) \in D(A).$$

So, this will be $H^1_0(\Omega)$ this will be $L^2(\Omega)$ and therefore, the disk space V.

So, this belong to $H^1_0(\Omega) \times L^2(\Omega) = V$. So, this is well defined and now you can star is the same as saying that U dashed of t is equal to AU that is a double star. You have d by dt of uv that is U dashed of t and then you have minus of equals $AU(t)$. So, if you take the other side v delta u, v delta u is nothing but the AU.

So, $U = (u, v)$ now and then you of 0 is nothing but fg and you call we put f and g we said f is in $H^2_0(\Omega) \cap H^2(\Omega)$ and $H^1_0(\Omega)$ and $g \in H^1_0(\Omega)$ and therefore, that belongs to $D(A)$. And then if u is in $D(A)$ so, $U = (u, v)$ then what is

$$(AU, U) = \int_\Omega \nabla v . \nabla u \, dx + \int_\Omega (\Delta u) v \, dx = 0.$$

but then this u is in $H^2(\Omega)$ in $H^1_0(\Omega)$ so, this is equal to minus integral on omega grad u dot grad v there are no boundary terms and therefore, this is equal to 0.

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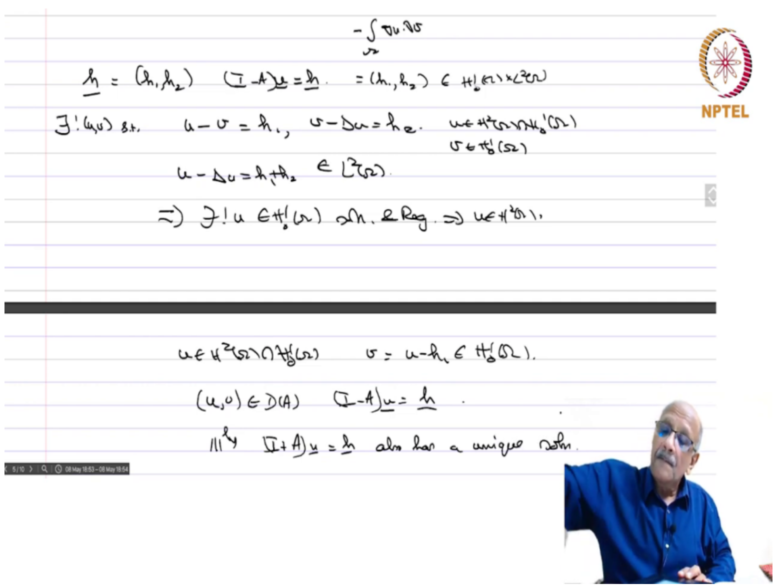

Now, you let $h = (h_1, h_2)$ and you consider $(I - A)u = h$. So, what is the t equation so, I u minus v equals h_1 and you have v minus delta u equals h_2 and u belong to $H^2(\Omega)$ in sections $H^1_0(\Omega)$ and you want v in $H^1_0(\Omega)$. So, does that exist uv such that you have this.

So, if you add these two equations you get

$$u - \Delta u = h_1 + h_2.$$

and that belongs to $L^2(\Omega)$. So, then there implies there exists unique u in $H^1_0(\Omega)$ solution and regularity implies u is in $H^2(\Omega)$ so there exists u in $H^2(\Omega) \cap H^1_0(\Omega)$such that) $u - \Delta u = h_1 + h_2$. and now you said v equal to u minus h_1 which will belong to $H^1_0(\Omega)$.

So, $h = (h_1, h_2)$ remember this is belongs to $H^1_0(\Omega) \times L^2(\Omega)$. So, this belongs $H^1_0(\Omega)$ and therefore we have that u v belongs to $D(A)$ and you have $I - A(U) = v$. And similarly I plus u equal to h you can check this also has a unique solution.

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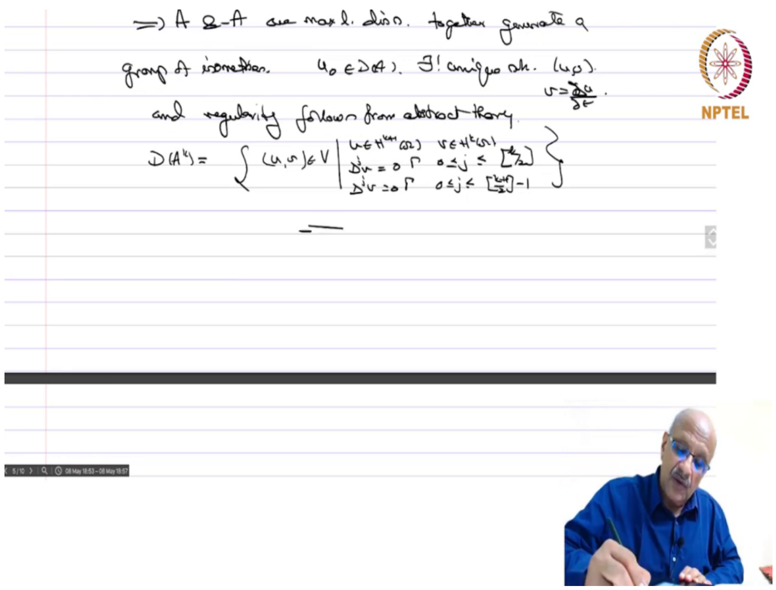

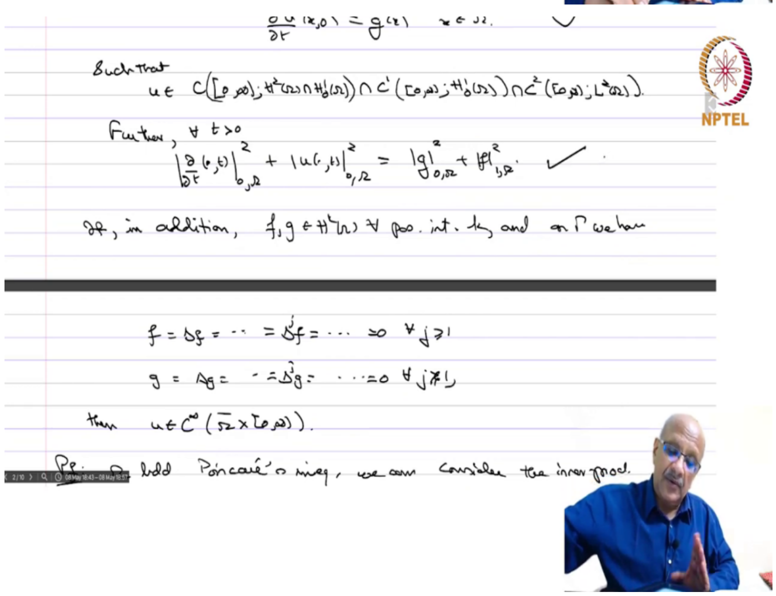

So, this means that A and $-\ A$ are maximum dissipate and together generate group of isometries. So, if u naught belongs to $D(A)$ then there exists a unique solution and regularity v will be du by dt and regularity follows from the abstract theory after the Helio sida theorem we saw that if you is in $D(A^k)$ then u naught is in $D(A^k)$ then it is all that.

So, you notice that what is

$D(A^k)$ here the set of all $(u, v) \in V$ such that u is in $H^{k+1}(\Omega)$, v is in $H^k(\Omega)$ and delta j of u equal to 0 in Γ_0 less than equal to j less than equal to the integral part of k by 2 and delta j of v equals 0 on Γ_0 less than equal to j less than equal to k plus 1 by 2 integral part minus 1.

This requires some checking, but then one can show this. So, then so given the conditions you have here now, the fact that you have an isometry tells you that you have the conservation law which I wrote down this comes from the fact that you have an isometry. So, that is just initial they are all (())(16:31) are all isometries and therefore, we have the theorem and the result follows.

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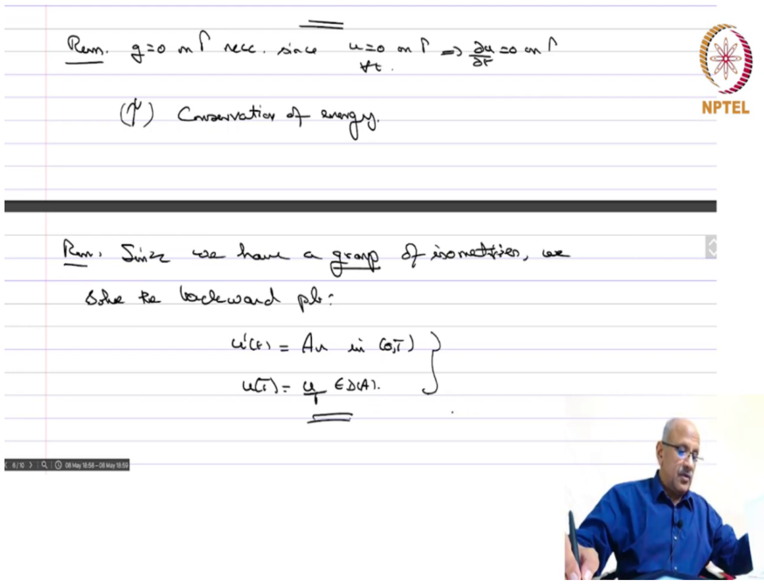

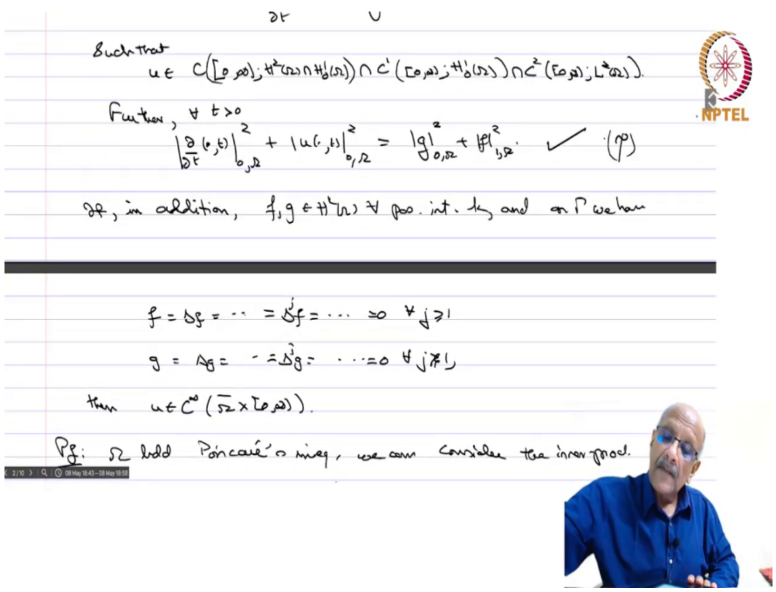

So,

Remark: $g = 0, \ on\, \Gamma_0$ necessary since $u = 0\ on\, \Gamma_0$ and that implies that $\frac{\partial u}{\partial t} = 0$ on Γ for all t this is true and therefore, this implies so, this means that g must also be 0 on gamma and then this relationship dagger is called conservation of energy and then remark again

Remark: since we have a group of isometries we can solve the backward problem namely

$$\frac{\partial U}{\partial t} = AU, \ \ in\ (0, T),$$

$$U(T) = U_T$$

in D assuming it is in $D(A)$ and therefore, this problem has a solution and since you have a group of a isometries. So, simulation we do not since we have a group and like the heat equation where you could not solve it backwards here you can solve it backwards.

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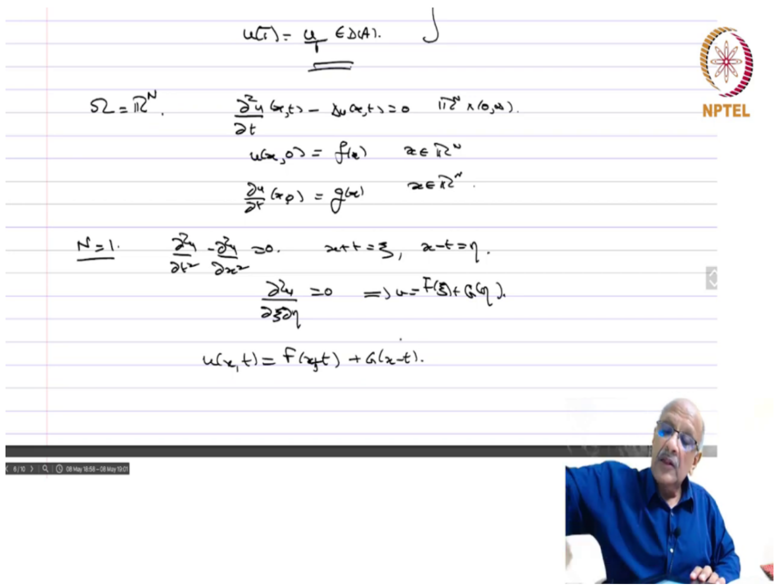

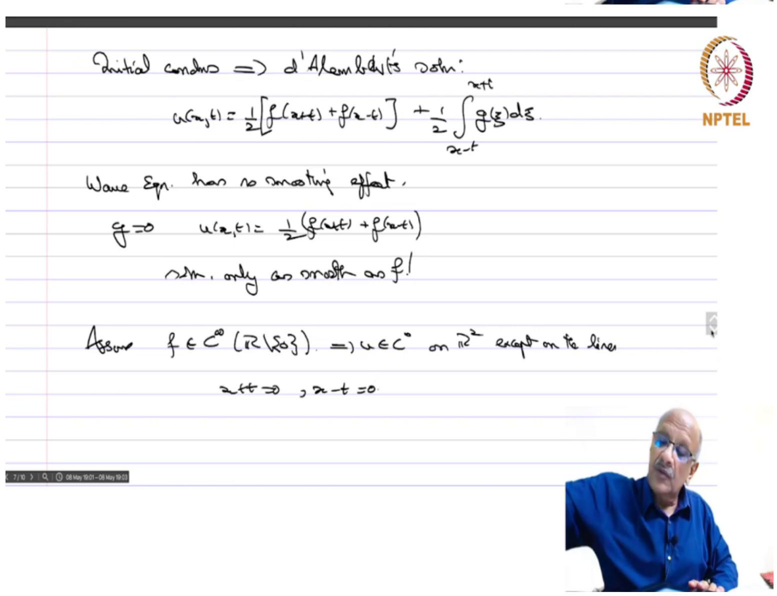

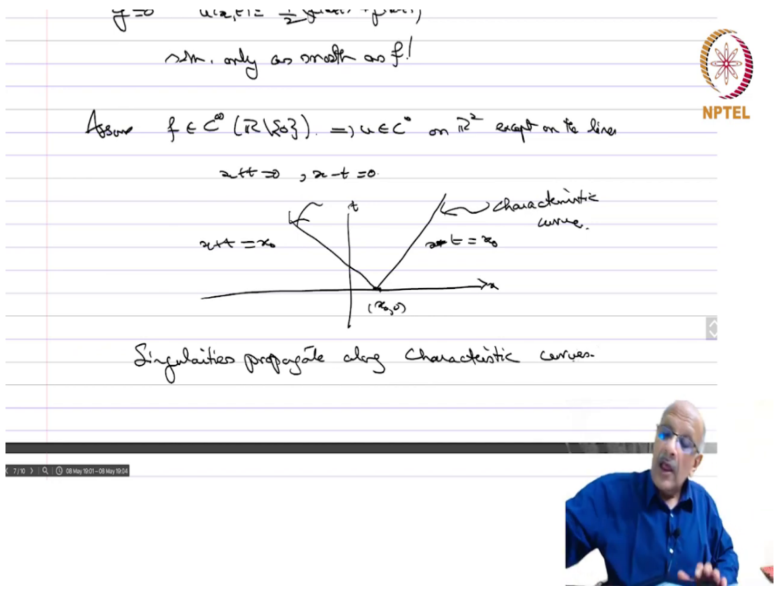

So, now just we saw in the case of the heat equation let us take omega to be $\mathbb{R}^N$. So, we have

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0 \quad in\ \mathbb{R}^N \times (0,\infty)$$

$$u(x,0) = f(x) \quad for\ \ x \in \mathbb{R}^N,$$

$$\frac{\partial u}{\partial t}(x,0) = g(x) \quad for\ \ x \in \mathbb{R}^N.$$

So, now let us I will concentrate on the case $N = 1$.

So, if you will have

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

So, then if you put $x + t = \xi$ and $x - t = \eta$ said you get

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

this implies that $u(x,t) = F(\xi) + G(\eta),\ so$

$$u(x,t) = F(x+t) + G(x-t).$$

And now if you impose the initial conditions implies we have the d Alembert's solution

$$u(x,t) = \frac{1}{2}(f(x + t) + f(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} g(\xi)\, d\xi$$

so, this is the solution. So, in the one dimensional case we can immediately see that wave equation has no smoothing effect for instance.

If you take g equal to 0 then you get $u(x,t) = \frac{1}{2}(f(x + t) + f(x - t))$ So, the solution is only as smooth as f it is not various in the heat equation instantaneously it became a C^{∞} function here if you have this solution does not change at all and also assume f belongs to C infinity on $C^{\infty}(\mathbb{R}\setminus\{0\})$ then the same place u is $C^{\infty}(\mathbb{R}^2)$ except on the lines $x - t = 0$ and $x + t = 0$. So, these are called the characteristic curves so, you have here x and here you have t and then you have a point x naught 0 and then you have these lines here x plus t equal to con $x - t = x_0$ and $x + t = x_0$ these are the constant lines. So, if you have a singularity at this point the singularity will persist on these things this are called the characteristic curves. So, singularities propagate along characteristic curves.

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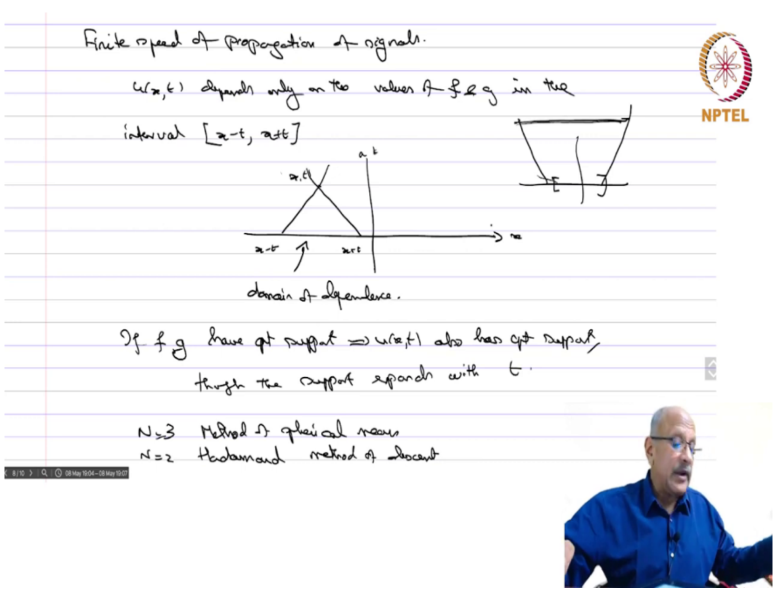

Another important difference between the wave equation and the heat equation is finite speed of propagation of signals. So, this function if you have unlike the thing so, o of xt depends only on the values of $f, g \in [x - t, x + t]$. So, if you look at the formula you can see that immediately it depends on the values of f at $x + t$, $x - t$ and g on the $[x - t, x + t]$ now nothing else depends.

So, this is called the domain of dependence. So, if you have x and t and you have a point x t here, then you draw the characteristic curves which pass through these points. So, this will be $x - t$ and this will be x plus t and then this is called the domain of dependence then if f and g how compact support this implies $u(x, t)$ it also has compact support though the support expands with t.

So, if you had for instance f and g are confined supported here, then you will have to take the characteristic curves coming. So, $u(x, t)$ will be nonzero only in this region, but that keeps expanding and therefore, you will have the solution. Now, one can write down explicitly the solution in N equals 3.

So, this is called the method of spherical means. And for $N = 2$ this is called the Hadamarad method of descent which means you write it for $N = 3$ and then you restrict it to the case where it is only two dimensional and one can study the properties of this solution of the wave equation. These are described in the book topics in functional analysis you can take a look and more or less the essential properties we have discussed already here. And so, one can wind up this section with that